

Strict majority bootstrap percolation in the r -wheel*

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Abstract

In the strict Majority Bootstrap Percolation process each passive vertex v becomes active if at least $\lceil \frac{\deg(v)+1}{2} \rceil$ of its neighbors are active (and thereafter never changes its state). We address the problem of finding graphs for which a small proportion of initial active vertices is likely to eventually make all vertices active. We study the problem on a ring of n vertices augmented with a “central” vertex u . Each vertex in the ring, besides being connected to u , is connected to its r closest neighbors to the left and to the right. We prove that if vertices are initially active with probability $p > 1/4$ then, for large values of r , percolation occurs with probability arbitrarily close to 1 as $n \rightarrow \infty$. Also, if $p < 1/4$, then the probability of percolation is bounded away from 1.

Keywords: bootstrap percolation, interconnection networks.

1. Introduction

Consider the following deterministic process on a graph $G = (V, E)$. Initially, every vertex in V can be either *active* or *passive*. A passive vertex v becomes active iff at least k of its neighbors are already active; once active, a vertex never changes its state. This process is known as *k -neighbor bootstrap percolation* [4]. If at the end of the process all vertices are active, then we say that the initial set of active vertices *percolates*. We wish to determine the minimum ratio of initially active vertices needed to achieve percolation with high probability. More precisely, suppose that the elements of the initial set of active

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vertices $A \subseteq V$ are chosen independently with probability p . The problem is finding the least p for which percolation of A is likely to occur.

Since its introduction by Chalupa et al. [4], the bootstrap percolation process has mainly been studied in the d -dimensional grid $[n]^d = \{1, \dots, n\}^d$ [1]. The precise definition of *critical probability* that has been used is the following:

$$p_c([n]^d, k) = \inf\{p \in [0, 1] : \mathbb{P}_p(A \text{ percolates } [n]^d) \geq 1/2\}.$$

In [1] it is proved that, for every $d \geq k \geq 2$, $p_c([n]^d, k) = \left(\frac{\lambda(d,k)+o(1)}{\log_{(k-1)} n}\right)^{d-k+1}$, where $\lambda(d, k) < \infty$ are equal to the values of specific definite integrals for every $d \geq k \geq 2$. In the (*simple*) *Majority Bootstrap Percolation* (*simple MBP*) process (introduced in [2]) each passive vertex v becomes active iff at least $\lceil \frac{\deg(v)}{2} \rceil$ of its neighbors are active, where $\deg(v)$ denotes the degree of vertex v in G . Note that for $[n]^d$, the critical probability for simple MBP percolation corresponds to $p_c([n]^d, d)$, which goes to 0 as $n \rightarrow \infty$.

Here we introduce the *strict Majority Bootstrap Percolation* (*strict MBP*) process: each passive vertex v becomes active iff at least $\lceil \frac{\deg(v)+1}{2} \rceil$ of its neighbors are active. Note that if $\deg(v)$ is odd, then strict and simple MBP coincide. For $[n]^d$ the critical probability in strict MBP $p_c([n]^d, d+1)$ goes to 1. This holds because, in this case, any unit hypercube starting with its 2^d corners passive will stay passive forever.

A natural problem is finding graphs for which the critical probability in the strict MBP is small. Results by Balogh and Pittel [3] imply that the critical probability of the strict MBP for random 7-regular graphs is 0.269. In [6], two families of graphs for which the critical probability is also small (but higher than 0.269) are explored. The idea behind these constructions is the following. Consider a regular graph of even degree G . Let $G * u$ denote the graph G augmented with a single universal vertex u . The strict MBP dynamics on $G * u$ has two phases. In the *first phase*, assuming that vertex u is not initially active, the dynamics restricted to G corresponds to the strict MBP. If more than half of the vertices of G become active, then the universal vertex u also becomes active, and the *second phase* begins. In this new phase, the dynamics restricted to G follows the simple MBP (and full activation becomes much more likely to occur).

The two augmented graphs studied in [6] were the *wheel* $\text{WH}_n = u * R_n$ and the *toroidal grid plus a universal vertex* $\text{TWH}_n = u * R_n^2$ (where R_n is the ring on n vertices and R_n^2 is the toroidal grid on n^2 vertices). For a family of graphs $\mathcal{G} = (G_n)_n$, the following parameter was defined (as before, A denotes the initial set of active vertices):

$$p_c^+(\mathcal{G}) = \inf \left\{ p \in [0, 1] : \liminf_{n \rightarrow \infty} \mathbb{P}_p(A \text{ percolates } G_n \text{ in strict MBP}) = 1 \right\}.$$

Consider the families $\mathcal{WH} = (\text{WH}_n)_n$ and $\mathcal{TWH} = (\text{TWH}_n)_n$. It was proved in [6] that $p_c^+(\mathcal{WH}) = 0.4030\dots$. For the toroidal case it was shown that $0.35 \leq p_c^+(\mathcal{TWH}) \leq 0.372$. Computing the critical probability for the wheel is trivial. Nevertheless, if we increase the *radius* of the vertices, then the situation becomes much more complicated. More precisely, let $R_n(r)$ be the ring where every vertex is connected to its r closest vertices to the left and to its r closest vertices to the right. Here we study the strict MBP process in a generalization of the wheel that we call *r-wheel* $\text{WH}_n(r) = u * R_n(r)$. Our main result is the following:

Theorem 1. *The limit of $p_c^+(\mathcal{WH}(r))$, as $r \rightarrow \infty$, exists and equals $1/4$.*

2. Preliminary results

We start by showing that we can reduce our problem to the issue of whether a single fixed (non-universal) vertex eventually becomes active.

Lemma 2. *Let $0 < p < 1$ be the probability for a vertex to be initially active. Let r be a positive integer. Denote by $p_W(n, r, p)$ the percolation probability of the r -wheel and denote by $p_R(n, r, p)$ the probability that the strict majority on $R_n(r)$ ends up with (strictly) more active than passive vertices. Then,*

$$\begin{aligned} \liminf_{n \rightarrow \infty} p_R(n, r, p) &\leq \liminf_{n \rightarrow \infty} p_W(n, r, p), \\ \limsup_{n \rightarrow \infty} p_W(n, r, p) &\leq p + (1 - p) \cdot \limsup_{n \rightarrow \infty} p_R(n, r, p). \end{aligned}$$

Proof. Note that for $\epsilon > 0$ we can choose n large enough so that the probability that at least one block of r consecutive vertices is initially active is larger than $1 - \epsilon$, in which case percolation occurs iff the universal vertex becomes active during the evolution. We deduce the first inequality by taking ϵ arbitrarily small.

Note now that the universal vertex is active when the dynamics stabilizes only if it was either already active initially (probability p) or if it was initially passive and the dynamics on the ring $R_n(r)$ produces more than $n/2$ active vertices. \square

The vertices of the ring R_n will be denoted as $0, 1, \dots, n-1$, starting at some arbitrary vertex (arithmetic over vertex indices will always be modulo n). The positive integer r will be called the *radius*.

Lemma 2 shows that we can study the ring $R_n(r)$ and its dynamics to derive results about the r -wheel. Now, consider the 0-1 random variable $X_i(n, r)$ giving the state of vertex i after stabilization of the dynamics ($X_i(n, r) = 0$ if the state is passive, and $X_i(n, r) = 1$ if it is active). Next, we show how to bound $p_R(n, r, p)$ in terms of $\mathbb{E}_p(X_0(n, r))$.

Lemma 3. *Let $0 < p < 1$, $n \in \mathbb{N}^+$, and r a fixed radius. Then,*

$$2\mathbb{E}_p(X_0(n, r)) - 1 \leq p_R(n, r, p) \leq 2\mathbb{E}_p(X_0(n, r)).$$

Proof. By definition $p_R(n, r, p) = \mathbb{P}_p(\sum_i X_i(n, r) > n/2)$. By Markov's inequality we then have $\mathbb{P}_p(\sum_i X_i(n, r) > n/2) \leq \frac{2}{n}\mathbb{E}_p(\sum_i X_i(n, r))$. Using linearity of expectation and the fact that all $X_i(n, r)$ are equally distributed (symmetry of the ring), we deduce $p_R(n, r, p) \leq 2\mathbb{E}_p(X_0(n, r))$. The lower bound is obtained in the same way considering again Markov's inequality but for the (again positive) random variable $n - \sum_i X_i(n, r)$. More precisely:

$$p_R(n, r, p) = 1 - \mathbb{P}_p\left(n - \sum_i X_i(n, r) > n/2\right) \geq 1 - \frac{2}{n}\mathbb{E}_p\left(n - \sum_i X_i(n, r)\right) \quad \square$$

3. Lower bound on $p_c^+(\mathcal{WH}(r))$

We will assume $n > 2r + 1$ and that the initial state of the universal vertex u is passive. Let $0 < p < 1/2$ and $q = 1 - p$. The starting configuration $\sigma = (\sigma_0, \dots, \sigma_{n-1})$, where vertex j is initially active (respectively passive) if and only if $\sigma_j = 1$ (respectively $\sigma_j = 0$), occurs with probability $p^{\sum_j \sigma_j} q^{n - \sum_j \sigma_j}$. We write X_0 instead of $X_0(n, r)$. Conditioning on σ_0 ,

$$\mathbb{P}_p(X_0 = 1) \leq p + \mathbb{P}_p(X_0 = 1 | \sigma_0 = 0). \quad (1)$$

We say there is a *wall* located $\ell > 0$ vertices to the left of vertex 0 if $\sigma_{-\ell} = 1$, $\sigma_{-\ell-1} = \sigma_{-\ell-2} = \dots = \sigma_{-\ell-(r+1)} = 0$. Similarly, we say there is a wall located at $\ell > 0$ vertices to the right of vertex 0 if $\sigma_\ell = 1$, $\sigma_{\ell+1} = \sigma_{\ell+2} = \dots = \sigma_{\ell+(r+1)} = 0$. Let L (respectively R) be the smallest positive ℓ such that there is a wall located ℓ vertices to the left (respectively right) of vertex 0 (if a wall does not occur, let $L = R = n$). For $0 < \Delta < n$ to be fixed later, and since L and R are identically distributed, we have that:

$$\begin{aligned} & \mathbb{P}_p(X_0 = 1 | \sigma_0 = 0) \\ & \leq 2 \cdot \mathbb{P}_p(X_0 = 1, R \geq \Delta | \sigma_0 = 0) + \mathbb{P}_p(X_0 = 1 \wedge L, R < \Delta | \sigma_0 = 0) \end{aligned} \quad (2)$$

Summarizing, to bound $\mathbb{E}_p(X_0) = \mathbb{P}_p(X_0 = 1)$ we can bound the two terms in the right hand side of (2).

Lemma 4. *For $0 < p < 1$ and positive integers a, r ,*

$$\mathbb{E}_p(R | \sigma_0 = \sigma_1 = \dots = \sigma_{a-1} = 0, \sigma_a = 1) \leq q^{-r}(aq^r + 1/(pq)).$$

Proof. Consider a Markov chain with states labeled $0, 1, \dots, r+1$ where for all $s \leq r$, the probability of going from state s to 0 (respectively s to $s+1$) is p (respectively q), and once state $r+1$ is reached, the Markov chain stays there forever. For $s \in \{0, \dots, r+1\}$, let N_s be the number of steps it takes the Markov chain to reach state $r+1$ when it starts at state s . Note that

$$\mathbb{E}(R | \sigma_0 = \sigma_1 = \dots = \sigma_{a-1} = 0, \sigma_a = 1) \leq a + \mathbb{E}(N_0).$$

Moreover, $\mathbb{E}(N_{r+1}) = 0$, and $\mathbb{E}(N_s) = 1 + q \cdot \mathbb{E}(N_{s+1}) + p \cdot \mathbb{E}(N_0)$ for all $0 < s \leq r$. Thus, for all $0 \leq s \leq r+1$,

$$\mathbb{E}(N_0) = \sum_{j=1}^s \frac{1}{q^j} + \mathbb{E}(N_s) = \sum_{j=1}^{r+1} \frac{1}{q^j} \leq \frac{1}{pq^{r+1}}.$$

Putting everything together yields the result. \square

Corollary 5. *For $0 < p < 1$ and positive integers r, Δ ,*

$$\mathbb{P}_p(X_0 = 1, R \geq \Delta | \sigma_0 = 0) \leq \frac{1}{\Delta} \cdot q^{-r}(rq^r + 1/(pq)).$$

Proof. If vertex 0 eventually becomes active, it must be the case that initially it did not belong to a block of $r+1$ consecutive passive vertices. Thus, if $X_0 = 1$ and $\sigma_0 = 0$, then there must exist a positive integer a such that $a \leq r$,

$\sigma_1 = \sigma_2 = \dots = \sigma_{a-1} = 0$, and $\sigma_a = 1$. For brevity, we will denote this particular array of outcomes for the σ 's as C_a . By Markov's inequality,

$$\mathbb{P}_p(X_0 = 1, R \geq \Delta | \sigma_0 = 0) \leq \sum_{a=1}^r \mathbb{P}_p(R \geq \Delta | C_a) \mathbb{P}_p(C_a) \leq \frac{1}{\Delta} \max_{a=1, \dots, r} \mathbb{E}_p(R | C_a).$$

The desired conclusion follows from Lemma 4. \square

Lemma 6. *For $0 < p < 1/2$ and positive integers r, Δ ,*

$$\mathbb{P}_p(X_0 = 1 \wedge L, R < \Delta | \sigma_0 = 0) \leq 2\Delta (4pq)^r.$$

Proof. Suppose the closest wall to the left (respectively right) of vertex 0 is at vertex $-a$ (respectively b). Furthermore, suppose vertex 0 is passive. Note that for vertex 0 to eventually become active, it must be the case that some passive vertex i for $-a < i < b$ must necessarily become active the first time the strict majority dynamics is applied. Hence, if S_i denotes the number of j 's, $j \neq i$ and $i - r \leq j \leq i + r$, for which vertex j initially takes the value 1, then

$$\begin{aligned} \mathbb{P}_p(X_0 = 1 \wedge L, R < \Delta | \sigma_0 = 0) &\leq \mathbb{P}_p(\exists i, -\Delta < i < \Delta \text{ s.t. } S_i \geq r + 1) \\ &\leq 2\Delta \max_{i: -\Delta < i < \Delta} \mathbb{P}_p(S_i \geq r + 1). \end{aligned}$$

However, a Chernoff bound tells us that, for $t = 1/2 - p \leq (r + 1)/(2r) - p$,

$$\mathbb{P}_p(S_i \geq r + 1) \leq \mathbb{P}_p(S_i \geq (p + t) \cdot 2r) \leq \left(\left(\frac{p}{p + t} \right)^{p+t} \left(\frac{q}{q - t} \right)^{q-t} \right)^{2r} \leq (4pq)^r$$

Putting everything together yields the conclusion. \square

Theorem 7. *For all $0 < p < 1/4$ there exists a large enough integer $r_0 = r_0(p)$ such that if $r \geq r_0$ and $n > 2r + 1$, then $\mathbb{E}_p(X_0(n, r)) < 1/4$.*

Proof. Let $r'_0 = r'_0(p)$ be such that $r \geq r'_0$ implies that $rq^r \leq 1/pq$, and let $C = 8r/(pq)$. By Corollary 5, $\mathbb{P}_p(X_0 = 1, R \geq Cq^{-r} | \sigma_0 = 0) \leq \frac{2}{Cpq} = \frac{1}{4r}$. By (1), and fixing $\Delta = Cq^{-r}$ in (2), and Lemma 6, we obtain that:

$$\mathbb{P}_p(X_0 = 1) \leq p + \frac{1}{2r} + 2Cq^{-r}(4pq)^r = p + \frac{1}{2r} + 2C(4p)^r.$$

Hence, for $p < 1/4$ there exists a large enough positive integer $r_0 = r_0(p) \geq r'_0(p)$ so that if $r \geq r_0$, then $p + \frac{1}{2r} + 2C(4p)^r < p + \frac{1}{r} < 1/4$. \square

Theorem 7, Lemma 3, and Lemma 2 yield the following:

Corollary 8. $\liminf_{r \rightarrow \infty} p_c^+(W\mathcal{H}(r)) \geq 1/4$.

4. Upper bound on $p_c^+(\mathcal{WH}(r))$

Consider a simplified process with three states on the one-dimensional integer lattice \mathbb{Z} : (i) w , a wall, (ii) s , a spreading state, and (iii) e , an empty lattice point. Let sites in state w and s remain so forever, and in consecutive rounds let sites in state e with at least one neighbor in state s update to state s . Let p_w , p_s and p_e be positive initial probabilities of states w , s and e respectively, where $p_w + p_s + p_e = 1$. Each lattice point is initially assigned a state, independent of the other lattice point states.

Lemma 9. *The probability that lattice point 0 is eventually in state s is greater than $1/(1 + p_w/p_s)$.*

Proof. Define s_L (respectively s_R) to be state of the closest lattice point on the left (respectively right) of 0 whose state is not e . Since $p_e = 1 - p_w - p_s < 1$, both s_L and s_R are well-defined with probability 1. Let E be the event that lattice point 0 is eventually in state s , and denote by P its probability of occurring. Note that for E to occur, either the lattice point 0 is initially in state s , or it is initially in state e and at least one of the lattice points s_L or s_R is initially in state s . Hence, recalling that $p_e = 1 - p_w - p_s < 1$,

$$\begin{aligned} P &= p_s + p_e \sum_{i \geq 0, j \geq 0} p_e^{i+j} (p_s^2 + 2p_w p_s) = p_s + \frac{p_s(p_s + 2p_w)p_e}{(1 - p_e)^2} \\ &\geq p_s \left(1 + \frac{1 - p_w - p_s}{p_w + p_s} \right) = \frac{1}{1 + p_w/p_s}. \end{aligned}$$

□

We now consider again the strict MBP process in the ring $R_n(r)$ and reduce it to the aforementioned three-state model as follows: Fix some length ℓ and partition the vertices of $R_n(r)$ into length ℓ blocks (i.e., sets of ℓ consecutive vertices, where $n = t\ell$). Let $W_{\ell,r}$ be the set of all possible blocks of length ℓ that contain $r + 1$ consecutive passive vertices. Also, let $S_{\ell,r}$ be the set of all blocks of length ℓ that do not contain $r + 1$ consecutive passive vertices and that, for any state configuration for vertices not contained in the block, all the vertices belonging to the block eventually become active when applying the strict majority dynamics. Any block in $W_{\ell,r}$ is a wall in $R_n(r)$ and any block in $S_{\ell,r}$ is a spreading state. Any other block is an empty state. Let $\mu(W_{\ell,r})$

(respectively $\mu(S_{\ell,r})$) be the probability that an arbitrary block belongs to $W_{\ell,r}$ (respectively $S_{\ell,r}$). The following lemma is not difficult to prove:

Lemma 10. *For $0 < p < 1$ and positive integers r, ℓ ,*

(i) $\liminf_{n \rightarrow \infty} \mathbb{E}_p(X_0(n, r)) \geq 1 / (1 + \mu(W_{\ell,r}) / \mu(S_{\ell,r}))$.

(ii) *If $\ell \geq r + 1$, then $\mu(W_{\ell,r}) \leq \ell q^{r+1}$.*

We will now find a lower bound for $\mu(S_{\ell,r})$. The goal is to prove that $\frac{\mu(W_{\ell,r})}{\mu(S_{\ell,r})}$ goes to 0 when $r \rightarrow \infty$. For that purpose we denote, for any 0-1 word v , by $|v|_0$ (respectively $|v|_1$) the number of occurrences of symbol 0 (respectively 1) in v , and denote the i -th character of v by v_i . We set $\ell = 2r + 1$ and consider the set T_r of binary words v of length ℓ satisfying the following properties: (1) $|v|_1 = r + 1, |v|_0 = r$; (2) $v_0 = v_{2r} = 1, v_r = 0$ and (3) the word $w = w_1 \cdots w_{r-1}$ of length $r - 1$ over alphabet $\{0, 1\}^2$ defined by $w_i = (v_i, v_{i+r})$ is a generalized Dyck word [5] associated to the weight function $\omega(a, b) = +1$ if $(a, b) = (0, 0)$, $\omega(a, b) = -1$ if $(a, b) = (1, 1)$ and $\omega(a, b) = 0$ otherwise. I.e., ω satisfies the following two conditions: (i) $\sum_{i=1}^j \omega(w_i) \geq 0$ for all $1 \leq j \leq r - 1$, and (ii) $\sum_{i=1}^{r-1} \omega(w_i) = 0$.

Lemma 11. *If r is a positive integer and $\ell = 2r + 1$, then $|T_r| \leq |S_{\ell,r}|$.*

Proof. Consider some $v \in T_r$ and denote by w the word of length $r - 1$ over the alphabet $\{0, 1\}^2$ as defined above. We first consider successively each vertex i of $R_n(r)$, for $i = r$ to $i = 2r - 1$, and apply the strict majority dynamics to it. Initially, the state of vertex i is v_i . During this first sequence of updates, we denote by n_i the number of 1s in the neighborhood of vertex i at the time this vertex is considered (i.e., we take into account updates of vertices $j < i$ which possibly occurred before in the sequence). Since $v \in T_r$, we have that $|v|_1 = r + 1$, so at the beginning of the process $i = r$ and $n_r = r + 1$. We claim that for all i with $r \leq i < 2r - 1$, when we consider vertex $i + 1$ in the process we have $n_{i+1} - n_i = \omega(w_{i+1-r})$ and the state of vertex i is 1. This claim is deduced by induction from the fact that $i - r$ is the only vertex of index less than $i + 1$ in the symmetric difference of the neighborhoods of vertices i and $i + 1$, hence $n_{i+1} - n_i = \delta_+ - \delta_-$ where $\delta_+ = 1$ if vertex i was updated from 0 to 1 in the previous step and 0 otherwise, and $\delta_- = 1$ if $v_{i-r} = 1$ and 0 otherwise. Using the induction hypothesis for all $j \leq i$, by the Dyck property of w , we

have $n_i = n_r + \sum_{j=1}^{i-r} \omega(w_j) \geq n_r = r + 1$. Hence, if vertex i was 0 before being considered ($v_i = 0$), it updates to 1 when considered. In any case vertex i is 1 once considered in the process. Moreover, we have $\delta_+ = 1$ iff $v_i = 0$ and we deduce that $\delta_+ - \delta_- = \omega(w_{i+1-r})$, thus establishing the claim.

Now, from the claim, we deduce that all vertices from $i = r$ to $i = 2r$ are active when the sequence of updates ends. Then, we consider a new sequence of updates from vertex $i = r - 1$ to vertex $i = 1$ (successively). Trivially, when vertex i is considered, its neighborhood contains at least $r + 1$ active vertices (because the neighborhood of vertex i contains the active vertices 0 and vertices $i + 1, \dots, i + r$). Hence, all vertices will be active at the end of this second sequence of updates. This completes the proof of the lemma. \square

Theorem 12. *If $p > 1/4$, then there is an $r_0 = r_0(p)$ such that for all $r \geq r_0$ we have: $\lim_{n \rightarrow \infty} \mathbb{E}_p(X_0(n, r)) = 1$.*

Proof. First, we claim that there is some positive rational function $\phi(\cdot)$ such that $|T_r| \geq \phi(r) \cdot 4^r$ for all r . To prove this, choose $r = 4k + 1$ (this is without loss of generality since $r \mapsto |T_r|$ is increasing). It is straightforward to associate injectively a word $v \in T_r$ to any word w of length $r - 1 = 4k$ over alphabet $\{0, 1\}^2$ which is a generalized Dyck word associated to the weight function ω as defined before. To obtain a lower bound on the number of such generalized Dyck words, we consider the subset U_k of words w of length $4k$ over the alphabet $\{0, 1\}^2$ and such that $|\{i : \omega(w_i) = +1\}| = |\{i : \omega(w_i) = -1\}| = k$ and $|\{i : \omega(w_i) = 0\}| = 2k$. The set U_k can be generated, up to a straightforward encoding, by considering classical Dyck words of length $2k$ (weights $+1/-1$) interleaved by binary words of size $2k$. Therefore, $|T_r| \geq D_{2k} \cdot \binom{4k}{2k} 2^{2k}$. Using classical results about Catalan numbers and Stirling's formula, for some positive rational functions ϕ_1 and ϕ_2 we have $|D_{2k}| \sim \phi_1(k) \cdot 4^k$ and $\binom{4k}{2k} \sim \phi_2(k) \cdot 4^{2k}$. It follows that there is some positive rational function ϕ such that $|T_r| \geq \phi(r) \cdot 4^r$. This establishes the claim.

By Lemma 11, for some positive rational function ϕ it holds that $\mu(S_{\ell, r}) \geq \phi(r) \cdot q^r p^{r+1} 4^r$. By Lemma 10 (ii), $\frac{\mu(W_{\ell, r})}{\mu(S_{\ell, r})}$ is asymptotically less than $\Phi(r) \cdot (4p)^{-r}$, where $\Phi(\cdot)$ is another positive rational function. If $p > 1/4$ then $\mu(W_{\ell, r})/\mu(S_{\ell, r})$ goes to 0 when $r \rightarrow \infty$. The theorem follows from Lemma 10 (i). \square

It follows that $\limsup_{r \rightarrow \infty} p_c^+(\mathcal{WH}(r)) \leq 1/4$ and, invoking Corollary 8, we conclude that $\lim_{r \rightarrow \infty} p_c^+(\mathcal{WH}(r)) = 1/4$.

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