

# Domino tilings and related models: space of configurations of domains with holes <sup>★</sup>

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## Abstract

We first prove that the set of domino tilings of a fixed finite figure is a distributive lattice, even in the case when the figure has holes. We then give a geometrical interpretation of the order given by this lattice, using (not necessarily local) transformations called *flips*.

This study allows us to formulate an exhaustive generation algorithm and a uniform random sampling algorithm.

We finally extend these results to other types of tilings (calisson tilings, tilings with bicolored Wang tiles).

*Key words:* tiling, height function, lattice

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## 1 Introduction

In the last ten years a lot of progress has been done in the study of tilings. Most remarkably, W. P. Thurston [23], using work of J. H. Conway and J. F. Lagarias [5], introduced the notion of height functions, which encode domino tilings and calisson tilings of a polygon  $P$ .

The notion of height function appears to be a very powerful tool for the study of tilings. It has notably been extended by different authors [12] [18] to study tiling algorithms for other sets of prototiles.

For domino tilings, height functions induce a structure of distributive lattice (in the sense of order theory, see for example [7]) on the set of tilings of a fixed polygon (see [4] [15] or [19] for a detailed study). This strong structure is a main point to explain a lot of results about domino tilings: tiling algorithm [23], rapidly mixing Markov chains for random sampling [13] [24], computation of the number of necessary flips (local transformations involving two dominoes) to pass from a fixed tiling to another fixed tiling [19], efficient exhaustive generation of tilings [8] [9].

Dominoes are of particular importance to theoretical physicists, for whom dominoes are models of *dimers*, which are diatomic molecules (such as dihydrogen), and each tiling is seen as a possible state of a solid or a fluid.

The present paper tries to generalize previous results to figures which are not polygons, *i. e.* figures with holes. To do it, generalizing previous ideas of [20] and [22], we first introduce an *equilibrium function* on edges of cells of the figure. We also need to introduce some structural notions, following works of J. Propp [15] and J. C. Fournier [10]: The *critical cycles*, which induce *forced components* and *generalized flips*. With these tools, we have a constructive way to obtain the structure of distributive lattice previously introduced by J. Propp [15]. Moreover, we have a geometrical interpretation of it.

Since our approach is constructive, it allows us to exhibit algorithms to compute the objects introduced. As a consequence, we obtain an exhaustive generation algorithm and a uniform random sampling algorithm.

We finish by proving that these ideas can be directly adapted for other types of tilings: Calisson tilings and tilings with bicolored Wang tiles.

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## 2 Figures in the plane grid

### 2.1 The plane grid

Let  $\Lambda$  be the plane grid of the Euclidean plane  $\mathbb{R}^2$ . A *vertex* of  $\Lambda$  is a point whose coordinates are both integers.

A vertex  $v = (x_1, y_1)$  is a *neighbor* of another vertex  $v = (x_2, y_2)$  if  $|x_1 - x_2| + |y_1 - y_2| = 1$ . Hence, each vertex  $v$  has four neighbors  $v + (1, 0)$ ,  $v - (1, 0)$ ,  $v + (0, 1)$  and  $v - (0, 1)$  which are canonically called the *East*, *West*, *North* and *South neighbor* of  $v$ , respectively. An *edge* of  $\Lambda$  is the closed segment of straight line between two adjacent vertices. A *cell* of  $\Lambda$  is a (closed) unit square whose corners are vertices. Two cells are *4-neighbors* (respectively *8-neighbors*) if they share an edge (respectively at least a vertex).

A directed graph  $G = (V, E)$  is *symmetric* if  $(v, v') \in E$  if and only if  $(v', v) \in E$  for all  $v, v' \in V$ . In this work we deal with the symmetric directed graph (denoted by  $\Lambda^+$ ) obtained from the planar grid  $\Lambda$  by replacing each edge  $vv'$  by two *arcs*  $(v, v')$  and  $(v', v)$ . For an arc  $a = (v, v')$  of  $\Lambda^+$  we denote by  $[a]$  its associated edge in  $\Lambda$ .

A (*directed*) *path*  $P$  in a directed graph  $G = (V, E)$  is a sequence of vertices  $(v_0, \dots, v_k)$  such that  $(v_i, v_{i+1})$  is an arc of  $G$  for every  $i = 0, \dots, k - 1$ . We denote by  $E(P)$  the multiset of all the arcs used by the path  $P$  and by  $V(P)$  the multiset of its vertices. We say that  $G$  is *connected* if any two vertices of  $V$  are linked by a path.

A path  $P = (v_0, \dots, v_k)$  with  $v_k = v_0$  is called a *cycle*. The cycle is *elementary* if  $v_i = v_j$  and  $i \neq j$  imply  $\{i, j\} = \{0, k\}$ . In a plane graph one has two kinds of elementary cycles: The *clockwise* cycles and the *counterclockwise* ones.

Let  $G = (V, E)$  be a symmetric directed graph. A function  $g : E \rightarrow \mathbb{Z}$  is *skew-symmetric* if  $g(v, v') = -g(v', v)$  for every  $(v, v') \in E$ . Given any function  $h : V \rightarrow \mathbb{Z}$  we define its associated difference function  $D(h) : E \rightarrow \mathbb{Z}$  by  $D(h)(v, v') = h(v') - h(v)$ , for all  $(v, v') \in E$ . Conversely, if  $G$  is connected, given a function  $g : E \rightarrow \mathbb{Z}$  which satisfies  $g(C) = 0$  for all cycle  $C$  of  $G$  and a vertex  $w_0$  of  $V$ , there exists a unique function  $h : V \rightarrow \mathbb{Z}$  such that  $h(w_0) = 0$  and  $D(h)(a) = g(a)$  for all  $a \in E$ .

Let  $E'$  be a multiset of arcs of  $G$ . We denote by  $g(E')$  the sum of the values  $g(a)$  over all the arcs  $a \in E'$  (each arc  $a$  is counted according to its corresponding multiplicity in the multiset). Then  $g(E') = \sum_{a \in E'} g(a)$ . For a path  $P$ , instead of  $g(E(P))$  we use the shorthand  $g(P)$ .

We assume that cells of  $\Lambda$  are colored as a checkerboard. We thus have black cells and white cells, and two cells sharing an edge have different colors. Let us define the *spin* function  $sp$  on the arcs of  $\Lambda^+$ . For each arc  $a = (v, v')$ , the spin of  $a$  is noted  $sp(a)$  and given by:

- $sp(a) = 1$  if an ant moving from  $v$  to  $v'$  has a white cell on its left side (and a black cell on its right side);
- $sp(a) = -1$  otherwise.

For each clockwise elementary cycle  $C$ , one has  $sp(C) = 4Dis(C)$  where  $Dis(C)$ , the *disequilibrium* of  $C$ , is the difference between the number of black cells and the number of white cells enclosed by  $C$ . The result is true for each cycle enclosing a single cell, and each elementary cycle can be decomposed into such square cycles.

## 2.2 Figures

A *figure*  $F$  of  $\Lambda$  is a 4-connected (*i. e.* whose cells are connected according to the 4-neighboring relation), finite union of cells of  $\Lambda$ . The unique infinite 8-connected component of  $\mathbb{R}^2 \setminus F$  is denoted by  $H_\infty$ . The other ones are called the *holes* of  $F$ . The set of all edges in the boundary of  $F$  is denoted by  $E_b(F)$ . The set of edges in  $H_\infty \cap F$  is called the *outer-boundary* of  $F$  and denoted by  $E_{ob}(F)$ . Analogously, we denote by  $V_b(F)$  the set of all the vertices on the boundary of  $F$ .

Because of the two types of connectivity for cells, we replace (until the end of the paper) each vertex  $v$  of  $F$  such that each edge issued from  $v$  is on the boundary of  $F$ , by two vertices  $v_1$  and  $v_2$ , each of them connected to exactly two neighbors of  $v$  (see Figure 1).

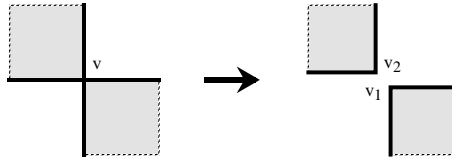


Fig. 1. Vertex duplication according to 4-connectivity of  $F$  and 8-connectivity of  $\mathbb{R}^2 \setminus F$ .

A figure defines a symmetric directed graph  $G_F = (V_F, E_F)$  such that  $V_F$  is the set of corners of cells of  $F$  (once duplication is done), and  $E_F$  is the set of arcs  $a$  such that  $[a]$  is a side of a cell of  $F$ . From this point of view, the clockwise and counterclockwise contours of each hole are elementary cycles of  $G_F$ .

For each elementary clockwise cycle of  $F$  (*i. e.* whose arcs are in  $E_F$ ) we

define  $Dis_F(C)$  as the difference between the number of black cells of  $F$  and the number of white cells of  $F$  enclosed by  $C$ .

### 2.3 Equilibrium function

Informally, we can say that we want to work as if  $F$  had no hole. To this end, the informal idea is to introduce values on edges which make holes disappear. Precisely, this is done by the use of *equilibrium functions* as defined below:

**Definition 2.1** *An equilibrium function (denoted by  $eq$ ) is a skew-symmetric function from  $E_F$  to  $\mathbb{Z}$  such that  $sp(C) + eq(C) = 4Dis_F(C)$ , for every clockwise cycle  $C$  of  $F$ .*

For figures without holes, it suffices to take  $eq = 0$ .

Notice that, through the decomposition of cycles, a function  $eq$  is an equilibrium function if and only if the following conditions holds.

- $eq(C) = 0$  for each elementary cycle  $C$  around a cell of  $F$ .
- $eq(C) = -sp(C)$  for each cycle  $C$  which follows clockwise the boundary of a hole of  $F$ .

More generally, for every cycle  $C$  of  $F$ ,  $eq(C)$  is a number which does not depend on the chosen equilibrium value. We prove in Section 4 that every figure has an equilibrium function which can be efficiently computed.

We also need some auxiliary functions deduced from the function  $eq$ .

**Definition 2.2** *The functions  $eq_r$ ,  $\mathbf{t}$  and  $\mathbf{b}$  are functions from the set of arcs of  $\Lambda^+$  to  $\mathbb{Z}$ , defined by:*

- (1)  $eq_r(a) = eq(a) - sp(a)$  for all  $a \in E_F$ .
- (2)  $\mathbf{t}(a) = eq(a) - sp(a) + 2$  for all  $a \in E_F \setminus E_b(F)$ .
- (3)  $\mathbf{b}(a) = eq(a) - sp(a) - 2$  for all  $a \in E_F \setminus E_b(F)$ .
- (4)  $\mathbf{t}(a) = \mathbf{b}(a) = eq(a) + sp(a)$  for all  $a \in E_b(F)$ .

Note that for any arc  $a$  in  $E_F$ ,  $\mathbf{t}(a) - \mathbf{b}(a)$  is either 0 or 4.

## 3 The lattice of tilings

In this section we associate three classes of objects to a figure: Tilings, height functions and acyclic orientations. Our goal is the study of tilings, and height

functions and acyclic orientations are some powerful tools to carry on this study.

### 3.1 Tilings and height functions

A *domino* is a figure formed by two cells sharing an edge, which is called the *central axis* of the domino. A *tiling*  $T$  of a figure  $F$  is a set of dominoes included in  $F$ , with pairwise disjoint interiors (*i. e.* there is no overlap), such that the union of the tiles of  $T$  equals  $F$  (*i. e.* there is no gap). Each tiling  $T$  of a figure  $F$  is completely determined by the set of central axes of the tiles. The characteristic function of a tiling  $T$  defined from the set of arcs of  $F$  into  $\mathbb{Z}$  is given by  $\chi_T(a) = 1$  if  $[a]$  is a central axis of a tile of  $T$  and  $\chi_T(a) = 0$  otherwise.

Let  $T$  be a tiling of  $F$ . The *height difference* in  $T$ , noted  $g_T$ , is the skew-symmetric function defined by

$$\forall a \in E_F \quad g_T(a) = eq_r(a) + 2sp(a)(1 - 2\chi_T(a))$$

Then  $g_T(a) \in \{\mathbf{b}(a), \mathbf{t}(a)\}$  for every  $a \in E_F$ .

Let us define  $g_F(a)$  as  $eq(a) + sp(a)$  for every  $a \in E_F$ . It can be seen that for each pair  $(T, T')$  of tilings of  $F$  and every  $a \in E_F$ ,  $g_T(a) - g_{T'}(a) = -4sp(a)(\chi_T(a) - \chi_{T'}(a))$ . Thus, if  $g_T = g_{T'}$ , then  $T = T'$ . Hence, the function  $g_T$  is a tool to encode the tilings. Moreover, for every arc  $a \in E_b(F)$  and every tiling  $T$  of  $F$  we necessarily have  $g_T(a) = g_F(a)$ . Thus  $g_T(a)$  does not depend on  $T$  for  $a \in E_b(F)$ . Additionally,  $g_T(a) - g_{T'}(a) \in \{-4, 0, 4\}$ , for every  $a \in E_F$ .

**Proposition 3.1** *Let  $T$  be a tiling of a figure  $F$ . For each cycle  $C$  of  $F$ , one has  $g_T(C) = 0$ .*

This proposition is a generalization of a theorem by J. H. Conway [5] on tilings of polygons.

*Proof.* (sketch) It suffices to prove the result for elementary cycles since the height difference of each cycle is the sum of the height differences of the elementary cycles which compose it. This is done by induction on the number of cells of  $\Lambda$  enclosed by the cycle.

The case of a cycle following the boundary of a hole is easily treated from the definition of equilibrium functions. We also verify that the proposition holds for elementary cycles of length 4 around a cell.

We now use induction. If we are not in one of the cases treated above, then the area enclosed by the cycle can be cut by a path in  $F$ , which induces two new cycles, each of them enclosing less cells of  $\Lambda$  than the original cycle. Thus, by the induction hypothesis, the height difference of both induced cycles is null, from which it is easily deduced that the height difference of the original cycle is null.  $\square$

Proposition 3.1 guarantees the correctness of the definition below.

**Definition 3.2** *We fix a vertex  $w_0$  of  $H_\infty$ . For each tiling  $T$ , the height function induced by  $T$  (denoted by  $h_T$ ) is the function from the set  $V_F$  of vertices of cells of  $F$  (once necessary vertex duplications have been done) to the set  $\mathbb{Z}$  of integers, defined by  $h_T(w_0) = 0$  and  $D(h_T) = g_T$ .*

We now give a characterization of height functions of tilings.

**Proposition 3.3** *We denote by  $\mathcal{H}_F$  the class of all the functions  $h : V_F \rightarrow \mathbb{Z}$  satisfying the following properties:*

- $h(w_0) = 0$ ;
- for each arc  $a$  of  $E_F$ ,  $D(h)(a) \in \{\mathbf{b}(a), \mathbf{t}(a)\}$ .

*For each tiling  $T$ , the function  $h_T$  belongs to  $\mathcal{H}_F$ . Conversely, for each  $h \in \mathcal{H}_F$  there exists a tiling  $T$  such that  $h = h_T$ .*

*Proof.* The first statement follows directly from the definition of  $g_T$ . Let  $h$  belong to  $\mathcal{H}_F$  and let  $C$  be a cycle around a cell. Then  $D(h)(C) - eq_r(C) = sp(C)$  since  $D(h)(C) = eq(C) = 0$ . Clearly,  $|sp(C)| = 4$ . Since  $D(h)(a) \in \{\mathbf{b}(a), \mathbf{t}(a)\}$ , there are three arcs of  $C$  such that  $D(h)(a) - eq_r(a) = 2sp(a)$  and exactly one arc satisfying  $D(h)(a) - eq_r(a) = -2sp(a)$ . Thus, the set  $T$  of all dominoes whose central axis  $[a]$  is such that  $D(h)(a) - eq_r(a) = -2sp(a)$  is a tiling  $T$  of  $F$ . The equality  $h(v) = h_T(v)$ , for each vertex  $v$  of  $F$ , is obvious by induction on the distance from  $w_0$  to  $v$ .  $\square$

The proposition above allows one to consider each tiling as a height function.

**Lemma 3.4** *For every pair of height functions  $h$  and  $h'$  and for each vertex  $v$  of  $F$ , one has  $h(v) - h'(v) = 0[4]$ . Moreover,  $h(v) - h'(v)$  does not depend on the chosen equilibrium function.*

*Proof.* Obvious by induction on the length of a shortest path from  $w_0$  to  $v$ .  $\square$

Let  $\leq$  be the canonical order on functions:  $h \leq h'$  if and only if  $h(v) \leq h'(v)$  for all  $v \in V_F$ .

**Proposition 3.5** *Let  $h$  and  $h'$  belong to  $\mathcal{H}_F$ . The functions  $\inf(h, h')$  and  $\sup(h, h')$  belong to  $\mathcal{H}_F$ .*

In the vocabulary of order theory (see for example [3], [7]) the above proposition can be restated as follows:  $(\mathcal{H}_F, \leq)$  is a distributive lattice.

*Proof.* Let  $h_1 = \inf(h, h')$ . We shall prove that for every arc  $(v, v') \in E_F$ ,  $h_1(v') - h_1(v) \in \{h(v') - h(v), h'(v') - h'(v)\}$  (the proof for  $\sup(h, h')$  is similar).

For the sake of contradiction, let us assume that there exists an arc  $a = (v, v')$  of  $F$  such that  $h_1(v) = h(v) < h'(v)$  and  $h_1(v') = h'(v') < h(v')$ . From Lemma 3.4, one has  $h'(v') \leq h(v') - 4$  and  $h(v) \leq h'(v) - 4$ . Then  $\alpha := h'(v') - h(v') + h(v) - h'(v)$  satisfies  $\alpha \leq -8$ . On the other hand,  $\alpha = h'(v') - h'(v) - (h(v') - h(v))$ . Since  $h$  and  $h'$  belong to  $\mathcal{H}_F$ , one obtains  $\alpha \geq \mathbf{b}(v, v') - \mathbf{t}(v, v') = -4$  which contradicts the hypothesis.  $\square$

We define the following order on the  $\Gamma_F$  of tilings of  $F$ :  $T \leq T'$  if  $h_T \leq h_{T'}$ . From Proposition 3.3,  $(\Gamma_F, \leq)$  is isomorphic to  $(\mathcal{H}_F, \leq)$ . Thus  $(\Gamma_F, \leq)$  is a distributive lattice.

### 3.2 Forced components

**Definition 3.6** *An elementary cycle  $C$  of  $G_F$  is critical if  $\mathbf{t}(C) = 0$ .*

*Such a cycle is strongly critical if, moreover,  $sp(a) = 1$ , for each arc  $a \in E(C) \setminus E_b(F)$ .*

*We say that  $v$  and  $v'$  are critically equivalent if there exists a critical cycle  $C$  such that  $v, v' \in V(C)$ . The equivalence classes of this equivalence relation are called the forced components of the figure  $F$ .*

*Let  $\hat{G}_F = (\hat{V}_F, \hat{E}_F)$  be the symmetric graph whose vertices are the forced components and where  $(U, U')$  is an arc of  $\hat{G}_F$  if there exists  $v \in U$  and  $v' \in U'$  such that  $(v, v')$  is an arc of  $G_F$ . This graph is called the graph of forced components of  $F$ .*

Notice that each boundary cycle of a hole of  $F$  is strongly critical. The boundary cycle of the outer boundary is critical if and only if  $F$  contains as many black cells as white cells. A strongly critical cycle can be deduced from each critical cycle by replacing each interior arc  $a$  with  $sp(a) = -1$  by a sequence  $a', a'', a'''$  of three arcs of positive spins.

It is easy to see that for any cycle  $C$  of  $G_F$ ,  $\mathbf{t}(C)$  and  $eq(C)$  do not depend on the chosen equilibrium function. Thus, the notion of critical cycle only depends on the shape of the figure.

**Definition 3.7** *Let  $T$  be a tiling of  $F$ . The graph of  $T$  is the spanning subgraph*



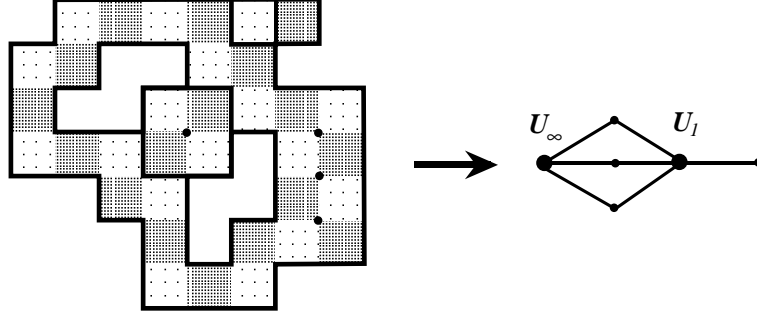


Fig. 2. The forced components of a figure and the graph of forced components ( $U_1$  denotes the component containing the contours of the two holes).

of  $G_F$ , denoted by  $G_T = (V_F, E_T)$ , where  $a \in E_T$  if and only if  $g_T(a) = \mathbf{t}(a)$ .

By definition, every arc in  $E_b(F)$  is an arc of  $G_T$ . Moreover,  $a \in E_T \setminus E_b(F)$  if and only if  $\chi_T(a) = 0$  and  $sp(a) = 1$ , or  $\chi_T(a) = 1$  and  $sp(a) = -1$ .

The following proposition is the reason why we are interested in critical cycles and forced components.

**Proposition 3.8** *Let  $C$  be a cycle of  $G_F$ .*

- *If  $C$  is critical, then for every tiling  $T$  the cycle  $C$  is a cycle of  $G_T$ . Conversely, if  $C$  is a cycle of  $G_T$  for some tiling  $T$ , then  $C$  is critical.*
- *If  $C$  is strongly critical, then for every tiling  $T$  the cycle  $C$  is a cycle of  $G_T$  which does not cut any tile of  $T$ . Conversely, if for some tiling  $T$   $C$  is a cycle of  $G_T$  that  $C$  cuts no tile of  $T$ , then  $C$  is strongly critical.*

*Proof.* If  $C$  satisfies  $\mathbf{t}(C) = 0$ , then for every tiling  $T$  of  $F$  one has  $\mathbf{t}(C) = g_T(C)$ . Therefore  $\mathbf{t}(a) = g_T(a)$  for every  $a \in E(C)$ , whence  $C$  is a cycle of  $G_T$ . Conversely, if  $C$  is a cycle of  $G_T$ , then  $g_T(a) = \mathbf{t}(a)$  for every  $a \in E(C)$ . Thus, from Proposition 3.1,  $\mathbf{t}(C) = g_T(C) = 0$  and  $C$  is a critical cycle.

For the second part, assume that  $C$  is strongly critical. For every  $a \in E(C)$  we have  $\mathbf{t}(a) = g_T(a)$ , which implies  $sp(a) = 1$ , from which one knows that  $\chi_T(a) = 0$ ; this means that  $C$  does not cut any tile of  $T$ . Conversely, let  $[a]$  be an interior edge which does not cut any tile of  $T$ . By definition,  $g_T(a) = sp(a) + eq(a)$ . Moreover, since  $a$  belongs to  $E_T$ , one has  $g_T(a) = \mathbf{t}(a) = eq(a) - sp(a) + 2$  and finally  $sp(a) = 1$ .  $\square$

**Corollary 3.9** *If  $F$  has a strongly critical cycle  $(v_0, v_1, \dots, v_p)$  such that for each integer  $0 \leq i < p$ ,  $[v_i, v_{i+1}]$  is an interior edge, then there exists no tiling of  $F$ .*

*Proof.* Let  $v_j = (x_j, y_j)$  be the vertex of this cycle with  $x_j + y_j$  maximal, and, moreover,  $x_j$  minimal with respect to the previous condition. One necessarily has  $v_{j-1} = v_j + (-1, 0)$ ,  $v_{j-2} = v_{j-1} + (0, -1)$  and (moreover)  $v_{j+1} = v_j +$

$(0, -1)$ . Now, follow the cycle until the first vertex  $v_{j+2k}$  such that  $v_{j+2k} = v_j + (k, -k)$  and  $v_{j+2k+2} \neq v_j + (k+1, -k-1)$  (which forces that  $v_{j+2k+1} = v_{j+2k} + (0, -1)$  and  $v_{j+2k+2} = v_{j+2k+1} + (-1, 0)$ ) (see Figure 3).

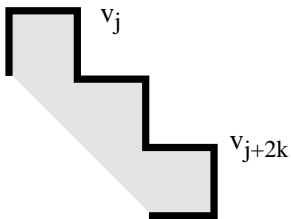


Fig. 3. Proof of Corollary 3.9.

Let  $T$  be a tiling of  $F$ ; at least one tile of  $T$  must be cut by an edge of the path formed from the part of the cycle from  $v_{j-2}$  to  $v_{j+2k+2}$ . But this is impossible, from Proposition 3.8. Thus there exists no tiling.  $\square$

From Corollary 3.9, if  $F$  can be tiled then there are three kinds of forced components: The component  $U_\infty$ , the *single components* which are reduced to a single vertex, and the *hole components* which contain the contour of at least one hole.

The following lemma establishes a useful relation between the height functions and the forced components.

**Lemma 3.10** *Let  $v$  and  $v'$  be critically equivalent vertices. For all  $h$  and  $h'$  in  $\mathcal{H}_F$ , one has  $h(v) - h'(v) = h(v') - h'(v')$ .*

*Proof.* Let  $C = (v_0, v_1, \dots, v_p)$  be a critical cycle passing through  $v$  and  $v'$ . One can assume without loss of generality that  $v = v_0$  and  $v' = v_k$ . One has  $h(v') - h(v) = \sum_{i=0}^{k-1} D(h)(v_i, v_{i+1}) = \sum_{i=0}^{k-1} \mathbf{t}(v_i, v_{i+1}) = h'(v') - h'(v)$ , which yields the result.  $\square$

Let us choose one vertex  $v_U$  in each forced component  $U$  of  $G_F$ . From Lemma 3.10,  $h \leq h'$  if and only if  $h(v_U) \leq h'(v_U)$  for all  $U \in \hat{V}_F$ .

We define a distance on  $\mathcal{H}_F$  by

$$\Delta(h, h') := \sum_{U \in \hat{V}_F} |h(v_U) - h'(v_U)|$$

Notice that the distance satisfies the following equalities:

$$\Delta(h, h') = \Delta(h, \inf(h, h')) + \Delta(\inf(h, h'), h')$$

$$\Delta(h, h') = \Delta(h, \sup(h, h')) + \Delta(\sup(h, h'), h')$$

### 3.3 Acyclic orientations and flips

In this part we prove that the lattice of height functions of a tileable figure  $F$  is isomorphic to a lattice of a subclass of orientations of the graph of forced components. These lattices have been precisely studied by J. Propp [15].

**Definition 3.11** *A directed graph  $G$  is an orientation of  $F$  if  $G$  is an orientation of  $\hat{G}_F$  such that  $|E(C) \cap E(G)| = -\frac{1}{4}\mathbf{b}(C)$  for all cycle  $C$  of  $\hat{G}_F$ . We denote by  $\mathcal{G}_F$  the class of all the acyclic orientations of  $F$ .*

*Let  $h$  be a height function and let  $T$  be the corresponding tiling. Let  $G_h = (\hat{V}_F, E)$  be the graph of strongly connected components of  $G_T$ , i. e. the directed graph defined by  $(U, U') \in E$  if and only if there exists an arc of  $G_T$  from a vertex of  $U$  to a vertex of  $U'$ .*

It is well known (see [2] for example) that the graph of strongly components of any directed graph is acyclic. Hence,  $G_h$  is an acyclic orientation of  $\hat{G}_F$ .

**Proposition 3.12** *Let  $h$  be an element of  $\mathcal{H}$ . Then  $G_h$  belongs to  $\mathcal{G}_F$ . Conversely, for each  $G = (V, E) \in \mathcal{G}_F$  there exists  $h \in \mathcal{H}$  such that  $G = G_h$ .*

*Proof.* Let  $C$  be a cycle of  $\hat{G}_F$ . Then  $0 = D(h)(C) = D(h)(E(C) \cap E) + D(h)(E(C) \setminus E)$ . Since  $D(h)(E(C) \cap E) = \mathbf{t}(E(C) \cap E)$  and, for any arc  $a$  of  $E_F$ ,  $\mathbf{t}(a) = \mathbf{b}(a) + 4$ , one obtains:  $D(h)(E(C) \cap E) = \mathbf{b}(E(C) \cap E) + 4|E(C) \cap E|$ . Moreover, from the definition of  $G_h$  one obtains  $D(h)(E(C) \setminus E) = \mathbf{b}(E(C) \setminus E)$ . Finally,  $0 = 4|E(C) \cap E| + \mathbf{b}(C)$ .

Conversely, let  $G = (V, E)$  belong to  $\mathcal{G}_F$ . Let  $g$  be the function defined by  $g(a) = \mathbf{t}(a)$  if  $a \in E$  and  $g(a) = \mathbf{b}(a)$  if  $a \in \hat{E}_F \setminus E$ . We prove that  $g(C) = 0$  for all cycle  $C$ . Clearly  $g(C) = g(E(C) \cap E) + g(E(C) \setminus E)$ . By definition of  $g$ , one has  $g(E(C) \cap E) = \mathbf{t}(E(C) \cap E)$  and  $g(E(C) \setminus E) = \mathbf{b}(E(C) \setminus E)$ . Since for any arc  $a$  of  $E_F$   $\mathbf{t}(a) = \mathbf{b}(a) + 4$ , one obtains that  $g(E(C) \cap E) = \mathbf{b}(E(C) \cap E) + 4|E(C) \cap E|$ . Then  $g(C) = \mathbf{b}(E(C) \cap E) + 4|E(C) \cap E| + \mathbf{b}(E(C) \setminus E) = \mathbf{b}(C) + 4|E(C) \cap E| = 0$ . Thus there exists  $h \in \mathcal{H}$  such that  $h(w_0) = 0$  and  $D(h) = g$ .  $\square$

**Definition 3.13** *Let  $G = (V, E)$  belong to  $\mathcal{G}_F$  and let  $U \neq U_\infty$  be in  $V$  without incoming (resp. outgoing) arcs. The graph obtained from  $G$  by an upward (resp. downward) flip in  $U$  is the acyclic directed graph  $G^U = (V, E^+)$  (resp.  $G^U = (V, E^-)$ ) where*

$$E^+ = E \setminus \{(U, U') : (U, U') \in E\} \cup \{(U', U) : (U, U') \in E\}$$

and

$$E^- = E \setminus \{(U', U) : (U', U) \in E\} \cup \{(U, U') : (U', U) \in E\}$$

An upward or downward flip in  $U$  corresponds to reversing all the arcs incident to  $U$ . If  $U$  is reduced to a single vertex  $v$ , the flip is said to be *local*. If  $U$  contains the contour of a hole, we say that it is a *hole flip*.

**Proposition 3.14** *For any  $G = (V, E) \in \mathcal{G}_F$ ,  $G^U = (V, E')$  belongs to  $\mathcal{G}_F$ .*

*Proof.* Let  $C$  be a cycle of  $\hat{G}_F$ . Since the number of arcs  $(U, U')$  used by  $C$  is equal to the number of arcs  $(U', U)$  used by  $C$ , one has  $|E(C) \cap E| = |E(C) \cap E'|$ .  $\square$

Let us denote by  $h^U$  the unique function in  $\mathcal{H}_F$  such that  $G_{h^U} = G^U$ . We have  $h^U = h$  in  $V \setminus U$  and  $|h^U - h| = 4$  in  $U$ . The two corresponding tilings differ only around  $U$ .

In particular, in  $\Gamma_F$  a local flip in  $U = \{v\}$  (see Figure 4) is the replacement in  $T$  of the pair of dominoes which cover the  $2 \times 2$  square centered in  $v$  by the other pair which can cover the same square.

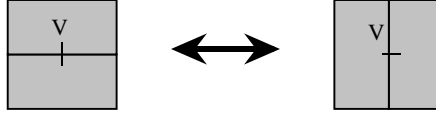


Fig. 4. A local flip.

The upward flips defined above canonically induce an order on the set  $\mathcal{G}_F$ . Given  $G$  and  $G'$  in  $\mathcal{G}_F$ , we say that  $G \leq_{flip+} G'$  if and only if there exists a sequence  $(U_0, \dots, U_{p-1})$  of vertices of  $\hat{G}_F$  and a sequence  $(G_0, G_1, \dots, G_p)$  of graphs of  $\mathcal{G}_F$  such that  $G_0 = G$ ,  $G_p = G'$  and, for each integer  $0 \leq i < p - 1$ ,  $G_{i+1}$  is obtained from  $G_i$  by an upward flip.

**Proposition 3.15** *Let  $h$  and  $h'$  belong to  $\mathcal{H}_F$ . Then  $h \leq h'$  if and only if  $G_h \leq_{flip+} G_{h'}$ . Moreover, in this case, one can pass from  $G_h$  to  $G_{h'}$  by a sequence of  $\Delta(h, h')/4$  flips.*

*Proof.* The direct part of the proposition is proved by induction on the quantity  $\Delta(h, h')$ . The result is obvious if  $\Delta(h, h') = 0$  (i. e.  $h = h'$ ).

Now, let us assume that  $\Delta(h, h') \neq 0$  and  $h \leq h'$ . We will prove that there exists a forced component  $U$  such that, for each vertex  $v_U$  of  $U$ ,  $h(v_U) < h'(v_U)$  (which implies  $h(v_U) \leq h'(v_U) - 4$ ) and an upward flip can be done from  $G_h$  on  $U$ .

Let  $U_0$  be a component such that  $h(v_{U_0}) < h'(v_{U_0})$ . If an upward flip can be done from  $G_h$  on  $U_0$ , then we are done. Otherwise, there exists an arc  $(v_1, v_0)$  in  $G_h$ , with  $v_0$  in  $U_0$  and  $v_1$  in another forced component  $U_1$ . We have:  $D(h)(v_1, v_0) = \mathbf{t}(v_1, v_0)$ , which gives  $D(h)(v_0, v_1) = \mathbf{b}(v_0, v_1)$ .

From Lemma 3.10, we have:  $h'(v_{U_1}) - h(v_{U_1}) = h'(v_1) - h(v_1)$  and  $h'(v_{U_0}) -$

$h(v_{U_0}) = h'(v_0) - h(v_0)$ . Moreover,  $h'(v_1) - h(v_1) = D(h')(v_0, v_1) + h'(v_0) - h(v_0) - D(h)(v_0, v_1)$ . Thus  $h'(v_{U_1}) - h(v_{U_1}) = h'(v_{U_0}) - h(v_{U_0}) + D(h')(v_0, v_1) - \mathbf{b}(v_0, v_1)$ , which yields  $h'(v_{U_1}) - h(v_{U_1}) \geq h'(v_{U_0}) - h(v_{U_0}) > 0$ .

Either an upward flip can be performed on  $U_1$ , or the same argument can be repeated from  $U_1$  to obtain an arc  $(v_1, v_2)$  from  $U_1$  to another forced component  $U_2$ ,  $D(h)(v_1, v_2) = \mathbf{b}(v_1, v_2)$ . By repeating the process, there are two possibilities: Either one obtains a forced component  $U_i$  on which an upward flip can be done, or an infinite sequence  $(U_i)_{i \in \mathbb{N}}$  is obtained. But since  $F$  is finite, the second possibility would imply that there exists a finite subsequence  $(U_j, U_{j+1}, \dots, U_k)$  for which  $U_j = U_k$ , which is a contradiction since  $G_h$  is acyclic.

We have now proved the existence of a forced component  $U$  such that an upward flip can be done from  $G_h$  around  $U$  to obtain a function  $h^U$ . Notice that  $\Delta(h^U, h') = \Delta(h, h') - 4$ . This proves that by induction one can pass from  $h$  to  $h'$  with  $\Delta(h, h')/4$  flips.

The undirect part of the first part of the proposition is obvious. □

**Corollary 3.16** *The function  $\varphi : \mathcal{H}_F \rightarrow \mathcal{G}_F$  defined by  $\varphi(h) = G_h$  is an order isomorphism between  $(\mathcal{H}_F, \leq)$  and  $(\mathcal{G}_F, \leq_{flip+})$ .*

**Corollary 3.17** *Let  $h$  and  $h'$  belong to  $\mathcal{H}_F$ . The number of successive flips needed to pass from  $G_h$  to  $G_{h'}$  is  $\Delta(h, h')/4$ . Moreover, the components  $U$  on which flips are done in such a sequence are those such that  $h(v_U) \neq h'(v_U)$ .*

*Proof.* A flip changes  $\Delta(h, h')/4$  by one unit. Thus  $\Delta(h, h')/4$  is a lower bound and the bound is reached if each flip lets the quantity decrease. Thus the bound can be reached only if the components  $U$  on which flips are done are precisely those such that  $h(v_U) \neq h'(v_U)$ .

Conversely, we have seen that for  $h \leq h'$ , one can pass from  $G_h$  to  $G_{h'}$  by a sequence of  $\Delta(h, h')/4$  flips. For the general case we use  $\mathit{inf}(h, h')$ . Passing through  $G_{\mathit{inf}(h, h')}$ , one can pass from  $G_h$  to  $G_{h'}$  by a sequence of  $\Delta(h, \mathit{inf}(h, h'))/4 + \Delta(\mathit{inf}(h, h'), h')/4 = \Delta(h, h')/4$  successive flips. □

**Corollary 3.18** *For each pair of tilings  $(T, T')$ , it is possible to pass from  $T$  to  $T'$  by a sequence of local flips if and only if  $h_T = h_{T'}$  for all the vertices on the boundary of  $F$ . Moreover, in this case the number of local flips needed is  $\sum_{v \in E_F} |h_T(v) - h_{T'}(v)|$ .*

*Proof.* This is a special case of the previous corollary. □

### 3.3.1 Freeness and rigidity of tilings

Informally, our tools allow one to see what is forced and what one has to choose in order to tile a figure.

**Lemma 3.19** *Let  $h$  belong to  $\mathcal{H}_F$ . The function  $h$  is maximal if and only if for each forced component  $U$  there exists a path  $P$  from  $U_\infty$  to  $U$  in  $G_h$ .*

*The function  $h$  is minimal if and only if for each forced component  $U$  there exists a path  $P$  from  $U$  to  $U_\infty$  in  $G_h$ .*

*Proof.* If there exists  $U'$  such that there is no path from  $U_\infty$  to  $U'$  in  $G_h$ , then, by taking a longest  $P'$  which finishes in  $U'$ , one clearly deduces the existence of a component  $U \neq U_\infty$  of  $G_h$  with no incoming arcs. Thus an upward flip can be done in  $U$  and  $h$  is not maximal.

Conversely, if  $h \neq h_{\max}$  then it is possible to pass from  $h$  to  $h_{\max}$  by a sequence of upward flips. The first component  $U$  on which a flip is done in such a sequence has no outgoing arc.

The proof for  $h_{\min}$  is similar. □

**Proposition 3.20** *For each component  $U$  such that  $U \neq U_\infty$ , one has  $h_{\min}(v_U) \neq h_{\max}(v_U)$ .*

*Proof.* Assume that  $h_{\min}(v_U) = h_{\max}(v_U)$ . Take an sequence of upward flips from  $h_{\min}$  to  $h_{\max}$ . Since no flip is done on  $U$ , no upward flip can be done on each component on a path of  $G_{h_{\min}}$  from  $U$  to  $U_\infty$ . Thus this path is also a path of  $G_{h_{\max}}$ . By concatenating it with a path of  $G_{h_{\max}}$  from  $U_\infty$  to  $U$ , a cycle appears in  $G_{h_{\max}}$ , which is a contradiction. □

**Corollary 3.21** *A vertex  $v$  of  $V_F$  belongs to  $U_\infty$  if and only if  $h(v) = h'(v)$  for all  $h, h' \in \mathcal{H}_F$ .*

*An arc  $a$  links two vertices of the same component if and only if  $D(h)(a) = D(h')(a)$ , for all  $h, h' \in \mathcal{H}_F$ .*

*Proof.* If  $v$  belongs to  $U_\infty$ , then  $h(v) - h'(v) = h(w_0) - h'(w_0) = 0$  from Lemma 3.10. Otherwise,  $h_{\min}(v) \neq h_{\max}(v)$  according to the previous Proposition.

If an arc  $a$  links two vertices of the same component, then  $D(h)(a) = D(h')(a)$  from Lemma 3.10. Conversely, if an arc  $a$  does not link two vertices of the same component, then one of its vertices is in a component  $U$  such that  $U \neq U_\infty$ . Take an sequence of upward flips from  $h_{\min}$  to  $h_{\max}$ . A flip of the sequence is done on  $U$ , which implies that there exists a pair  $(h, h')$  of functions such that  $D(h)(a) \neq D(h')(a)$ . □

## 4 Effective construction and algorithms

### 4.1 Forced components

When  $F$  can be tiled (which is the interesting case), the graph of forced components can be constructed in polynomial time: Given a tiling  $T$ , we have to construct the graph of strongly connected components of  $G_T$ , which can be done in linear time (see [21] or [6]).

We know from matching theory that there exists an  $O(n^{3/2})$  algorithm [11] to obtain such a tiling  $T$ , where  $n$  denotes the area of  $F$ . N. Thiant [22] gives an algorithm which is linear in the area enclosed by  $F$  (*i. e.* the sum of the area of  $F$  and the areas of the holes).

### 4.2 Construction of an equilibrium function

An equilibrium function can be exhibited using *cut lines* (see also [20]) as follows (see Figure 5):

For each hole  $H_i$  of  $F$ , we (arbitrarily) fix a vertical segment  $L_i = [p_i, p'_i]$  (which is called a *cut line* issued from  $H_i$ ) of  $\mathbb{R}^2$  such that  $p_i$  is the central point of a highest cell of  $H_i$ ; there exists a positive integer  $n_i$  such that  $p'_i = p_i + (0, n_i)$ , the vertex  $p'_i$  is not in  $F$ , and, for each integer such that  $0 < n'_i < n_i$ , the point  $p_i + (0, n'_i)$  is the central point of a cell of  $F$ . Hence, the point  $p'_i$  is the central point of a cell of another 8-connected component,  $H_j$  of  $\mathbb{R}^2 \setminus F$ , with  $j \neq i$  (and, possibly,  $j = \infty$ ).

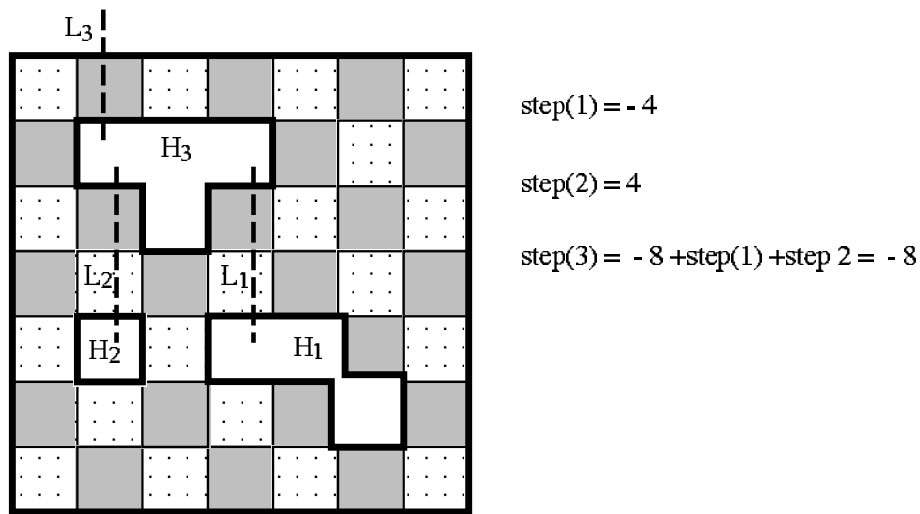


Fig. 5. Computation of an equilibrium function by step values.

We say that  $H_j$  is the (immediate) predecessor of  $H_i$ . This construction induces a directed tree whose vertices are 8-connected components of  $\mathbb{R}^2 \setminus F$ . This tree is rooted in  $H_\infty$ . We inductively define the *step value* of a hole  $H_i$  (denoted by  $step(i)$ ) as follows: Let  $C_i = (v_{i,0}, v_{i,1}, \dots, v_{i,p_i})$  be a cycle which (clockwise) follows the boundary of  $H_i$ . We state:

$$step(i) = \sum_{j|H_j \text{ has } H_i \text{ for predecessor}} step(j) - sp(C_i)$$

We can now define the equilibrium value of an arc  $(v, v')$  by:

- $eq(v, v') = step(i)$  if  $v'$  is the East neighbor of  $v$  and the line segment  $[v, v']$  crosses the cut line  $L_i$ ;
- $eq(v, v') = -step(i)$  if  $v'$  is the West neighbor of  $v$  and the line segment  $[v, v']$  crosses the cut line  $L_i$ ;
- $eq(v, v') = 0$  if the line segment  $[v, v']$  crosses no cut line.

This function satisfies the conditions of the definition: It is obvious for cycles surrounding single cells, and if  $C_i$  is a cycle which clockwise surrounds the hole  $H_i$ , one has  $eq(C_i) = step(i) - \sum_{j|H_j \text{ has } H_i \text{ for predecessor}} step(j) = -sp(C_i)$ .

The equilibrium function defined above has a specific property which can be used for algorithmic arguments: For each pair  $(v, v')$  of vertices of  $V_F$ , there exists a path  $P$  from  $v$  to  $v'$  such that for each arc  $a$  of  $P$ ,  $eq(a) = 0$ . This implies that there exists a spanning tree  $T_F$  of  $G_F$  such that for each arc  $a$  of  $T_F$  (seen as a symmetric graph),  $eq(a) = 0$ .

For each arc  $a$  of  $E_F$ ,  $|eq(a)| \leq 4n$  (this upper bound can be reached by taking a cycle which uses  $a$  and cuts no line, except  $a$ ). A precise study shows that such an equilibrium function can be constructed in  $O(n \log n)$  time units, where  $n$  denotes the number of cells of  $F$ .

### 4.3 Minimal tiling

For figures without holes, W. P. Thurston [23] has exhibited an algorithm that builds the minimal tiling. For the general case, this algorithm can be generalized as follows.

**Initialization:** For each vertex  $v$ , the algorithm uses a variable value  $h(v)$  and two fixed values  $low(v)$  and  $sup(v)$ .

From previous results, we know that for all  $a \in E_{ob}(F)$  and all tiling  $T$ ,  $g_T(a) = g_F(a)$ . Then the value  $h(v)$  can be computed for all the vertices on



the boundary of  $H_\infty$ . If a contradiction appears, then stop. Otherwise, set  $low(v) = sup(v) := h(v)$  for all  $v$  on the boundary of  $H_\infty$ .

We first construct a spanning tree  $T_F$  on  $G_F$  rooted in  $w_0$ . Then, for all vertices  $v$  not on the boundary of  $H_\infty$ , we set  $sup(v) = \mathbf{t}(T_F v)$  and  $low(v) = \mathbf{b}(T_F v)$ , where  $T_F v$  denotes the unique path from  $w_0$  to  $v$  in  $T_F$ .

We set  $h(v) = low(v)$  for all vertices  $v$  not on the boundary of  $H_\infty$ .

The algorithm also uses a set  $V$  consisting of vertices  $v$  such that there exists a neighbor  $v'$  of  $v$  satisfying  $h(v) + \mathbf{t}(v, v') < h(v')$ . This set is computed during the initialization.

**Main loop:** While  $V$  is not empty:

- Pick a vertex  $v$  in  $V$  and update  $h(v)$  by adding 4 units. If  $h(v) > sup(v)$  after updating, then stop (there is no tiling).
- Update  $V$  by adding the neighbors  $v''$  of  $v$  such that  $h(v'') + \mathbf{t}(v'', v) < h(v)$  and remove  $v$  if necessary.

**Proposition 4.1** *Given a figure  $F$  formed of  $n$  cells, the above algorithm stops after at most  $16n^2$  time passages through the loop.*

*Moreover, the algorithm stops with  $V$  empty if and only if there exists a tiling. In this case, when the algorithm stops, one has  $h = h_{T_{min}}$ .*

*Proof.* First remark that, for each arc  $(v, v')$  of  $E_F$ ,  $h(v') - h(v) = \mathbf{t}(v, v')$  in  $\mathbb{Z}/4\mathbb{Z}$ . This is true during the initialization: We introduce the cycle  $C$  formed by the concatenation of (the opposite path of)  $T_F v'$ ,  $T_F v$  and  $(v, v')$ . We have  $\mathbf{t}(C) = 0[4]$  and this, together with the relations between  $\mathbf{t}$  and  $\mathbf{b}$ , gives the result. Moreover, this property is preserved by the loop. Thus, for each passage through the loop, the sum  $\sum_{v \in V_F} (sup(v) - h(v))$  decreases by at least 4 units. Moreover, on each vertex  $v$ ,  $sup(v) - low(v) \leq 4|E(T_F v)|$ , where  $|E(T_F v)|$  denotes the number of arcs of  $T_F v$ . Notice that  $|E(T_F v)| \leq E_F \leq 4n$  and  $V_F \leq 4n$ , which yields that  $\sum_{v \in V_F} (sup(v) - h(v)) \leq 64n^2$  so the algorithm stops after at most  $16n^2$  passages through the loop.

When  $V$  becomes empty, each arc  $(v, v')$  of  $E_F$  satisfies the hypothesis of Proposition 3.3. Thus there exists a tiling  $T$  such that  $h_T = h$ . Moreover  $h \leq h_{T_{min}}$  (this is true during the initialization, and this property is preserved by the loop), so  $h_{T_{min}} = h$ .

If the algorithm finds a vertex  $v$  such that  $h(v) > sup(v)$ , then there is no tiling since, otherwise, for each tiling  $T$ ,  $h_T(v) > sup(v)$ , which is a contradiction (clearly, from the definition of  $sup$ ,  $h_T(v) \leq sup(v)$ ). This finishes the proof.

□

If we take into account the implementation, the cost of the algorithm is at most  $O(n^2)$  time units, as follows:

All the values are encoded in unary numeration, which permits to add constant numbers in constant time. We use the equilibrium function described in part 4.2 and the spanning tree  $T_F$  is chosen in such a way that for each arc  $a$  of  $T_F$ ,  $eq(a) = 0$ . Thus, the initialization costs  $O(n^2)$  time units, to compute  $h(v)$  for each vertex and  $h(v') - h(v)$  for each arcs.

Each passage through the loop uses  $O(1)$  time units since it consists in a fixed number of additions of 4 units and sign tests. This gives the time complexity.

Of course, a similar algorithm can be designed to construct the maximal tiling of  $F$ . Notice that, in the general case, this algorithm is not the best known to construct a tiling, but we will see below that it is useful to have the extremal ones.

#### 4.4 Exhaustive generation

An exhaustive generation can be done, extending ideas of [9] to figures with holes. Let  $(U_1, U_2, \dots, U_q)$  be a fixed total order of forced components of  $F$  (except  $U_\infty$ ). We define a total order  $<_{lex}$  on tilings of  $F$  as follows: Given two tilings  $T$  and  $T'$ , we have  $T <_{lex} T'$  if there exists an integer  $1 \leq i \leq q$  such that  $h_T(v) < h_{T'}(v)$  for each vertex  $v$  of  $U_i$  and  $h_T(v) = h_{T'}(v)$  for each vertex  $v$  of  $U_j$  with  $1 \leq j < i$ .

The order  $<_{lex}$  is a linear extension of  $<_{height}$ , *i. e.* given two tilings  $T$  and  $T'$  such that  $T <_{height} T'$ , we have  $T <_{lex} T'$ .

**Proposition 4.2** *Let  $T$  be a (non-maximal) tiling of  $F$  and let  $T_{succ}$  denote the successor of  $T$  in the lexicographic order; let  $i$  denote the largest integer such that an upward flip is possible in  $U_i$ .*

*The tiling  $T_{succ}$  is the lowest tiling (for  $<_{height}$ ) such that  $h_{T_{succ}}(v) = h_T(v) + 4$  for each vertex  $v$  of  $U_i$ , and  $h_T(v) = h_{T_{succ}}(v)$  for each vertex  $v$  of  $U_j$  with  $1 \leq j < i$ .*

*Proof.* Let  $T'$  be a tiling such that  $h_{T'}(v) = h_T(v) + 4$  for each vertex  $v$  of  $U_i$  and  $h_{T'}(v) = h_T(v)$  for each vertex  $v$  of  $U_j$  with  $1 \leq j < i$ . By definition, one has  $T_{succ} \leq_{lex} T'$ , since  $T \leq_{lex} T'$ .

Moreover, assume that  $h_T = h_{T_{succ}}$  in  $U_i$ . Thus, by Corollary 3.16, one can pass from  $T$  to  $sup(T, T_{succ})$  by a sequence of upward flips, which contradicts the definition of the integer  $i$ .  $\square$

This proposition enables one to generate all the tilings of  $F$  as follows:

**Initialization:** Construct the graph  $G_{forc}$  of forced components, the tiling  $T_{min}$ , and the orientation  $o(T_{min})$ . Output the tiling  $T_{min}$ .

The algorithm uses a variable tiling  $T$  stored in memory, which for initialization is equal to  $T_{min}$ .

**Main loop:** Compute the successor of  $T$  as follows:

- Find the last component  $U_i$  on which an upward flip can be done (if no upward flip is possible, then stop).
- Construct the minimal tiling  $T'$  such that for each vertex  $v$  of  $\cup_{j=1}^{i-1} U_j$ ,  $h_T(v) = h_{T'}(v)$  and for each vertex  $v$  of  $U_i$ ,  $h_{T'}(v) = h_T(v) + 4$ .
- Replace  $T$  by  $T'$ ,  $o(T)$  by  $o(T')$ , output the tiling and go back to the beginning of the loop.

The second item of the main loop can be done in  $O(n^2)$  time units using an algorithm derived from the algorithm of construction of the minimal tiling (it suffices to change the initialization, fixing an appropriate value of  $h(v)$  for  $v$  in  $\cup_{j=1}^i U_j$ ).

Thus, once the initialization is done, the maximal waiting time between two consecutive tilings is  $O(n^2)$  time units. The memory space is  $O(n^2)$  since for each vertex  $v$ , one has to store  $h_T(v)$  (using unary numeration).

#### 4.5 Uniform random sampling

Consider the following process: Given a tiling  $T$ , randomly choose a forced component  $C$  and a direction (upwards or downwards). If a flip can be done in  $C$  according to the chosen direction, then make this flip; otherwise, do not change  $T$ . Trivially, this Markovian random process is ergodic and converges to the uniform distribution.

Moreover, the method of “coupling from the past” [17] can be applied since the process is monotonic and one has a method to construct the maximal and minimal tilings. We thus have a randomized algorithm to sample domino tilings uniformly at random. The space required is polynomial.

It has been previously proved [13] [24] that this process is rapidly mixing for figures without holes. Does it remain true in the general case ? To our knowledge, the question is still open.

About algorithms, the reader can also easily verify that, given a pair  $(T, T')$

of tilings, one can

- compute in linear time if one can pass from  $T$  to  $T'$  by a sequence of local flips (it suffices to compare  $h_T$  and  $h_{T'}$  on the boundary of  $F$ );
- compute in polynomial time (with low degree) a shortest path of flips to go from  $T$  to  $T'$  (using the cyclic orientations and the distance) and the length of such a path.

## 5 Extension to other types of tilings

### 5.1 Calissons

The same study can easily be done for calisson (*i. e.* tiles formed by two neighboring cells of the triangular lattice) tilings to get similar results. In this case, local flips are induced by the two tilings of hexagons formed by six triangular cells. There are only two small differences, detailed below:

- One has two types of connectivity for triangular cells (3-connectivity for cells which share an edge, and 12-connectivity for cells which share a vertex). Thus, some vertices have to be duplicated or triplicated (see Figure 6).

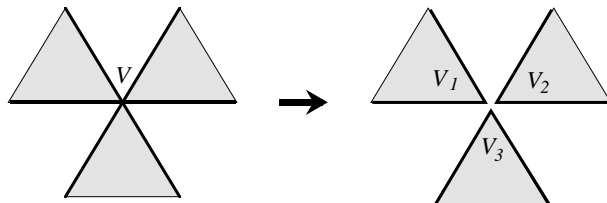


Fig. 6. Example of “triplication”.

- For the proof of corollary 3.9, one has to consider a part of the critical cycle with vertices  $v = (x, y)$  such that  $y$  is maximal.



Fig. 7. A part of a critical cycle which implies that there is no tiling.

### 5.2 Bicolored Wang tiles

The case of dominoes is a particular case of tilings with Wang tiles (*i. e.*  $1 \times 1$  squares with colored edges, see [14] for details). They give rise to a tiling if the colors on the edges of neighbor squares are compatible.

An instance of the problem of tiling by bicolored Wang tiles is given by a finite figure and a coloration of the edges which are on its boundary. Hence, a domino tiling is a tiling by Wang tiles with one red edge and three blue edges, all the edges on the boundary of the figure being blue.

### 5.2.1 Eulerian orientations

The same study can be done for tilings with “balanced Wang tiles” such that each square has two blue edges and two red edges. In this case, one has  $D(h_T)(v, v') = sp(v, v') + eq(v, v')$  if the corresponding edge is blue and  $D(h_T)(v, v') = -sp(v, v') + eq(v, v')$  otherwise. This is recognizable as the height function for Eulerian orientations of the dual lattice, called the *six-vertex ice model* by physicists [1]. This is also equivalent to the height function for three-colorings of vertices of the square lattice, and to alternating-sign matrices [16]. The results are similar to those obtained for dominoes.

### 5.2.2 Examples with finite height functions

For the case of “odd tiles” (*i. e.* tiles with an odd number of blue edges and an odd number of red edges as in [14]): One has to take a height function in  $\mathbb{Z}/8\mathbb{Z}$  such that  $D(h_T)(v, v') = sp(v, v') + eq(v, v')$  if the corresponding edge is blue and  $D(h_T)(v, v') = -3sp(v, v') + eq(v, v')$  otherwise. With our technique of equilibrium value, it is easily proved that the set of the tilings of a fixed figure has a structure of boolean lattice (or hypercube), even if the figure has holes.

The case of “even tiles” (*i. e.* tiles with an even number of blue edges and an even number of red edges) is very similar, with  $val_T(v, v') = (sp(v, v'), 0)$  if the corresponding edge is blue and  $val_T(v, v') = (sp(v, v'), sp(v, v'))$  otherwise. These values are taken in  $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ .

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