The complexity of approximating the oriented diameter of chordal graphs*

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Abstract

The oriented diameter of a bridgeless connected undirected (b.c.u.) graph G is the smallest diameter among all the diameters of strongly connected orientations of G. We study algorithmic aspects of determining the oriented diameter of a chordal graph. We

- construct a linear time approximation algorithm that, for a given chordal b.c.u. graph G, finds a strongly connected orientation of G with diameter at most one plus twice the oriented diameter of G;
- prove that, for every $k \geq 2$ and $k \neq 3$, to decide whether a chordal (split for $k = 2$) b.c.u. graph G admits an orientation of diameter k is NP-complete;
- show that, unless $P = NP$, there is neither a polynomial-time absolute approximation algorithm nor an α-approximation algorithm that computes the oriented diameter of a b.c.u. chordal graph for $\alpha < \frac{3}{2}$.

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1 Introduction

When the linkage structure of a communication networks is modeled by a graph, its diameter corresponds to the maximum number of links over which a message between two nodes must travel. In cases where the number of links in a path is roughly proportional to the time delay or signal degradation encountered by messages sent along the path, the diameter is then involved in the complexity analysis for the performance of the networks.

A variety of interrelated diameter problems are discussed in the literature (see [4] for a survey and Chapter 2 in [2]), including the problem of finding orientations for undirected or mixed graphs to minimize diameters. This problem has a long history. In 1939 Robbins proved that an undirected graph G admits a strongly connected orientation (one where every two vertices are mutually reachable by directed paths) if and only if G is connected and bridgeless [21]. More recently, Chung, Garey, and Tarjan provided a linear-time algorithm for testing whether a graph has a strongly connected orientation and finding one if it does [5]. From a graph theoretical point of view this problem has been studied for particular graphs: the torus, the Cartesian product of complete graphs, complete bipartite graphs, and others [12, 14, 15, 16, 17, 27]. This problem also appears in the study of network routing, broadcasting, gossip and one-way street arrangements [4, 7, 13, 22, 23, 24, 25].

Chvátal and Thomassen studied how the diameter of a bridgeless connected undirected graph G and the diameter of a strongly connected orientation of G are related [6]. They considered the following problems:

Oriented Diameter Problem (ODP): Given a bridgeless connected undirected graph G, find a strongly connected orientation H with the smallest diameter.

Oriented Diameter Decision Problem (ODPD(k)): Given a bridgeless connected undirected graph G, decide whether there exists a strongly connected orientation H with diameter at most k.

For every $k \geq 2$ and $k \neq 3$, Chvátal and Thomassen showed that ODPD(k) is NP-complete for general graphs. Therefore, there is a natural interest to study complexity issues of ODPD(k)

for different graph classes.

Chordal graphs form a very well investigated class of graphs (see [9] for classical results and [3] for more recent ones). They have well understood properties and many NP-hard problems such as Coloring, Clique, Independent Set can be solved in polynomial time when the input is restricted to chordal graphs. However, some other problems remain NP-complete even when restricted to chordal graphs (for instance Pathwidth [11] and Bandwidth [20]). For some of these problems the nice properties of chordal graphs has been useful for constructing approximation algorithms (for instance Bandwidth [10, 18]).

To the best of our knowledge, no work on the algorithmic aspects of ODP and ODPd for chordal graphs has been done. Even though, the work of Chvátal and Thomassen suggests the existence of a 3-approximation algorithm for ODP when the input is restricted to graphs where every edge belongs to a triangle (which holds for chordal graphs). Indeed, they proved that for every bridgeless connected graph G there is a strongly connected orientation H such that, if an edge $\{u, v\}$ belongs to a cycle in G of length k, then (u, v) or (v, u) belongs to a directed cycle in H of length at most $(k-2)2^{\lfloor (k-1)/2 \rfloor} + 2$.

This paper is motivated by the searching of both approximation algorithms and hardness results for ODP when restricted to chordal graphs.

1.1 Definitions

Let G be either an undirected graph or a directed graph with vertex set $V(G)$ and edge set $E(G)$. By $\{u, v\}$ we denote the undirected edge with ends in u and v and by (u, v) we denote the directed arc, directed from u towards v. The distance $d_G(u, v)$ between two vertices u and v of G is the length of the shortest path (the shortest directed path if G is directed) between u and v in the graph G (from u to v if G is directed). If there is no path from u to v then we put $d_G(u, v) = +\infty$. The diameter of a graph G, denoted by $\mathrm{diam}(G)$, is defined to be the maximum distance between two vertices of G. Thus $\mathrm{diam}(G) = \max\{d_G(u, v) : u, v \in V(G)\}$. When G is a directed graph we denote by $\vec{d}_G(u, v) = \max\{d_G(u, v), d_G(v, u)\}$.

Let G be a connected undirected graph. An *orientation* of G is a directed graph whose arcs

correspond to assignments of directions to the edges of G. An orientation H of G is *strongly connected* if every two vertices in H are mutually reachable in H ($\mathrm{diam}(H) < +\infty$). An edge e of G is called a *bridge* if $G - e$ is not connected. The graph G is *bridgeless* if $G - e$ is connected for every edge e, i.e. there is no bridge in G. The oriented diameter of G is defined as follows.

$$OD(G) = \min\{\mathrm{diam}(H) \colon H \text{ is an orientation of } G\}.$$

It was proved by Robbins in 1939 that G is not connected or has a bridge if and only if there is no strongly connected orientation of G. In that case, $OD(G) = +\infty$. Further we consider only bridgeless connected graphs.

An algorithm $\mathcal{A}$ is an (α, k)-*approximation algorithm* for ODP if for every graph G it runs in polynomial time and outputs an orientation H of G such that $\mathrm{diam}(H) \leq \alpha OD(G) + k$.

An $(\alpha, 0)$-approximation algorithm for ODP is called an α-*approximation algorithm* for ODP and a $(1, k)-$approximation algorithm for ODP is called an *absolute approximation algorithm* for ODP (see [1]).

A *chord* of a cycle C in G is an edge not in C that has both ends in C. A *chordless* or *induced cycle* in G is a cycle of length more than three that has no chord. A graph G is *chordal* if it contains no chordless cycles.

1.2 Our Contribution

In Section 2 we show that for every chordal graph G there exists a linear time computable orientation H such that, for every pair of vertices u and v, $\vec{d}_H(u, v) \leq 2d_G(u, v) + 1$. Therefore, for every chordal graph G there is an orientation H such that $\mathrm{diam}(H) \leq 2\,\mathrm{diam}(G) + 1$. On one hand this result implies that for chordal graphs $OD(G) \leq 2\,\mathrm{diam}(G) + 1$. We prove that the bound is sharp by constructing an infinite sequence of chordal graphs such that, for every graph G in this sequence, any strongly connected orientation of G has diameter $2\,\mathrm{diam}(G) + 1$. On the other hand, since $\mathrm{diam}(G) \leq OD(G)$, we deduce that ODP is $(2, 1)-$approximable.

In Section 3 we prove that $\mathrm{ODPD}(2)$ remains NP-complete in the subclass of chordal graphs called split graphs. We also prove that, for $k \geq 4$, $\mathrm{ODPD}(k)$ remains NP-complete for chordal

graphs. Moreover, we prove two non approximability results: first, for every $\alpha < \frac{3}{2}$ ODP is not $\alpha-$approximable in the class of split graphs; second, there is no absolute approximation algorithm for ODP when restricted to chordal graphs.

2 Positive results

Our algorithmical contribution is stated in the following theorem.

Theorem 1. *There is a linear time* $(2,1)$*-approximation algorithm for* ODP *in the class of chordal graphs.*

Since for every u and v we have $d_G(u,v) \leq \mathrm{diam}(G) \leq OD(G)$ and $\mathrm{diam}(H) = \max_{u,v \in V} \vec{d}_H(u,v)$, Theorem 1 follows from the next much stronger result.

Theorem 2. *There is a linear time algorithm such that, given a chordal graph G, it computes an orientation H of G satisfying, for every pair of vertices u and v, $\vec{d}_H(u,v) \leq 2d_G(u,v) + 1$.*

Notice that Theorem 2 is an improvement of the bound of Chvátal and Thomassen applied to chordal graphs. Moreover, as stated in Theorem 3, this bound is the best possible.

Theorem 3. *For every $n \geq 1$ there exists a chordal graph G^n such that* $\mathrm{diam}(G^n) = 2n+1$ *and* $\mathrm{diam}(H) = 2\,\mathrm{diam}(G^n) + 1$ *for every stronlgy connected orientation H of G^n.*

Proof. In Figure 1 we show a chordal graph G^2 of diameter 5 for which there is no orientation with diameter smaller than $2 \cdot 5 + 1$. This construction can be easily generalized to larger graphs. To obtain G^{n+1} from G^n it suffices to add, for every vertex x in G^n of degree 2, two new vertices x' and x'' and connect them in such a way that x, x' and x'' form a copy of K_3. Then $\mathrm{diam}(G^{n+1}) = \mathrm{diam}(G^n) + 2$. Moreover, every strongly connected orientation H of G^n orients every copy of K_3 in a cyclic way. Therefore, for every strongly connected orientation H^{n+1} of G^{n+1} and every strongly connected orientation H^n of G^n we have that $\mathrm{diam}(H^{n+1}) = \mathrm{diam}(H^n) + 4$. Assuming inductively that $\mathrm{diam}(H^n) = 2\,\mathrm{diam}(G^n) + 1$ we get that $\mathrm{diam}(H^{n+1}) = 2\,\mathrm{diam}(G^{n+1}) + 1$ $\square$

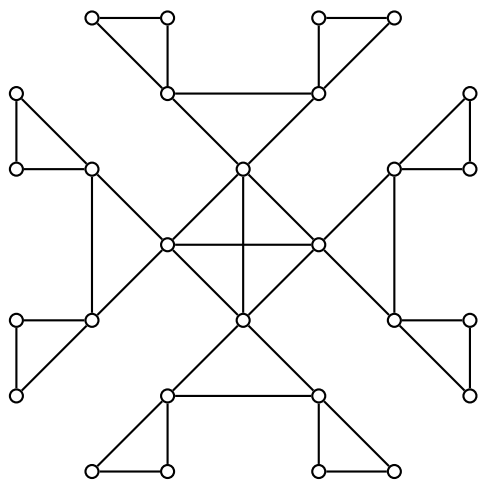

Figure 1: Connected bridgeless chordal graph of diameter 5.

The rest of this section is devoted to the proof of Theorem 2. The proof of this theorem is indirect. First we prove that every 2-connected chordal graph has a special orientation that can be obtained in linear time (Lemma 7). Then we use this orientation to prove the theorem for 2-connected chordal graphs (Lemma 8) and only then we prove Theorem 2.

Let us begin with some definitions. For a given chordal graph G and an orientation H of its edges we say that an arc in H is *good* if it belongs to a directed triangle and it is *bad* otherwise. A good orientation is an orientation leaving every arc good. Let K_n be the complete graph with n vertices.

In order to orient chordal graphs we first need to construct good orientations of complete graphs K_n for $n \geq 5$.

Lemma 4. *For every $n \geq 3$, K_n has a good orientation if and only if $n \neq 4$. Moreover, for every $n \geq 5$, every good orientation of K_n can be extended to a good orientation of K_{n+1} and this extension can be constructed in linear time.*

Proof. We first consider the cases $n = 3, 4$ and 5. It is easy to see that K_3 has a good orientation and that K_4 has no good orientations. Nevertheless, K_4 has an orientation with exactly one

bad arc. An orientation of K_5 is obtained from an orientation of K_4 with exactly one bad arc (v_1, v_2) by orienting the edges $\{v_2, v_5\}$ and $\{v_1, v_5\}$ as (v_2, v_5) and (v_5, v_1) and the edges $\{v_3, v_5\}$ and $\{v_4, v_5\}$ in such a way that they form a directed triangle with the arc associated with the edge $\{v_3, v_4\}$. This orientation of K_5 is good.

Let us assume that $n \geq 5$. In order to construct a good orientation H_{n+1} for K_{n+1} we use a good orientation H_n of K_n. Let us think that K_{n+1} is obtained from K_n by adding a new vertex v.

If n is even then a good orientation of K_{n+1} can be obtained from H_n by forming $\frac{n}{2}$ directed triangles using all the edges adjacent to v. The orientation of every triangle is induced by the arc of H_n. Clearly, this orientation can be done in $O(n)$ steps.

Suppose that n is odd. Since K_4 has no good orientations, for any $n \geq 5$ and every orientation H of K_n there are three vertices in K_n inducing a triangle that is not strongly connected. Let a, b and c be such vertices for the orientation H_n. W.l.o.g. we may think that the arcs in H are of the form (a, b), (a, c) and (b, c). We orient the edges $\{v, a\}$, $\{v, b\}$ and $\{v, c\}$ as (v, a), (c, v) and (b, v) in H_{n+1}. The remaining $n-3$ edges adjacent to v are in $(n-3)/2$ triangles in K_{n+1}, each of the triangles having one arc in H_n. Since $n-3$ is even we orient these edges as in the previous case.

So to obtain the orientation of K_{n+1} one should choose four arbitrary vertices in K_n and find three vertices that do not induce a strongly connected triangle. This can be done in a constant number of steps. And the orientation of the remaining $n-3$ edges can be done in $O(n)$ steps. □

In terms of diameter we have the following corollary which will be used in Section 3.

Corollary 5. *For every $n \geq 3$ there exists an orientation of K_n with diameter 2 if $n \neq 4$, and with diameter 3 if $n = 4$.*

A vertex v in a graph G is called *simplicial* if the graph induced by its neighborhood $N_G(v)$ is a clique. Chordal graphs have been characterized as those having a *perfect elimination ordering* (p.e.o.) [8]. This is a vertex ordering $\{v_1, \ldots, v_n\}$ such that for every $i \in \{1, \ldots, n\}$, the vertex v_i is simplicial in $G[v_i, \ldots, v_n]$ (where $G[S]$ denotes the graph induced by the vertex set S). We

say that $\{v_1, \ldots, v_n\}$ is a *perfect construction ordering* (p.c.o.) if $\{v_n, \ldots, v_1\}$ is a p.e.o. A p.c.o. of a chordal graph can be found in linear time using the LexBFS algorithm [26]. Moreover, the first vertex can be chosen arbitrarily. Since LexBFS is a special version of BFS it follows that every graph $G_i := G[v_1, \ldots, v_i]$ is connected.

For a p.c.o $\{v_1, \ldots, v_n\}$ the idea of our construction is to orient all the edges incident to v_i in $G[v_1, \ldots, v_i]$ sequentially (following the p.c.o.).

Let $\delta = \{v_1, \ldots, v_n\}$ be a p.c.o. of a chordal graph G. We say that a vertex v_i is *δ-super-simplicial* if $N(v_i) \cap \{v_{i+1}, \ldots, v_n\} = \emptyset$. Notice that every δ-super-simplicial vertex is simplicial in G but not vice versa.

A connected graph G is said to be 2-*connected* if for every vertex v, the graph $G - \{v\}$ is connected.

We need the following technical lemma about super-simplicial vertices in 2-connected graphs.

Lemma 6. *Let $\delta = \{v_1, \ldots, v_n\}$ be a p.c.o. of a 2-connected chordal graph G. If a vertex v_i is not δ-super-simplicial, then there are $k > i > l$ such that $\{v_k, v_i, v_l\}$ is a clique in G.*

Proof. Let v_p and v_q, $p > i > q$, be vertices adjacent to v_i (they exist since G_i is connected and v_i is not a δ-super-simplicial vertex). If $\{v_p, v_q\} \in E(G)$ then $\{v_p, v_q, v_i\}$ induce a clique and the lemma is proved. If $\{v_p, v_q\} \notin E(G)$ then the vertices v_p, v_q, v_i belong to a cycle C in G (G is 2-connected). We choose C to have the shortest length among all cycles containing v_p, v_q, v_i. Notice that the length of C is at least 4. The cycle contains at least one vertex which is before v_i (in δ) and at least one vertex that is after v_i. Therefore, there are two adjacent vertices $v_{p'}$, $v_{q'}$ with $p' > i > q'$. Since C is the shortest cycle, the only chords in this cycle are the edges adjacent to v_i. The chordality of G implies that v_i is adjacent to both $v_{p'}$ and $v_{q'}$ and the lemma is proved. □

Lemma 7. *There exists a linear time algorithm such that, given a 2-connected chordal graph G and a p.c.o. $\delta = \{v_1, \ldots, v_n\}$ of $V(G)$, it computes an orientation H with the following properties.*

(a) Every maximal clique in G has at most one bad arc in H.

(b) If (u, v) is a bad arc in H then u is a δ-super-simplicial vertex of G.

(c) For every $v \in V(G)$, $\vec{d}_H(v, v_1) \leq 2d_G(v, v_1)$.

(d) Every clique in H has diameter at most 3.

Proof. Iteratively, for $k = 3, \ldots, n$, we construct an orientation H_k of $G_k = G[v_1, \ldots, v_k]$ with the following properties.

(P1) Every bad arc belongs to a maximal clique (in H_k) of size four.

(P2) At most one arc is bad in each maximal clique.

(P3) If (u, v) is a bad arc in H_k then u is either a δ-super-simplicial vertex in G or the vertices u, v are used in some step $j > k$ to form a new clique, i.e. $u, v \in N_{G_j}[v_j]$ for some $j > k$.

Let $k \geq 2$ and let us denote $N_{G_{k+1}}(v_{k+1}) = \{u_1, \ldots, u_r\}$. Since G is 2-connected, it follows that $r \geq 2$. Therefore G_3 is a triangle, and obviously its cyclic orientation H_3 satisfies properties (P1), (P2) and (P3).

Now we show how to extend, for every $k = 3, \ldots, n-1$ the orientation H_k to an orientation H_{k+1} of G_{k+1} satisfying properties (P1), (P2) and (P3).

1. If $r > 4$ then by (P1) we have that every arc in $H_k[u_1, ..., u_r]$ is good. We use Lemma 4 in order to get a good orientation of $G[u_1, ..., u_r, v_{k+1}]$.

2. For $r = 4$ we use the construction of a good orientation of K_5 given in Lemma 4 to get a good orientation of $G[u_1, ..., u_4, v_{k+1}]$. For $r = 2$ we orient new edges in a directed triangle following the orientation given to $\{u_1, u_2\}$. In both cases the bad arc in H_k (if any) belongs to one of the directed triangles in $H[u_1, ..., u_r, v_{k+1}]$. In other words, this arc becomes good in H_{k+1}.

3. For $r = 3$ we consider three cases. Let us define $H' := H_k[u_1, u_2, u_3]$. By property (P2) we know that in H' there is at most one bad arc.

   (i) If in H' there is one bad arc, let us say (u_1, u_2), then we direct the new edges obtaining the following arcs: (u_2, v_{k+1}) and (v_{k+1}, u_1). Moreover, if $(u_1, u_3) \in H'$ then we

add (u_3, v_{k+1}) to H_{k+1}. Otherwise (u_3, u_2) belongs to H' and we add (v_{k+1}, u_3) to H_{k+1}. Then the arcs (u_2, v_{k+1}), (v_{k+1}, u_1) and (u_1, u_2) are in a directed triangle and the arc between v_{k+1} and u_3 is also in a directed triangle. Therefore all arcs in $H_{k+1}[u_1, u_2, u_3, v_{k+1}]$ are good.

(ii) If H' has no bad arcs and v_{k+1} is not δ-super-simplicial then by Lemma 6 at least one edge, say, with ends in $\{v_{k+1}, u_1\}$, is used in a step $j > k$. Then we direct edges $\{v_{k+1}, u_2\}$ and $\{v_{k+1}, u_3\}$ to form a directed triangle with the arc (u_2, u_3) (or (u_3, u_2)) and we add the bad arc (v_{k+1}, u_1).

(iii) The last case is when H' has no bad arcs and v_{k+1} is δ-super-simplicial. To have property (c) we direct the edges $\{v_{k+1}, u_2\}$ and $\{v_{k+1}, u_3\}$ in order to form a directed triangle with the arc (u_2, u_3) (or (u_3, u_2)), where the vertex u_2 has, among all the u_i's, the minimum distance in G to v_1. Finally, we add the bad arc (v_{k+1}, u_1).

Notice that a bad arc appears in $H_{k+1}[u_1, \ldots, u_r, v_{k+1}]$ only when $r = 3$ and $H_{k+1}[u_1, u_2, u_3]$ has no bad arcs. In this case it is (v_{k+1}, u_1). It is easy to see that the orientation H_{k+1} satisfies properties (P1), (P2) and (P3).

Clearly the orientation $H := H_n$ satisfies properties (a) and (b). We prove that H satisfies property (c) by induction in k. Since the p.c.o is the ordering obtained for a LexBFS started at v_1 we know that $d_G(v_1, v_k) = d_{G_k}(v_1, v_k)$. As before, let $N_{G_{k+1}}(v_{k+1}) = \{u_1, \ldots, u_r\}$. Let us assume that for all $v \in G_k$

$$\vec{d}_H(v_1, v) \leq 2d_G(v_1, v) \tag{1}$$

If there are no bad arcs in H connecting v_{k+1} with $u_1, \ldots, u_r$ then we have that $\vec{d}_H(u_i, v_{k+1}) \leq 2$ for all $i = 1, \ldots, r$ and (1) holds for G_{k+1}.

If v_{k+1} is connected to some u_i by a bad arc in H then $r = 3$, $H[u_1, u_2, u_3]$ has no bad arcs and v_{k+1} is a δ-super-simplicial vertex of G. Moreover $d_G(v_1, v_{k+1}) = d_G(v_1, u_2) + 1$ where u_2 is, among all the u_i's, the vertex having minimum distance to v_1 in G.

By the construction of H, there exists a directed triangle that contains v_{k+1} and u_2 which implies that $\vec{d}_H(u_2, v_{k+1}) \leq 2$. Therefore $\vec{d}_H(v_1, v_{k+1}) \leq 2d_G(v_1, u_2) + 2 = 2d_G(v_1, v_{k+1})$. Property (d) follows from (P2).

Finally, we claim that for every k the orientation of the arcs adjacent to v_{k+1} during the extension of the orientation H_k to H_{k+1} can be performed in $O(|N(v_{k+1})|)$. We assume that the set of δ-super-simplicial vertices is known. (Clearly this set can be computed in linear time $O(\sum_{v \in V(G)} |N(v)|) = O(|E(G)|)$.)

If we are in the cases 1 or 2 then the orientation of arcs can be performed in $O(|N(v_{k+1})|)$ steps by Lemma 4. If we are in case 3 then subcase (i) is performed in a constant number of steps. For subcases (ii) we should be able to find a vertex from $\{u_1, u_2, u_3\}$ which has a common neighbor with v_{k+1} in $\{v_{k+2}, \ldots, v_n\}$. Every neighbor of v_{k+1} in $\{v_{k+2}, \ldots, v_n\}$ has at most 3 neighbors in $\{v_1, v_2, \ldots, v_k\}$ and such a vertex can be found in $O(|N(v_{k+1})|)$ steps. The subcase (iii) takes constant number of steps.

Therefore, the complexity of the algorithm is

$$O(\sum_{1 \leq k \leq n} |N(v_k)|) = O(|E(G)|)$$

□

Lemma 8. *There exists a linear time algorithm such that, given a 2-connected chordal graph G, it computes an orientation H satisfying, for every pair of vertices u and v, $\vec{d}_H(u,v) \leq 2d_G(u,v) + 1$.*

Proof. Given G the algorithm first computes (in linear time) a p.c.o. and then the orientation H (in linear time) given by Lemma 7. We prove that H has the desired property.

Take $u, v \in V(G)$ and let P be a shortest $(u,v)-$path in G. If $d_G(u,v) = 1$ then u, v are in some clique C, $|C| \geq 3$. From Property (d) we have $\vec{d}_H(u,v) \leq 3$.

Suppose that $d_G(u,v) > 1$. Let us assume that $P = (u, x, \ldots, y, v)$ (where x and y could be equal). Then $d_H(u,v) \leq d_H(u,x) + d_H(x,y) + d_H(y,v)$. Clearly, the inner vertices of P cannot be simplicial; therefore each arc in H associated to some inner edge of P is contained in a directed triangle in H. Thus $d_H(x,y) \leq 2d_G(x,y)$. In order to finish the proof we need to prove that $d_H(u,x) + d_H(y,v) \leq 5$. Since every clique in H has diameter at most three it suffices to show that $d_H(u,x) = 2$. If $(u,x) \in H$ then $d_H(u,x) = 1$. Otherwise, $(x,u) \in H$.

Since x is not simplicial (x, u) is a good arc and then $d_H(u, x) = 2$. A similar analysis shows that $d_H(v, u) \leq d_H(v, y) + d_H(y, x) + d_H(x, u) \leq 2d_G(u, v) + 1$ (by proving that $d_H(v, y) \leq 2$). $\square$

Proof of Theorem 2. The set of 2-connected components has a tree-like structure T. By the classical result of Tarjan [28], the tree-like structure of the 2-connected components can be computed in linear time. We choose one 2-connected component C_0 as a root of T. Notice that once a root has been defined the notion of father and sons of a 2-connected component is well defined. For every 2-connected component C we define its *father cut vertex* as the unique vertex in C which belongs to its father.

In each 2-connected component other than the root we compute a p.c.o. starting in its father cut vertex and orient it as in Lemma 7. For C_0 we compute a p.c.o starting at any vertex and orient it as in Lemma 7. Let H be the orientation of G so obtained. In each 2-connected component the construction of H is done in linear time by Lemma 7.

Let u and v be two vertices of G and let P be a $(u, v)-$shortest path in G. If P has no cut vertices then u and v lie in the same 2-connected component. From Lemma 8 and the construction of H we have $\vec{d}_H(u, v) \leq 2d_G(u, v) + 1$. Otherwise, let $u_1, \ldots, u_r$ be the cut vertices in P and C_i the 2-connected component containing u_i and u_{i+1} for $i = 1, \ldots, r-1$. Notice that for at most one i_0 neither u_{i_0} nor u_{i_0+1} are father cut vertices of C_{i_0}. From the construction of H we know that $\vec{d}_H(u_i, u_{i+1}) \leq 2d_G(u_i, u_{i+1})$ for all $i = 1, \ldots, r$ not equal to i_0 (in this case $\vec{d}_H(u_{i_0}, u_{i_0+1}) \leq 2d_G(u_{i_0}, u_{i_0+1}) + 1$). Therefore, $\vec{d}_H(u, v) \leq 2d_G(u, v) + 1$. $\square$

3 Negative results

Our first step is to prove the NP-completeness of ODPD(k) for chordal graphs. In fact we will prove two results: the NP-completeness of ODPD(2) for split graphs and the NP-completeness of ODPD(k) for chordal graphs for every $k \geq 4$.

A graph G is a split graph if its vertex set $V(G)$ can be partitioned into sets C and I such that C is a clique and I is an independent set. Split graphs form a subclass of chordal graphs. Our proof, inspired by the one of Chvátal and Thomassen [6], relies on the NP-completeness

of the 2-coloring problem for hypergraphs [19]. Let us recall that a hypergraph $\mathcal{H}$ is called 2-colorable if its vertices can be colored red and blue in such a way that every edge includes at least one vertex of each color.

Lemma 9. *For every $t \geq 0$ and for every hypergraph $\mathcal{H}$ there exists a polynomial time computable chordal graph $G^t_{\mathcal{H}}$ (split graph for $t = 0$) such that if $\mathcal{H}$ is 2-colorable then $OD(G^t_{\mathcal{H}}) = 2(t+1)$ and if $\mathcal{H}$ is not 2-colorable then $OD(G^t_{\mathcal{H}}) = 3(t+1)$*

Proof. We first consider the case $t = 0$. For a given hypergraph $\mathcal{H}$, we will construct a split graph $G^0_{\mathcal{H}} = G_{\mathcal{H}}$ such that if $\mathcal{H}$ is 2-colorable then $OD(G_{\mathcal{H}}) = 2$ and if $\mathcal{H}$ is not 2-colorable then $OD(G_{\mathcal{H}}) = 3$. Let $\mathcal{H}$ be a hypergraph with vertex set V of size n and edge set E of size m.

The clique C of $G_{\mathcal{H}}$ contains $n + 2m + 5$ vertices. More precisely, $C = W \cup Y$ with $W = V \cup \{\gamma, \delta, \eta\}$ and $Y = \{\alpha, \beta\} \cup E_1 \cup E_2$ where E_1 and E_2 are copies of the edge set E of $\mathcal{H}$. The independent set I of $G_{\mathcal{H}}$ contains $m + 1$ vertices. More precisely, $I = \{x\} \cup E$.

Now let us explain how to connect the vertices of I with those of C. The vertex x is connected to all the vertices of W. A vertex $e \in E$ is connected to a vertex $v \in V$ if and only if $v \in e$ (in the hypergraph $\mathcal{H}$). Finally, every vertex $y \in Y$ is connected to every vertex e of E.

We construct an orientation H of $G_{\mathcal{H}}$ with the required properties in two steps. In the first step we orient all the edges not connecting V with I. The orientation of these edges will not depend on the 2-colorability of $\mathcal{H}$. In the second step we orient the remaining edges: those connecting V with I.

Step 1. Notice first that the graph induced by $E \cup E_1 \cup E_2$ contains a copy of $K_{m,m,m}$. Gutin has proven that for all $m \geq 1$ there is an orientation of $K_{m,m,m}$ with diameter 2 [12]. We use this orientation for the edges between E, E_1 and E_2. The orientation of the edges inside E_1 and inside E_2 is irrelevant. We assume that $n \geq 5$ and we orient the edges inside V as in Lemma 4. Then the diameter of the graph $G_{\mathcal{H}}[V]$ is 2.

The rest of the edges not connecting V and $I = \{x\} \cup E$ are oriented as indicated in the following $0-1$ matrix. A value 1 in the position (P, Q) means that all the edges connecting the vertices of P with those of Q are oriented from P towards Q and we denote it by $P \to Q$.

	x	γ	η	δ	V	α	β	E	E_1	E_2
x	-	1	1	-	**?**	-	-	-	-	-
γ	-	-	-	1	1	1	1	-	-	-
η	-	1	-	-	1	-	-	-	1	1
δ	1	-	1	-	-	1	1	-	-	-
V	**?**	-	-	1	-	-	-	**?**	1	1
α	-	-	1	-	1	-	-	1	1	-
β	-	-	1	-	1	1	-	-	-	1
E	-	-	-	-	**?**	-	1	-	-	-
E_1	-	1	-	1	-	-	1	-	-	-
E_2	-	1	-	1	-	1	-	-	-	-

Step 2. We first prove that, no matter the 2-colorability of the hypergraph $\mathcal{H}$, we can orient the remaining edges to obtain an orientation H of the whole graph with diameter less than or equal to 3. Using the orientation $V \to \{x\}$ and $V \to E$ we obtain an orientation H satisfying $d_H(u,v) \leq 3$, for all $u, v \in V(G_{\mathcal{H}})$. To see this it is enough to exhibit a $(u,v)-$directed path of length at most three, for all $u, v \in V(G_{\mathcal{H}})$. Here we only show the property for the case $u = x$ and for the case $u \in E$. The remaining cases can be verified easily. Since $x \to \{\gamma, \eta\}$, $\gamma \to \{\delta, V, \alpha, \beta\}$ and $\eta \to \{E_1, E_2\}$, we have that $d_H(x,v) \leq 2$ for every $v \notin E$. For a vertex $u \in E$ we proceed analogously: since the orientation of the copy of $K_{m,m,m}$ in $E_1 \cup E_2 \cup E$ has diameter 2, there exists an arc (u, u') in H with $u' \in E_1 \cup E_2$. Since $E_1 \cup E_2 \to \{\gamma, \delta\}$, $E \to \beta$ and $\beta \to \{\eta, V, \alpha\}$, we have that $d_H(u,v) = 2$, for all $u \in E$ and all $v \neq x$. Moreover, if $d_H(u,v) = 3$ then u and v respectively belong to $\{x\}$ and E.

In order to conclude that if $\mathcal{H}$ is 2-colorable then $OD(G_{\mathcal{H}}) = 2$ and if $\mathcal{H}$ is not 2-colorable then $OD(G_{\mathcal{H}}) = 3$, it is enough to prove that

$$\mathcal{H} \text{ is 2-colorable} \Leftrightarrow OD(G_{\mathcal{H}}) = 2.$$

($\Leftarrow$) Let H be an orientation of $G_{\mathcal{H}}$ of diameter two. The way we color every vertex v

of the hypergraph $\mathcal{H}$ is the following: if according to H, the edge connecting x with v is oriented towards v, then we color it red. Otherwise we color it blue. Since for every vertex $e \in E$ the distance $d_H(x, e) = d_H(e, x) = 2$, it follows that every edge e in $\mathcal{H}$ contains a red and a blue vertex.

($\Rightarrow$) Now let us suppose that $\mathcal{H}$ is 2-colorable. Let us denote by R and B the set of red and blue vertices in V. We orient all the edges in $G_{\mathcal{H}}$ except those with ends in V and E or with one of their ends being x as given by the partial orientation obtained in Step 1. In order to achieve $\mathrm{diam}(H) = 2$ we only have to reach x from E with paths of length two and vice versa. We orient all the edges between R and E towards E and all the edges between B and E towards B. Finally, we orient $\{x, v\}$ as (x, v) if $v \in R$ and as (v, x) otherwise. Since $\mathcal{H}$ is 2-colorable there is a directed path of length 2 in H from x to every $e \in E$ and from every $e \in E$ to x.

For $t > 0$ the chordal graph $G^t_{\mathcal{H}}$ is constructed from $t+1$ copies of the split graph $G^0_{\mathcal{H}} = (C, \{e_1, \ldots, e_m, x\})$ where the vertex x of the ith copy is identified with some element in E in the $(i-1)$th copy, for $i = 1, \ldots, t$ (see Figure 2). It is easy to see that $G^t_{\mathcal{H}}$ is a chordal graph and that if $\mathcal{H}$ is 2-colorable then $OD(G^t_{\mathcal{H}}) = 2(t+1)$ and if $\mathcal{H}$ is not 2-colorable then $OD(G^t_{\mathcal{H}}) = 3(t+1)$. $\square$

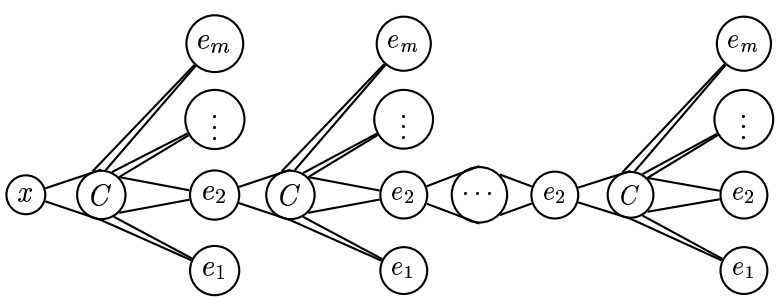

Figure 2: Path-like structure of $G^t_{\mathcal{H}}$.

Theorem 10. ODPD(2) *is NP-complete for split graphs and, for every* $k \geq 4$, ODPD(k) *is NP-complete for chordal graphs.*

Proof. By using Theorem 9 we can reduce, in polynomial time, the 2-coloring hypergraph problem to ODPD(k). For $k = 2$ we take $t = 0$ and for $k \geq 4$ we take $t = \lfloor \frac{k}{2} \rfloor - 1$. □

Now we prove two results concerning the hardness of approximating the oriented diameter.

Theorem 11. *Let $\alpha < \frac{3}{2}$. Unless $P = NP$,* ODP *has no α-approximation algorithm for split graphs.*

Proof. Let $A(G)$ be the orientation assigned to a graph G by an α-approximation algorithm for ODP. If $\mathcal{H}$ is 2-colorable then $G_{\mathcal{H}}$ has diameter 2. Thus $\text{diam}(A(G_{\mathcal{H}})) \leq 2\alpha < 3$. On the other hand, if $\mathcal{H}$ is not 2-colorable then every orientation of the graph $G_{\mathcal{H}}$ has diameter at least 3. Whence $\text{diam}(A(G_{\mathcal{H}})) \geq 3$. Therefore, since $G_{\mathcal{H}}$ can be constructed in polynomial time, by knowing $\text{diam}(A(G_{\mathcal{H}}))$ we could decide the 2-colorability of $\mathcal{H}$. □

Theorem 12. *Unless $P = NP$, there is no absolute approximation algorithms for* ODP *when restricted to chordal graphs.*

Proof. Let us assume that there exist K and an absolute approximation algorithm A for ODP such that $\text{diam}(A(G)) \leq OD(G) + K$. By using this algorithm we could decide the 2-coloring problem for hypergraphs. Let $\mathcal{H}$ be a hypergraph and $t \geq K$. From Lemma 9 there exists a chordal graph $G^t_{\mathcal{H}}$ computable in polynomial time such that, if $\mathcal{H}$ is 2-colorable, then G^t_H has diameter $2t + 2$. Thus $\text{diam}(A(G^t_{\mathcal{H}})) \leq 2t + 2 + K \leq 3t + 2$. And, if H is not 2-colorable then $G^t_{\mathcal{H}}$ has diameter $3t + 3$. Thus $\text{diam}(A(G^t_{\mathcal{H}})) \geq 3t + 3$. □

4 Concluding remarks

In this paper we have provided a linear time $(2, 1)$-approximation algorithm for the problem of finding the oriented diameter of chordal graphs. On the other hand, we have proved that for every $\alpha < 3/2$, finding an orientation with diameter at most α times the oriented diameter is NP-hard. The challenge is to decrease the gap between these lower and upper bounds. But even the existence of a 2-approximation algorithm is an interesting open problem.

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