k-Pseudosnakes in large grids*

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Abstract

We study the problem of finding maximum induced subgraphs of bounded maximum degree k—so-called "k-pseudosnakes"— in D-dimensional grids with all side lengths large. We prove several asymptotic upper bounds and give several lower bounds based on constructions. The constructions turn out to be asymptotically optimal for every D when $k = 0, 1, D, 2D-2, 2D-1$ and $2D$.

1 Introduction

Many combinatorial optimization problems can be reformulated as the searching of a maximum independent set of a graph G, i.e. a set S of vertices inducing an edgeless subgraph. What happens if this total isolation of the vertices is not required, but instead we ask that every vertex of S is adjacent to at most k vertices of S in G?

Following the terminology of [5], we call an induced subgraph $G[S]$ of a graph G a k-*pseudosnake* if its maximum degree is at most k. The maximum size of the vertex set of a k-pseudosnake in a graph G is denote by $\alpha_k(G)$. In [7] it is shown that $\alpha_k(G) \geq \sum_{v \in G}(d_G(v) + \frac{1}{k})^{-1}$. Some authors have studied the relation of k-pseudosnakes with other properties such as the chromatic number [1] and the domination number [8]. For the particular graph consisting on the product of D complete graphs on n vertices K_n^D, good bounds have been obtained in [3]: $\alpha_k(K_n^D) \leq (1 + \frac{1}{D-1})n^{D-1}$ for $n \geq 2$ and $D \geq 2$, while $\alpha_k(K_n^D) \geq n^{D-1} + n^{\frac{D}{2}}$ for $n \geq 3$ and $D \geq 2$.

Nevertheless, most of the work has been devoted to the problem of searching 2-pseudosnakes in highly regular graphs (and in fact many results have been obtained for the even more restricted case of 2-pseudosnakes with induced degree exactly 2 [2, 6, 13, 4, 11, 14]). A basic sharp result is known for the D-dimensional hypercube Q^D. In [5] it is proved that $\alpha_2(Q^3) = 6$, $\alpha_2(Q^4) = 9$ and $\alpha_2(Q^D) = 2^{D-1}$for $D \geq 5$.

Since the largest 2-pseudosnake in the D-dimensional hypercube contains half of the vertices, the following is a natural question: How much the density of the 2-pseudosnake can increase if we consider grids instead of hypercubes? Notice that hypercubes are special kind of grids. Indeed, the grid P_n^D is the Cartesian product of D paths P_n on n vertices, and the hypercube Q^D is P_2^D.

Let us define the asymptotic density λ_k^D of k-pseudosnakes in D-dimensional grids as follows

$$\lambda_k^D = \overline{\lim}_{n\to\infty} \frac{\alpha_k(P_n^D)}{n^D}$$

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The goal of this paper is to find sharp bounds for λ_k^D. In Section 2 we study upper bounds (we include some results valid for arbitrary graphs). In particular, we prove that $\lambda_2^3 \le 0.5373$ and $\lambda_2^4 \le 0.5128$. In Section 3 we exhibit constructions that yield lower bounds. In particular, we prove that $\lambda_2^3 \ge 0.5092$ and $\lambda_2^4 \ge 0.5008$.

2 Upper Bounds

Before concentrating on grids, let us establish some inequalities that hold for general graphs.

Proposition 1. *For every graph G and every integer $k \ge 1$*

$$\alpha_{k+1}(G) \ge \alpha_k(G) \ge \begin{cases} \frac{3}{5}\alpha_{k+1}(G) - 3 & \text{for } k \le 5 \\ \frac{k+1-\ln(k+2)}{k+2}\alpha_{k+1}(G) - 1 - \ln(k+2) & \text{for } k > 5 \end{cases}$$

Proof: The first inequality is obvious since every k-pseudosnake is also a $k+1$-pseudosnake. For the second one, let S be a $(k+1)$-pseudosnake in G. We add some complete graph K_{k+2} to $G[S]$, and join vertices of S to sufficiently many vertices of K_{k+2} such that each vertex of S has degree $k+1$ in the resulting graph G'. Certainly this graph has minimum degree $k+1$, so we can apply theorems in [9] and [10], assuring that G' has a dominating set U of cardinality at most $\frac{2(k+2+|S|)}{5}$ for $k \le 5$ and $\frac{(k+2+|S|)(1+\ln(k+2))}{k+2}$ for $k > 5$. By the construction of G', deleting U in $G[S]$ results in a graph of maximum degree at most k. This graph has at least $\frac{3|S|-2k-4}{5}$ vertices for $k \le 5$ and at least $\frac{|S|(k+1-\ln(k+2))-(k+2)(1+\ln(k+2))}{k+2} \ge \frac{k+1-\ln(k+2)}{k+2}\alpha_{k+1} - 1 - \ln(k+2)$ vertices for $k > 5$. □

Proposition 2. *Let G be a graph where $\alpha|V|$ of the vertices have degree smaller than $\Delta(G)$. Then*

$$\alpha_k(G) \le \frac{\Delta(G) + \alpha(\Delta(G) - \delta(G))}{2\Delta(G) - k}|V|.$$

Proof: Let S be a k-pseudosnake in G. Let m denote the number of edges between S and $V \setminus S$. Let $S' = \{x \in S : d_G(x) < \Delta(G)\}$. Since each $x \in S$ has at least $d_G(x) - k$ neighbors in $V \setminus S$, we get

$$\begin{aligned} m &\ge \sum_{x \in S}(d_G(x) - k) \\ &\ge |S'|(\delta(G) - k) + |S \setminus S'|(\Delta(G) - k) \\ &= |S|(\Delta(G) - k) + |S'|(\delta(G) - \Delta(G)) \\ &\ge |S|(\Delta(G) - k) + \alpha|V|(\delta(G) - \Delta(G)). \end{aligned}$$

On the other hand, each vertex y in $V \setminus S$ has at most $d_G(y)$ neighbors in S. Thus $m \le (|V| - |S|)\Delta(G)$. Putting both inequalities together, we get $|S| \le \frac{\Delta(G)+\alpha(\Delta(G)-\delta(G))}{2\Delta(G)-k}|V|$.
□

Previous properties yield the following results for grids.

Proposition 3. *For every integers $D \ge 1$ and $1 \le k \le 2D$,*

1. $\lambda_{k+1}^D \ge \lambda_k^D$
2. $\lambda_k^D \ge \begin{cases} \frac{3}{5}\lambda_{k+1}^D & \text{for } k \le 5 \\ \frac{k+1-\ln(k+2)}{k+2}\lambda_{k+1}^D & \text{for } k > 5 \end{cases}$

3. $\lambda_k^D \geq \lambda_k^{D+1}$

Proof: (1) and (2) follow from Proposition 1. For (3), take some maximum k-pseudosnake S in some grid P_n^{D+1} By the pigeonhole principle, some of the sets $S_i := \{(x_1, x_2, \ldots, x_{D+1}) \in S : x_{D+1} = i\}$ has at least $\frac{|S|}{n}$ elements and it induces a k-pseudosnake in P_n^D. $\square$

It is easy to see that $\lambda_k^D = \overline{\lim}_{n\to\infty} \alpha_k(C_n^D)$ where C_n is the cycle with n vertices. Therefore, we can assume that our graph is not a grid but a torus. In particular, from Proposition 2 we have the following.

Corollary 4. *For all integers $D \geq 1$ and $1 \leq k \leq 2D$, $\lambda_k^D \leq \frac{2D}{4D-k} = \frac{1}{2-\frac{k}{2D}}$.*

By making use of the special structure of our graphs, the previous bound can be improved for all cases $k < D$. We need the following terminology: Any given k-pseudosnake S in a graph G partitions the set of edges into the set C of those edges contained in $G[S]$, the set T of those edges touching exactly one vertex in S and the set N of those edges with no end in S.

In our graphs we call two edges *parallel* if they are opposite edges of some 4-cycle.

Theorem 5. *For every $k < D$ we have that $\lambda_k^D \leq \frac{2(D-1)D}{4D(D-1)-k(k-1)} = \frac{1}{2-\frac{k}{2D}\frac{k-1}{D-1}}$.*

Proof: Let S be a k-pseudosnake in the torus $G = C_n^D$. The graph G is $2D$-regular and the incident edges with a vertex $x \in S$ are either contained in S ($e \in C$) or touched by S ($e \in T$). More precisely,

$$2D|S| = \sum_{x\in S} d_G(x) = 2|C| + |T|.$$

Since N, T, C is a partition of the edges in V we have that $D|V| = |N| + |T| + |C|$. Therefore

$$\frac{|S|}{|V|} = \frac{2|C| + |T|}{2|N| + 2|T| + 2|C|} = \frac{1}{1 + \frac{2|N|+|T|}{2|C|+|T|}}$$

Since S induces a k-pseudosnake in G, every vertex x of S meet at most k members of C. Since members of C are counted twice in this way, we obtain $2|C| \leq k|S|$. Moreover, every vertex x of S meet at least $2D - k$ members of T which implies that $(2D - k)|S| \leq |T|$. Both inequalities combined yield

$$2a|C| \leq |T|$$

where $a = \frac{2D}{k} - 1$.

Let xy be an edge in C, i.e. $x, y \in S$. If t denotes the number of parallel edges of xy in T, and c the number of parallel edges of xy in C, then $2k \geq d_{G[S]}(x) + d_{G[S]}(y) \geq 2 + t + 2c$. Since each edge of G has $2(D-1)$ parallel edges we have that every edge in C has at least $2(D-1) - t - c \geq 2(D-1) - (2k-2) = 2(D-k)$ parallel edges in N. Therefore $2(D-k)|C| \leq 2(D-1)|N|$ and

$$b|C| \leq |N|$$

where $b = \frac{D-k}{D-1} = 1 - \frac{k-1}{D-1} < 1$.

Let $e = \frac{a+b}{a+1}$. Then $e < 1$ and it holds that $e = b + a(1-e)$. We get the following bounds

$$2|N| + |T| = 2|N| + (1-e)|T| + e|T| \geq 2b|C| + 2a(1-e)|C| + e|T| = e(2|C| + |T|)$$

Therefore $\frac{2|N|+|T|}{2|C|+|T|} \geq e$ and then

$$\frac{|S|}{|V|} \leq \frac{1}{1+e} = \frac{1}{2-(1-e)} = \frac{1}{2 - \frac{1}{a+1}(1-b)} = \frac{1}{2 - \frac{k}{2D}\frac{k-1}{D-1}} = \frac{2D(D-1)}{4D(D-1) - k(k-1)}$$

□

The main idea in the proof of Theorem 5 was to concentrate on the N-parallels of a given edge f. Now we are going to refine this method by having a closer look on the parallels in C for those edges in N parallel to f.

For instance, for $D = 3$ and $k = 2$, although every edge in N may have up to 4 parallels in C, this is impossible for both N-parallels of $f \in C$ in the situation given in Figure 1. (In the other situation where f has just two N-parallels shown in Figure 1, both these parallels may have 4 C-parallels each, so we need to look into this a little more carefully, see the proof of Theorem 7.)

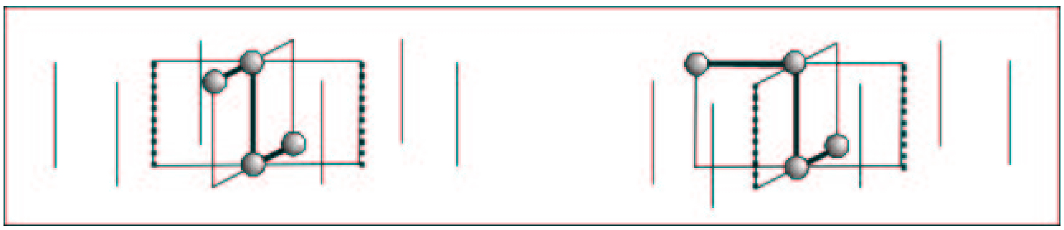

Figure 1: a) The two N-parallels to f can not have four C-parallels. b) The two N-parallels to f can have four C-parallel.

The following Theorem improves the bound of Theorem 5 for the cases $k \leq \frac{2}{3}D$.

Theorem 6. *Let $k < D$. Then*

$$\lambda_k^D \leq \frac{1}{2 - \frac{k}{2D}\varphi(D,k)}.$$

where $\varphi(D,k) = \frac{k^2}{(2(D-1)-k)^2+k^2}$.

Proof: The bound is similar to those already obtained in Corollary 4 and Theorem 5. The reader should remark that we only have to prove that

$$|N| \geq \left(1 - \frac{k(k-1)}{2(D-k)^2 + k(k-1)}\right)|C|$$

For an edge f let us denote P_f the set of all parallel edges to f. We use the abbreviations $N_f := P_f \cap N$, $C_f := P_f \cap C$, $T_f := P_f \cap T$, $\lambda := |N_f|$ and $\mu := 2(D-1)$.

Since for every $f \in C$ and $h \in N$ we have that $h \in N_f$ if and only if $f \in C_h$, we deduce that

$$|N| = \sum_{h,h\in N}\sum_{f,f\in C_h}\frac{1}{|C_h|} = \sum_{f,f\in C}\sum_{h,h\in N_f}\frac{1}{|C_h|}$$

If for all $f \in C$ we have that $\sum\limits_{h,h\in N_f}\frac{1}{|C_h|} \geq b$, where b does not depend on f, we would obtain $|N| \geq b|C|$.

Since for every set of n positive numbers we have that $(\sum\limits_{i=1}^{n} x_i)(\sum\limits_{i=1}^{n} x_i^{-1}) \geq n^2$, in order to find a lower bound for $\sum\limits_{h,h\in N_f}\frac{1}{|C_h|}$ we have to find an upper bound for $\sum\limits_{h,h\in N_f}|C_h|$. This quantity is the number of edges in C parallel to some parallel edge of f.

Since an edge g in T_f has at most $k-1$ parallels in C, among all edges parallel to g and to some h in N_f, at least $\lambda - 1 - (k-1)$ are not in C. Since an edge in C_f has at most $k-2$

parallels in C, at least $\lambda - 1 - (k-2)$ edges parallel to g and to some $h \in N_f$ are not in C. Clearly, these quantities are positive for $k \leq \frac{2}{3}(D+1)$ since $\lambda \geq \mu - 2(k-1)$. Moreover, we have $|T_f| + |C_f| = \mu - \lambda$. Therefore

$$\begin{aligned}\sum_{h, h \in N_f} |C_h| &\leq \mu\lambda - |T_f|(\lambda - 1 - (k-1)) - |C_f|(\lambda - 1 - (k-2)) \\ &\leq \mu\lambda - (\mu - \lambda)(\lambda - k) \\ &= \lambda^2 + k(\mu - \lambda)\end{aligned}$$

Therefore we get

$$\sum_{h, h \in N_f} \frac{1}{|C_h|} \geq \frac{\lambda^2}{\lambda^2 + k(\mu - \lambda)}$$

The right hand side term is monotonically increasing for $\lambda = \mu - 2(k-1), ..., \mu$. Whence

$$|N| \geq \frac{(\mu - 2(k-1))^2}{(\mu - 2(k-1))^2 + 2k(k-1)}|C| = (1 - \frac{k(k-1)}{2(D-k)^2 + k(k-1)})|C|$$

□

For $D = 3, k = 2$, we can even improve the bound of $6/11 \approx 0.545$ obtained in Theorem 5 and Theorem 6 slightly:

Theorem 7. $\lambda_2^3 \leq 36/67 \approx 0.537$.

Proof: We will show that for every two incident edges f and f' in C we have

$$s(f, f') := \sum_{h, h \in N_f} \frac{1}{|C_h|} + \sum_{h, h \in N_{f'}} \frac{1}{|C_h|} \geq \frac{7}{6}$$

If $|N_f| + |N_{f'}| \geq 5$ then $s(f, f') \geq \frac{5}{4} \geq \frac{7}{6}$, since $|C_h| \leq 4$. Otherwise $|N_f| + |N_{f'}| = 4$. In this situation f and f' are edges of some 4-cycle. If there are two adjacent edges h and h' with $h \in N_f$ and $h' \in N_{f'}$ then $|C_h| \leq 3$ and $|C_{h'}| \leq 3$ which gives $s(f, f') \geq \frac{7}{6}$. So we can assume that h and h' are not adjacent for any pair h, h' with $h \in N_f$ and $h' \in N_{f'}$.

To study this situation we need some notations. Let us denote by h_1, h_2, h_3, h_4 the parallel edges to f and by h'_1, h'_2, h'_3, h'_4 those parallel to f'. Moreover, let us assume that $h_1, h_2 \in N_f$, $h'_1, h'_2 \in N_{f'}$, h_3 is the edge adjacent to f' and h'_3 is the edge adjacent to f. Additionally, let us assume that the edge h_1 belongs to the plane defined by f and h_3 and that h'_1 belongs to the plane defined by f' and h'_3. Remember that we are assuming that h_2 and h'_2 are not adjacent. Let g be the edge different from f in $P_{h_2} \cap P_{h_3}$. If $g \in C$ then the edge r' different from f' in $P_{h'_1} \cap P_{h'_4}$ can not belong to C, due to the degree constrain in the vertex incident with g and h'_4. Therefore $|C_{h_2}| + |C_{h'_1}| \leq 7$. Similarly we can show that $|C_{h'_1}| + |C_{h_2}| \leq 7$ which implies that $s(f, f') \geq \frac{7}{6}$.

Finally we have that $2|N| = \sum_{f, f' \in C} s(f, f') \geq \frac{7}{6}|C|$. Thus

$$|N| \geq \frac{7}{12}|C|$$

and we obtain the bound $\lambda_2^3 \leq \frac{1}{2 - \frac{2}{6}\frac{5}{12}} = \frac{36}{67}$ as indicated in the proof of Theorem 5. □

3 Lower bounds

We start this section by showing some fairly simple constructions which imply that, at least for certain cases, the previously given bounds are sharp.

Let us label the vertices of the torus $G = C_n^D$ by $(x_1, x_2, \ldots, x_D)$ with $0 \leq x_i \leq n-1$ and $i = 1, \ldots, D$. Three canonical induced subgraphs of the D-dimensional torus will turn out to be optimal solutions for certain k's. The first graph is G itself and we have $\lambda_{2D}^D = 1$. The second graph is the checkerboard subgraph. Let S be the set of those vertices $(x_1, x_2, \ldots, x_D)$ of G for which $\sum_{i=1}^D x_i$ is odd. This set is independent and contains asymptotically half of the vertices of the torus. The optimality of this construction for $k = 0$ and $k = 1$ follows from Theorem 5. More precisely,

Corollary 8. $\lambda_0^D = \lambda_1^D = 1/2$ *for* $D \geq 2$.

The third subgraph, $G[S]$, is induced by the set S of all the vertices $(x_1, x_2, \ldots x_D)$ for which the sum $\sum_{i=1}^D x_i$ of its coordinates equals 0 or 1 modulo 3. Surely $|S|/|V| \to 2/3$ for $n \to \infty$. Now consider $x = (x_1, x_2, \ldots, x_D) \in S$. If $\sum_{i=1}^D x_i = 0 \pmod 3$, then each neighbor $(x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_D)$ is in S, but each neighbor $(x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_D)$ is not. Thus x has exactly one neighbor in S in each dimension. The case $\sum_{i=1}^D x_i = 1 \pmod 3$ is analogous. Using Corollary 4 we get

Corollary 9. *For every D we have* $\lambda_D^D = \frac{2}{3}$.

For getting more general bounds let us fix $p > D$ and $a = (1, 2, \ldots, D)$. Let us define the linear function $f(x) = xa^t = \sum_{i=1}^D ix_i$, where $x = (x_1, x_2, \ldots, x_D) \in G = C_n^D$ and all the calculations are taken modulo p. For each $0 \leq i \leq p-1$, we define S_i as the set of vertices corresponding to $f^{-1}(\{i\})$. Since $p > D$, every S_i is independent. On the other hand, all the sets of the partition $S_0, S_1, \ldots, S_{p-1}$ have "almost" the same cardinality. Moreover, for the n's which are multiple of p, all the sets satisfy $|S_i| = \frac{n^D}{p}$. To see this, notice that if we fix $x_2, \ldots, x_D$ then the value of x_1 will determine to which class the vertex $x = (x_1, x_2, \ldots, x_D)$ belongs. We will consider the particular cases for which $p = 2D+1$ and $p = D+1$.

Theorem 10.

1. *For every integer k, $1 \leq k \leq 2D$,* $\lambda_k^D \geq \frac{k+1}{2D+1}$.
2. *For every integer k, $1 \leq k \leq D$,* $\lambda_{2k}^D \geq \frac{k+1}{D+1}$.

Proof: For the first inequality let $p = 2D+1$. Let n be a multiple of $2D+1$ and let $G = C_n^D$. Let us consider the graph $G[\bigcup_{i=0}^k S_i]$. Its maximum degree is at most k. In fact, let $x \in S_i$ and let $i' \neq i$. In order to find a vertex $x' \in S_{i'}$ such that $xx' \in E(G)$, we have to solve the equation $i' = i + y \pmod{2D+1}$, where $y \in \{1, \cdots, D\} \cup \{-1, \cdots, -D\}$. It's easy to notice that this equation admits only one solution. It follows that the number of vertices is $(k+1)n^D/2D+1$ and the first inequality is concluded.

We can improve the bound when the pseudosnake degree is bounded by an even number by taking $p = D+1$. In fact, in this case the maximum degree of the graph $G[\bigcup_{i=0}^k S_i]$ is at most $2k$: The equation $i' = i + y \pmod{D+1}$, where $y \in \{1, \cdots, D\} \cup \{-1, \cdots, -D\}$, admits two solutions. It follows that the number of vertices is $(k+1)n^D/D+1$ and the second inequality is concluded. □

Using again Corollary 4 we have

Corollary 11. $\lambda_{2D-1}^D = \frac{2D}{2D+1}$, *and* $\lambda_{2D-2}^D = \frac{D}{D+1}$.

We know from Theorem 10 that, for every $1 \le k \le D$, $\lambda_{2k}^D \ge \frac{k+1}{D+1}$. But we can remove one third of the vertices of the pseudosnake in such a way that the degree is reduced to half the original, i.e:

Theorem 12. *For every integer* $1 \le k \le D$, $\lambda_k^D \ge \frac{2}{3}(\frac{k+1}{D+1})$.

Proof: We are in the context of the second construction of Theorem 10. Let us consider the sets $A_l \subseteq \bigcup_{i=0}^k S_i$ of those x's for which $\sum_{i=1}^D x_i = l \pmod 3$ where $l = 0, 1, 2$. Without loss of generality we can assume that $|A_0| \le (\frac{1}{3})n^D$ (otherwise we choose the sum to be 1 or 2 instead of 0). All we need to prove is that for any vertex belonging to either A_1 or A_2, at least half of its neighbors belong to A_0. Therefore, by removing A_0, we reduce in one half the degree of the graph. Let $x = (x_1, x_2, x_3, \ldots, x_D) \in A_1 \cap S_r$. It follows that, for each $r' \ne r$, x has two neighbors in $S_{r'}$: $(x_1, \ldots, x_{r'-r} + 1, \ldots x_D) \in A_2$ and $(x_1, \ldots, x_{r'+r} - 1, \ldots x_D) \in A_0$. □

Since $\frac{2}{3}(\frac{k+1}{D+1}) \ge \frac{k+1}{2D+1}$ for all $D \ge 1$ previous result is an improvement of our bound in Theorem 10 for all $k \le D$.

We will now concentrate on torus-hypercube constructions. The idea is very simple: k-pseudosnakes for tori C_n^D yield k-pseudosnakes for tori $C_{\alpha n}^D$. Let us recall that Q^D denotes the D-dimensional hypercube.

Proposition 13. *Let G be a fixed graph. Let $D \ge 1$. Assume that S as well as its complement $\overline{S}$ induce k-pseudosnakes in $Q^D \times G$. Then there exist complementary k-pseudosnakes S' and $\overline{S'}$ in $C_6^D \times G$ such that $|S'| \ge |S| + (3^D - 1)2^{D-1}|V(G)|$.*

Proof: By induction on the dimension D. Let D=1. We know that both $S \subseteq \{0,1\} \times V(G)$ and its complement $\overline{S} = (\{0,1\} \times V(G)) \setminus S$ induce k-pseudosnakes in $Q^1 \times G$. It is useful to see $Q^1 \times G$ as "two layers of G". Let S_0 and S_1 be the part of S in each layer. In other words, $S_0 = \{x \in V(G) : (0, x) \in S\}$ and $S_1 = \{x \in V(G) : (1, x) \in S\}$. Notice that $|S_0| + |S_1| = |S|$. Since $C_6^D \times G$ is a six layers graph, the k-pseudosnake S' will be constructed as a six layers pseudosnake. More precisely, $S_0' = S_0$, $S_1' = S_1$, $S_2' = \overline{S_1}$, $S_3' = \overline{S_0}$, $S_4' = \overline{S_1}$, $S_5' = S_1$. It is easy to notice that both S' and $\overline{S'}$ are k-pseudosnakes. On the other hand, $|S'| = |S_0| + |S_1| + |\overline{S_1}| + |\overline{S_0}| + |\overline{S_1}| + |S_1| = 2|V(G)| + |S|$.

Let us assume the proposition true for D. Let S induce a k-pseudosnake in $Q^{D+1} \times G$ such that $\overline{S}$ is also a k-pseudosnake. Since $Q^{D+1} \times G = Q^D \times (Q^1 \times G)$ then, by the induction hypothesis, we know the existence of a k-pseudosnake S' in $C_6^D \times (Q^1 \times G)$ such that $|S'| \ge |S| + (3^D - 1)2^{D-1}|V(Q^1 \times G)|$ and with $\overline{S'}$ in $C_6^D \times (Q^1 \times G)$ also being a k-pseudosnake. Since $C_6^D \times (Q^1 \times G) = Q^1 \times (C_6^D \times G)$ we can apply the result for the case $D = 1$ and deduce the existence of a k-pseudosnake S'' in $C_6^{D+1} \times G$ with $\overline{S''}$ also being a k-pseudosnake and such that

$$\begin{aligned} |S''| &\ge |S'| + 2|V(C_6^D \times G)| \\ &\ge |S| + (3^D - 1)2^{D-1}|V(Q^1 \times G)| + 2|V(C_6^D \times G)| \\ &= |S| + (3^D - 1)2^{D-1}2|V(G)| + 2 \cdot 6^D|V(G)| \\ &= |S| + (3^{D+1} - 1)2^{D+1-1}|V(G)|. \end{aligned}$$

□

Therefore, lower bounds for k-pseudosnakes in Q^D would be highly desirable. For $k = 2$, everything is known:

Theorem 14 (Danzer/Klee [5]). $\alpha_2(Q^3) = 6$, $\alpha_2(Q^4) = 9$. *For every* $D \ge 5$, $\alpha_2(Q^D) = 2^{D-1}$.

Corollary 15. $\alpha_2(C_6 \times C_6 \times C_6) \ge 110$ *and* $\alpha_2(C_6 \times C_6 \times C_6 \times C_6) \ge 649$.

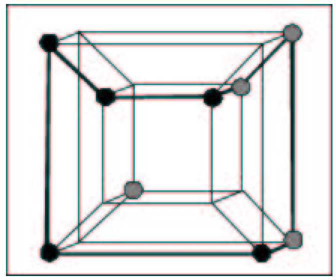

Figure 2: A 9-vertex pseudosnake in the cube Q^4.

Proof: We apply Proposition 13. For the first equation we use the 6-vertex 2-pseudosnake in Q^3, whose complement turns out to be also a 2-pseudosnake. For the second equation we take the 9-vertex 2-pseudosnake in Q^4, shown in Figure 2. □

Corollary 16. *$\lambda_2^3 \geq 110/216 \approx 0.509$, and $\lambda_2^4 \geq 649/1296 \approx 0.5008$.*

For arbitrary $k > 2$, k-pseudosnakes in hypercubes have not yet been investigated. But for every k-pseudosnake S in Q^D, the set $S' = \{0,1\} \times S$ is a $(k+1)$-pseudosnake in Q^{D+1} and $|S'| = 2|S|$. Therefore each useful lower bound for $\alpha_k(Q^D)$ yields lower bounds for all $\alpha_{k+t}(Q^{D+t})$. Whence $\alpha_{2+t}(Q^{3+t}) \geq 2^t{\cdot}6$ and $\alpha_{2+t}(Q^{4+t}) \geq 2^t{\cdot}9$, for each $t \geq 0$. The second family of inequalities can be improved. In fact, the construction of Figure 3 gives the lower bounds $\alpha_{3+t}(Q^{5+t}) \geq 2^t \cdot 20$ for each integer $t \geq 0$.

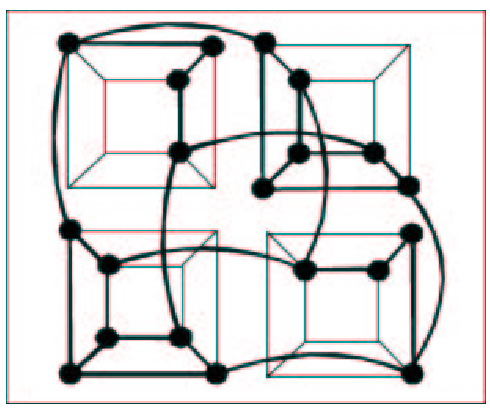

Figure 3: A 20-vertex 3-pseudosnake in Q^5.

Corollary 17. *For every $t \geq 0$, $\lambda_{2+t}^{3+t} \geq \frac{2^{t+1}+3*6^{2+t}}{6^{3+t}}$ and $\lambda_{3+t}^{5+t} \geq \frac{2^{t+2}+3*6^{4+t}}{6^{5+t}}$.*

The first bounds are much weaker than that in Theorem 12. The second bounds are very close to 0.5 but for three values still the best we have at the moment. We get $\lambda_3^5 \geq 3892/7776 \approx 0.5005$, $\lambda_4^6 \geq 23336/46656 \approx 0.50017$, $\lambda_5^7 \geq 139984/279936 \approx 0.500057$, and the others are again much weaker than the bound in Theorem 12.

4 Conclusion

The current lower and upper bounds for λ_k^D for $D \leq 7$ can be seen in the following table.

k	$D=1$	$D=2$	$D=3$	$D=4$	$D=5$	$D=6$	$D=7$
0	.5	.5	.5	.5	.5	.5	.5
1	.667	.5	.5	.5	.5	.5	.5
2	1	.667	.509 — .537	.5008 — .5128	.5 — .505	.5 — .502	.5 — .501
3		.8	.667	.533 — .571	.5005 — .5344	.5 — .516	.5 — .509
4		1	.75	.667	.556 — .588	.50017 — .556	.5 — .530
5			.857	.667 — .727	.667	.571 — .6	.500057 — .568
6			1	.8	.667 — .714	.667	.583 — .609
7				.889	.727 — .769	.667 — .706	.667
8				1	.833	.714 — .75	.667 — .7
9					.909	.769 — .8	.667 — .737
10					1	.857	.75 — .778
11						.923	.8 — .824
12						1	.875
13							.933
14							1

Notice that from Theorem 5 and Theorem 10 we have

$$\lambda^{D}_{2D-3} \geq \frac{2(D-1)}{2D+1}$$

and

$$\lambda^{D-1}_{2(D-1)-3} \leq \frac{2(D-1)}{4(D-1)-2(D-1)+3} = \frac{2(D-1)}{2D+1}$$

which explains some regularities in previous table.

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