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Epsilon-Eigenvalues of Multivalued Operators

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Abstract. Landau's theory of ε-eigenvalues for bounded linear operators is well suited to cope with difficulties arising from the lack of usual eigenvalues. The notion of ε-eigenvalue is studied here in the general setting of positively homogeneous multivalued operators. Besides containing a number of new results, the paper sheds a fresh light on many features of the original theory.

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1. Introduction

Throughout this paper H denotes a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and associated norm $\|\cdot\|$. The point spectrum of a bounded linear operator $A\colon H \to H$ is understood as the set of all *real* eigenvalues of A, that is to say,

$$\Lambda(A) := \{\lambda \in \mathbb{R} \mid Ax = \lambda x \text{ for some } x \neq 0\}.$$

Although this set provides a valuable information on the structure of A, the unfavorable case $\Lambda(A) = \emptyset$ occurs in some concrete applications. Thus, judging the behavior of A solely by its point spectrum may be problematic. To compensate the lack of eigenvalues, it is natural to bring exogenous or artificial eigenvalues into the picture. One possible way of enlarging the set $\Lambda(A)$ is as follows:

DEFINITION 1.1. Let $A\colon H \to H$ be a bounded linear operator. For $\varepsilon > 0$, one defines

$$\Lambda_\varepsilon(A) := \{\lambda \in \mathbb{R} \mid \|\lambda x - Ax\| \leqslant \varepsilon\|x\| \text{ for some } x \neq 0\}$$

as the set of *ε-eigenvalues* of A. The elements in

$$\Lambda_+(A) := \bigcap_{\varepsilon>0} \Lambda_\varepsilon(A)$$

are said to be *approximate eigenvalues* of A. The set $\Lambda_+(A)$ is called the *approximate point spectrum* of A.

Perhaps the simplest way of characterizing $\Lambda_+(A)$ is by means of the equivalence:

$$\lambda \in \Lambda_+(A) \Longleftrightarrow \begin{cases} \text{there is a sequence } \{x_n\}_{n\in\mathbb{N}} \subset H \text{ of unit} \\ \text{vectors such that } \|\lambda x_n - Ax_n\| \to 0 \quad \text{as } n \to \infty. \end{cases} \tag{1.1}$$

Most authors [2–4] use (1.1) as the definition of $\Lambda_+(A)$. The point spectrum is contained in the approximate point spectrum, but the converse statement is not true. Definition 1.1 makes sense also for $\varepsilon = 0$, but in such case one simply gets $\Lambda_0(A) = \Lambda(A)$. The introduction of the concept of ε-eigenvalue can be traced back to Landau [7, 8]. Complex ε-eigenvalues are allowed in Landau's work, but we restrict our attention only to the real field. To have an intuitive grasp of Definition 1.1, nothing is better than examining an illuminating special example:

EXAMPLE 1.1. Let φ: $[0, 1] \to \mathbb{R}$ be a strictly increasing continuous function. Consider the self-adjoint linear operator A: $L^2[0, 1] \to L^2[0, 1]$ given by

$$(Ax)(t) = \varphi(t)x(t) \quad \text{for a.e. } t \in [0, 1].$$

One sees immediately that $\Lambda(A) = \emptyset$. The number $\lambda \in \mathbb{R}$ is an ε-eigenvalue of A iff,

$$\int_0^1 [\lambda - \varphi(t)]^2 x^2(t)\, \mathrm{d}t \leqslant \varepsilon^2 \int_0^1 x^2(t)\, \mathrm{d}t$$

for some nonzero vector $x \in L^2[0, 1]$. A direct computation shows that

$$\Lambda_\varepsilon(A) =]\varphi(0) - \varepsilon, \varphi(1) + \varepsilon[\quad \forall \varepsilon > 0,$$

from where one gets $\Lambda_+(A) = [\varphi(0), \varphi(1)]$.

Much of the early research on ε-eigenvalues was centered on infinite dimensional applications to integral equations and differential equations. General information on this subject can be found, for instance, in the survey by Trefethen [17], or in the book by Gustafson and Rao [3]. As an alternative to Landau's theory of ε-eigenvalues, some authors have developed the theory of ε-pseudospectra. The ε-pseudospectrum

$$\sigma_\varepsilon(A) := \{\lambda \in \mathbb{R} \mid A - \lambda I \text{ is not invertible or } \|(A - \lambda I)^{-1}\| \geqslant \varepsilon^{-1}\}$$

corresponds to an enlargement of the spectrum

$$\sigma(A) := \{\lambda \in \mathbb{R} \mid A - \lambda I \text{ is not invertible}\}.$$

Both theories coincide for matrices, but some differences arise in the context of linear operators on infinite-dimensional spaces. Readers interested in the theory of ε-pseudospectra can consult Trefethen [17], Harrabi [5], Roch and Silbermann

[13], and the references therein. The purpose of this work is to further develop Landau's theory of ε-eigenvalues. Most of our results apply not only to

$$\mathcal{L}(H) := \{A\colon\ H \to H \mid A \text{ is linear and bounded}\},$$

but also to a broad class of multivalued operators. Although it is helpful to have some familiarity with the handling of multivalued operators, the paper is largely self-contained and the proofs are kept as elementary as possible. The notation that we employ is for the most part standard; a partial list is provided below:

$$\begin{aligned} B_H &:= \{x \in H \mid \|x\| \leqslant 1\},\\ S_H &:= \{x \in H \mid \|x\| = 1\},\\ \operatorname{dist}[z; C] &:= \text{distance from } z \text{ to } C,\\ \operatorname{co} C &:= \text{convex hull of } C,\\ \operatorname{cl} C &:= \text{closure of } C. \end{aligned}$$

For a multivalued operator $F\colon\ H \rightrightarrows H$, we write:

$$\begin{aligned} D(F) &:= \{x \in H \mid F(x) \neq \emptyset\},\\ \operatorname{Gr} F &:= \{(x, y) \in H \times H \mid y \in F(x)\}. \end{aligned}$$

2. Perturbational Approach

By analogy with the single-valued case, the point spectrum of a multivalued operator $F\colon\ H \rightrightarrows H$ is defined as

$$\Lambda(F) := \{\lambda \in \mathbb{R} \mid \lambda x \in F(x) \text{ for some } x \neq 0\}.$$

Extending Definition 1.1 to the multivalued setting is, however, a more delicate matter. To obtain a meaningful and useful extension, a minimal set of hypotheses must be imposed on F. Unless stated otherwise, we shall assume that

$$\begin{cases} F(x) \neq \emptyset \quad \text{for some } x \neq 0;\\ F \text{ is positively homogeneous, i.e. } F(sx) = sF(x) \quad \forall s > 0, x \in H. \end{cases} \tag{2.1}$$

For the sake of convenience, we write

$$\mathcal{M}(H) := \{F\colon\ H \rightrightarrows H \mid F \text{ satisfies (2.1)}\}.$$

Further assumptions on F will be invoked whenever the need arise. Without further ado, we introduce:

DEFINITION 2.1. For $\varepsilon > 0$, the set $\Lambda_\varepsilon(F)$ of *ε-eigenvalues* of $F \in \mathcal{M}(H)$ is defined by

$$\lambda \in \Lambda_\varepsilon(F) \iff \begin{cases} \text{there is } (x, y) \in \operatorname{Gr} F, \quad \text{with } x \neq 0, \text{ such that}\\ \|\lambda x - y\| \leqslant \varepsilon \|x\|. \end{cases} \tag{2.2}$$

The elements in

$$\Lambda_+(F) := \bigcap_{\varepsilon>0} \Lambda_\varepsilon(F)$$

are called *approximate eigenvalues* of F. One says that $\Lambda_+(F)$ is the *approximate point spectrum* of F.

It is not difficult to check that the set $\Lambda_+(F)$ remains unchanged if condition (2.2) is written with a strict inequality. By convention, one sets $\Lambda_0(F) = \Lambda(F)$. Of course, the single-valued case $F(x) = \{Ax\}$ yields exactly what has been introduced in Definition 1.1.

EXAMPLE 2.1. The simplest example of multivalued operator $F \in \mathcal{M}(H)$ is perhaps

$$F(x) := \{Ax \mid A \in \mathcal{A}\} \quad \forall x \in H,$$

with $\mathcal{A}$ being a nonempty set in $\mathcal{L}(H)$. This operator appears, for instance, in Ioffe's fan theory [6]. The ε-eigenvalue analysis of F reduces to that of each $A \in \mathcal{A}$. In fact, it suffices to observe that

$$\Lambda_\varepsilon(F) = \bigcup_{A\in\mathcal{A}} \Lambda_\varepsilon(A) \quad \forall \varepsilon \in \mathbb{R}_+.$$

The approximate eigenvalues of F must be handled more carefully. The equality

$$\Lambda_+(F) = \bigcup_{A\in\mathcal{A}} \Lambda_+(A)$$

holds when $\mathcal{A}$ is a compact set in $\mathcal{L}(H)$. If $\mathcal{A}$ is not compact, then one has only the lower estimate

$$\Lambda_+(F) \supset \bigcup_{A\in\mathcal{A}} \Lambda_+(A).$$

To see that this inclusion can be strict, consider the collection $\mathcal{A} = \{A_n \mid n \in \mathbb{N}\}$ of operators $A_n\colon L^2[0,1] \to L^2[0,1]$ given by

$$(A_n x)(t) := \left[\frac{1}{2n} + \left(1 - \frac{1}{2n}\right)t\right]x(t) \quad \text{for a.e. } t \in [0,1].$$

Since $\Lambda_+(A_n) = [\frac{1}{2n}, 1]$, the set $\bigcup_{n\in\mathbb{N}} \Lambda_+(A_n) =]0,1]$ is strictly included in $\Lambda_+(F) = [0,1]$.

EXAMPLE 2.2. As a more elaborate example of multivalued operator $F \in \mathcal{M}(H)$, consider the *linear complementarity process*

$$F(x) := \begin{cases} \{y \in H \mid Ax - y \in K^+, \langle Ax - y, x\rangle = 0\} & \text{if } x \in K, \\ \emptyset & \text{if } x \notin K. \end{cases}$$

Here $A \in \mathcal{L}(H)$, $K \subset H$ is a closed convex cone, and

$$K^+ := \{v \in H \mid \langle v, x\rangle \geqslant 0 \quad \forall x \in K\}$$

denotes the positive dual cone of K. The existence of eigenvalues for F has been studied in detail by the second author [10, 16]. One can easily see that $\lambda \in \mathbb{R}$ is an eigenvalue of F if and only if, there is a nonzero vector $x \in H$ such that

$$x \in K, \quad Ax - \lambda x \in K^+, \quad \text{and} \quad \langle Ax - \lambda x, x\rangle = 0.$$

The concept of ε-eigenvalue takes here the form:

$$\lambda \in \Lambda_\varepsilon(F) \Longleftrightarrow \begin{cases} \text{there are } x \neq 0 \text{ and } y \in H, \text{ such that} \\ x \in K,\ Ax - y \in K^+,\ \langle Ax - y, x\rangle = 0, \\ \text{and } \|\lambda x - y\| \leqslant \varepsilon \|x\|. \end{cases}$$

In the general multivalued setting, it is natural to ask whether the ε-eigenvalues of F correspond to usual eigenvalues of some 'perturbed' operator F_ε. This question can be answered in a straightforward manner:

PROPOSITION 2.1. *For each $\varepsilon \in \mathbb{R}_+$, let $F_\varepsilon\colon\ H \rightrightarrows H$ be defined by*

$$F_\varepsilon(x) := F(x) + \varepsilon\|x\|B_H \quad \forall x \in H. \tag{2.3}$$

If F belongs to $\mathcal{M}(H)$, then so does each F_ε. Moreover, $\Lambda_\varepsilon(F) = \Lambda(F_\varepsilon)$ $\forall \varepsilon \in \mathbb{R}_+$.

Proof. It is trivial. □

The operator F_ε can be seen as an ε-enlargement of F. Observe that

$$x \in H \rightrightarrows A_\varepsilon(x) := Ax + \varepsilon\|x\|B_H$$

preserves neither the linearity nor the single-valuedness of $A \in \mathcal{L}(H)$. This is one of the reasons why (2.1) emerges as a natural framework for building a consistent theory of ε-eigenvalues. The enlargement procedure (2.3) is not the only one that serve to generate ε-eigenvalues. There are, in fact, other interesting alternatives. This point can be made clear by formalizing the idea of admissible perturbation. Recall that the 'magnitude' of a positively homogeneous operator $P\colon\ H \rightrightarrows H$ can be evaluated by means of its *outer-norm*

$$\|P\|_+ := \sup_{x \in D(P)\cap B_H} \sup_{v \in P(x)} \|v\|.$$

Another option is to make use of the *inner-norm*

$$\|P\|_- := \sup_{x \in D(P)\cap B_H} \inf_{v \in P(x)} \|v\|.$$

When restricted to the vector space $\mathcal{L}(H)$, both functions $\|\cdot\|_+$ and $\|\cdot\|_-$ coincide with the usual operator norm:

$$\|P\|_+ = \|P\|_- = \|P\| := \sup_{\|x\|\leqslant 1} \|Px\| \quad \forall P \in \mathcal{L}(H).$$

An elaborate discussion on these functions can be found in [11, 15]. In what follows, we use the notation

$$\mathcal{K}(H) := \{P \colon H \rightrightarrows H \mid P \text{ is nonempty-valued and positively homogeneous}\}.$$

DEFINITION 2.2. $\mathcal{P} \subset \mathcal{K}(H)$ is called a *bundle of perturbations* if the following two axioms are fulfilled:

(a) *boundedness*: $\|P\|_+ \leqslant 1 \quad \forall P \in \mathcal{P}$;
(b) *thickness*: $B_H \subset \bigcap_{\|x\|=1} \bigcup_{P\in\mathcal{P}} P(x)$.

In the context of Definition 2.2, each element P in the bundle $\mathcal{P}$ is interpreted as a perturbation operator. The boundedness condition amounts to saying that all perturbations in $\mathcal{P}$ are nonexpansive with respect to the outer-norm

$$\|v\| \leqslant \|x\| \quad \forall (x, v) \in \operatorname{Gr} P,\ \forall P \in \mathcal{P}. \tag{2.4}$$

The thickness requirement stipulates that there are enough perturbations in the bundle in order to ensure a certain covering property:

$$\forall u \in B_H,\ \forall x \in S_H \quad \exists P \in \mathcal{P} \text{ such that } u \in P(x). \tag{2.5}$$

PROPOSITION 2.2. *Examples of bundles of perturbations include*:

(a) $\mathcal{P}_1 := \{P \in \mathcal{L}(H) \mid \|P\| \leqslant 1, \operatorname{rank}(P) \leqslant 1\}$;
(b) $\mathcal{P}_2 := \{P \in \mathcal{L}(H) \mid \|P\| \leqslant 1\}$;
(c) $\mathcal{P}_3 := \{P_\alpha \mid \alpha \in [0, 1]\}$, *with* $P_\alpha \in \mathcal{K}(H)$ *given by* $P_\alpha(x) = \alpha\|x\|S_H$.

Proof. $\mathcal{P}_2$ obviously satisfies the boundedness axiom. As a consequence, the smaller set $\mathcal{P}_1$ satisfies the same requirement. We now check that $\mathcal{P}_1$ is 'thick' enough. Given $u \in B_H$ and $x \in S_H$, one constructs the tensor product

$$h \in H \longrightarrow Ph = [u \otimes x](h) := \langle x, h\rangle u.$$

Clearly, $P \in \mathcal{P}_1$ and $u = Px$. This proves that $\mathcal{P}_1$ satisfies the thickness axiom. As a consequence, the larger set $\mathcal{P}_2$ satisfies this property as well. The case of $\mathcal{P}_3$ is easy to deal with, so it is left as exercise. □

No great intrinsic interest is claimed for Definition 2.2. The abstract notion of bundle of perturbations is used here to obtain a general representation formula for the set $\Lambda_\varepsilon(F)$.

THEOREM 2.1. *Let $F \in \mathcal{M}(H)$ and $\varepsilon \in \mathbb{R}_+$. The set $\Lambda_\varepsilon(F)$ admits the representation*

$$\Lambda_\varepsilon(F) = \bigcup_{P \in \mathcal{P}} \Lambda(F + \varepsilon P), \tag{2.6}$$

with $\mathcal{P} \subset \mathcal{K}(H)$ being an arbitrary bundle of perturbations.

Proof. The case $\varepsilon = 0$ is trivial and without interest. Consider any $\varepsilon > 0$. We begin by showing that

$$\text{boundedness axiom } \Rightarrow \bigcup_{P \in \mathcal{P}} \Lambda(F + \varepsilon P) \subset \Lambda_\varepsilon(F).$$

Take λ in the right-hand side of (2.6). Then one can find $P \in \mathcal{P}$ and $x \neq 0$ such that

$$\lambda x \in F(x) + \varepsilon P(x).$$

If one writes $\lambda x = y + \varepsilon v$, with $y \in F(x)$ and $v \in P(x)$, then $\|\lambda x - y\| = \varepsilon \|v\| \leqslant \varepsilon \|x\|$. This proves that $\lambda \in \Lambda_\varepsilon(F)$. Now we show that

$$\text{thickness axiom } \Rightarrow \Lambda_\varepsilon(F) \subset \bigcup_{P \in \mathcal{P}} \Lambda(F + \varepsilon P).$$

If $\lambda \in \Lambda_\varepsilon(F)$, then $\|\lambda x - y\| \leqslant \varepsilon$ for some $(x, y) \in \operatorname{Gr} F$, with $\|x\| = 1$. Since

$$u := \varepsilon^{-1}(\lambda x - y) \in B_H,$$

we know that $u \in P(x)$ for some $P \in \mathcal{P}$. In such a case, $\lambda x \in F(x) + \varepsilon P(x)$. This proves that $\lambda \in \Lambda(F + \varepsilon P)$ for some $P \in \mathcal{P}$. □

Some comments on Theorem 2.1 are in order. Consider the operator P_0 given by

$$P_0(x) = \|x\| B_H \quad \forall x \in H.$$

It is easy to check that the singleton $\mathcal{P}_0 := \{P_0\}$ fulfills both requirements of Definition 2.2. In fact, $\mathcal{P}_0$ is the only singleton satisfying boundedness and thickness at the same time. The choice of $\mathcal{P}_0$ in the representation formula (2.6) yields the result stated in Proposition 2.1. Another particular instance of (2.6) is

$$\Lambda_\varepsilon(A) = \bigcup_{\|E\| \leqslant \varepsilon} \Lambda(A + E) \quad \forall A \in \mathcal{L}(H). \tag{2.7}$$

The set appearing on the right-hand side of (2.7) has been studied in particular contexts by a number of authors [3, 5, 12, 13, 17].

3. Monotonicity Properties

It is plain to see that the mapping $\varepsilon \in \mathbb{R}_+ \rightrightarrows \Lambda_\varepsilon(F)$ is nondecreasing in the sense that

$$0 \leqslant \varepsilon_1 \leqslant \varepsilon_2 \Rightarrow \Lambda_{\varepsilon_1}(F) \subset \Lambda_{\varepsilon_2}(F).$$

The above monotonicity property can be sharpened as follows:

PROPOSITION 3.1. *Let $F \in \mathcal{M}(H)$. For any $\alpha \in \mathbb{R}_+$ and $\varepsilon \in \mathbb{R}_+$, one can write*

$$\Lambda_\alpha(F) + \varepsilon[-1, 1] \subset \Lambda_{\alpha+\varepsilon}(F). \tag{3.1}$$

In other words,

$$0 \leqslant \varepsilon_1 \leqslant \varepsilon_2 \Rightarrow \Lambda_{\varepsilon_1}(F) + (\varepsilon_2 - \varepsilon_1)[-1, 1] \subset \Lambda_{\varepsilon_2}(F). \tag{3.2}$$

Proof. Let $\mu = \lambda + \varepsilon s$, with $\lambda \in \Lambda_\alpha(F)$ and $s \in [-1, 1]$. For some $(x, y) \in \operatorname{Gr} F$, with $x \neq 0$, one has

$$\frac{\|\mu x - y\|}{\|x\|} = \frac{\|\lambda x - y + \varepsilon s x\|}{\|x\|} \leqslant \frac{\|\lambda x - y\|}{\|x\|} + \varepsilon|s| \leqslant \alpha + \varepsilon.$$

Hence, $\mu \in \Lambda_{\alpha+\varepsilon}(F)$. □

Remark. Formula (3.1) yields in particular $\operatorname{cl}[\Lambda_\varepsilon(F)] \subset \Lambda_{2\varepsilon}(F)$ $\forall \varepsilon > 0$, from where one obtains

$$\Lambda_+(F) = \bigcap_{\varepsilon>0} \operatorname{cl}[\Lambda_\varepsilon(F)].$$

As a consequence, approximate point spectra are always closed sets.

COROLLARY 3.1. *Let $F \in \mathcal{M}(H)$. Then,*

$$\Lambda(F) + \varepsilon[-1, 1] \subset \Lambda_\varepsilon(F) \quad \forall \varepsilon \in \mathbb{R}_+. \tag{3.3}$$

The next proposition provides an upper estimate for $\Lambda_\varepsilon(F)$. Recall that

$$W(F) := \left\{ \frac{\langle y, x\rangle}{\langle x, x\rangle} \mid (x, y) \in \operatorname{Gr} F,\ x \neq 0 \right\}$$

corresponds to the *numerical range* of $F\colon H \rightrightarrows H$ (see [9] for more details on this concept).

PROPOSITION 3.2. *For any $F \in \mathcal{M}(H)$, one has*

$$\Lambda_\varepsilon(F) \subset W(F) + \varepsilon[-1, 1] \quad \forall \varepsilon \in \mathbb{R}_+.$$

In particular, $\Lambda_+(F) \subset \operatorname{cl}[W(F)]$.

Proof. If $\lambda \in \Lambda_\varepsilon(F)$, then one has $\lambda x - y = \varepsilon\|x\|u$ for some $u \in B_H$ and $(x, y) \in \mathrm{Gr}\, F$, with $x \neq 0$. As a consequence,

$$\lambda = \frac{\langle y, x\rangle}{\langle x, x\rangle} + \varepsilon\frac{\langle u, x\rangle}{\|x\|}.$$

This yields of course the desired conclusion. □

In what follows, the least upper bound and the greatest lower bound of $\Lambda_\varepsilon(F)$ are respectively denoted by

$$u_F(\varepsilon) := \sup\{\lambda \in \mathbb{R} \mid \lambda \in \Lambda_\varepsilon(F)\} \quad \text{and} \quad l_F(\varepsilon) := \inf\{\lambda \in \mathbb{R} \mid \lambda \in \Lambda_\varepsilon(F)\}.$$

COROLLARY 3.2. *Let $F \in \mathcal{M}(H)$. Then,*

(a) $u_F\colon \mathbb{R}_+ \to \overline{\mathbb{R}}$ *is an increasing function. More precisely,*

$$0 \leqslant \varepsilon_1 \leqslant \varepsilon_2 \Rightarrow u_F(\varepsilon_1) + (\varepsilon_2 - \varepsilon_1) \leqslant u_F(\varepsilon_2);$$

(b) $l_F\colon \mathbb{R}_+ \to \overline{\mathbb{R}}$ *is a decreasing function. More precisely,*

$$0 \leqslant \varepsilon_1 \leqslant \varepsilon_2 \Rightarrow l_F(\varepsilon_2) + (\varepsilon_2 - \varepsilon_1) \leqslant l_F(\varepsilon_1).$$

Proof. It is straightforward from the second formula in Proposition 3.1. □

The strict monotonicity property (3.1) has further consequences. The theorem below provides a formula for the least upper bound of the set $\Lambda_+(F)$.

THEOREM 3.1. *Assume that $F \in \mathcal{M}(H)$ has a bounded numerical range. Then,*

$$\sup\{\lambda \in \mathbb{R} \mid \lambda \in \Lambda_+(F)\} = u_F(0^+) := \lim_{\varepsilon\to 0^+} u_F(\varepsilon). \tag{3.4}$$

Proof. The monotonicity of u_F guarantees the existence of the limit

$$u_F(0^+) := \lim_{\varepsilon\to 0^+} u_F(\varepsilon) = \inf_{\varepsilon>0} u_F(\varepsilon). \tag{3.5}$$

Observe that $u_F(0^+) \neq \infty$ because

$$u_F(\varepsilon) \leqslant \varepsilon + \sup[W(F)] < \infty \quad \forall \varepsilon > 0.$$

Since $\Lambda_+(F) \subset \Lambda_\varepsilon(F)$ for every $\varepsilon > 0$, one obtains immediately

$$\sup[\Lambda_+(F)] \leqslant u_F(0^+).$$

To prove the reverse inequality, consider first the case $\Lambda_+(F) \neq \emptyset$. It is clear that $u_F(0^+) \in \mathbb{R}$. From

$$u_F(\varepsilon) \in \mathrm{cl}[\Lambda_\varepsilon(F)] \quad \forall \varepsilon > 0,$$

one obtains

$$u_F(\varepsilon) \in \Lambda_\varepsilon(F) + r[-1, 1] \subset \Lambda_{\varepsilon+r}(F) \quad \forall \varepsilon, r > 0.$$

But, according to (3.5), for every $\alpha > 0$, there exists $\varepsilon_0 > 0$ such that

$$|u_F(0^+) - u_F(\varepsilon)| \leqslant \alpha \quad \forall \varepsilon \in]0, \varepsilon_0].$$

As a consequence,

$$u_F(0^+) \in \Lambda_{\varepsilon+r}(F) + \alpha[-1, 1] \subset \Lambda_{\varepsilon+r+\alpha}(F) \quad \forall \alpha, r > 0,\ \forall \varepsilon \in]0, \varepsilon_0].$$

This shows that

$$u_F(0^+) \in \Lambda_+(F),$$

from where one obtains the inequality

$$u_F(0^+) \leqslant \sup[\Lambda_+(F)].$$

We now consider the case $\Lambda_+(F) = \emptyset$. Since $\{\mathrm{cl}[\Lambda_\varepsilon(F)]\}_{\varepsilon>0}$ is a nested family of compact sets, one has $\Lambda_\varepsilon(F) = \emptyset$ for all $\varepsilon > 0$ sufficiently small. Therefore,

$$\sup[\Lambda_+(F)] = u_F(0^+) = -\infty.$$

This completes the proof. □

Remark. The boundedness of $W(F)$ yields, of course, a similar formula for the greatest lower bound of $\Lambda_+(F)$:

$$\inf\{\lambda \in \mathbb{R} \mid \lambda \in \Lambda_+(F)\} = l_F(0^+) := \lim_{\varepsilon\to 0^+} l_F(\varepsilon).$$

4. The Companion Function

A lot of information concerning the mapping $\varepsilon \in \mathbb{R}_+ \rightrightarrows \Lambda_\varepsilon(F)$ can be recovered by using a certain auxiliary function $c_F\colon \mathbb{R} \to \mathbb{R}$. As a complement to Definition 2.1, it is natural to introduce:

DEFINITION 4.1. The *companion function* $c_F\colon \mathbb{R} \to \mathbb{R}$ of $F \in \mathcal{M}(H)$ is defined by

$$c_F(\lambda) := \inf\left\{ \frac{\|\lambda x - y\|}{\|x\|} \mid (x, y) \in \mathrm{Gr}\, F,\ x \neq 0 \right\} = \inf_{x\neq 0} \frac{\mathrm{dist}[\lambda x; F(x)]}{\|x\|}.$$

The next proposition collects some basic facts concerning this function. Recall that

$$\lambda \in \mathbb{R} \to \Psi^*_{W(F)}(\lambda) := \sup\{\lambda s \mid s \in W(F)\}$$

stands for the *support function* of the set $W(F)$.

PROPOSITION 4.1. *The function c_F of $F \in \mathcal{M}(H)$ satisfies the following properties*:

(a) $c_F(\lambda) \geqslant 0\ \forall \lambda \in \mathbb{R}$;
(b) $c_F(0) = \inf_{x\neq 0} \frac{\mathrm{dist}[0; F(x)]}{\|x\|} = \inf_{\|x\|=1} \mathrm{dist}[0; F(x)]$;

(c) $c_F^2(\lambda) \geqslant \lambda^2 - 2\Psi^*_{W(F)}(\lambda) + c_F^2(0)\ \forall \lambda \in \mathbb{R}$;
(d) $|c_F(\lambda) - c_F(\mu)| \leqslant |\lambda - \mu|\ \forall \lambda, \mu \in \mathbb{R}$;
(e) *if* $\mathrm{co}[W(F)] \neq \mathbb{R}$, *then* $\mathrm{lip}(c_F) := \sup_{\substack{\lambda,\mu \in \mathbb{R} \\ \lambda \neq \mu}} \frac{|c_F(\lambda) - c_F(\mu)|}{|\lambda - \mu|} = 1$.

Proof. Parts (a) and (b) are immediate. To prove (c), observe that

$$\frac{\|\lambda x - y\|^2}{\|x\|^2} = \lambda^2 - 2\lambda \frac{\langle y, x\rangle}{\langle x, x\rangle} + \frac{\|y\|^2}{\|x\|^2} \quad \forall x \neq 0.$$

Part (d) follows from the inequality

$$\frac{\|\lambda x - y\|}{\|x\|} \leqslant |\lambda - \mu| + \frac{\|\mu x - y\|}{\|x\|} \quad \forall x \neq 0.$$

Part (e) is a consequence of (c) and (d). □

The companion function c_F has an important role to play in the analysis of the nested family $\{\Lambda_\varepsilon(F)\}_{\varepsilon > 0}$. As shown in the next proposition, the sublevel sets

$$\{c_F \leqslant \varepsilon\} := \{\lambda \in \mathbb{R} \mid c_F(\lambda) \leqslant \varepsilon\}$$
$$\{c_F < \varepsilon\} := \{\lambda \in \mathbb{R} \mid c_F(\lambda) < \varepsilon\}$$

provide upper and lower estimates for $\Lambda_\varepsilon(F)$.

PROPOSITION 4.2. *For any $F \in \mathcal{M}(H)$, one can write*

$$\{c_F < \varepsilon\} \subset \Lambda_\varepsilon(F) \subset \{c_F \leqslant \varepsilon\} \quad \forall \varepsilon > 0. \tag{4.1}$$

Proof. It is immediate. □

COROLLARY 4.1. *The approximate point spectrum of $F \in \mathcal{M}(H)$ is given by*

$$\Lambda_+(F) = \{\lambda \in \mathbb{R} \mid c_F(\lambda) = 0\}. \tag{4.2}$$

Proof. The function $c_F\colon \mathbb{R} \to \mathbb{R}$ is nonnegative and continuous. The general theory of sublevel sets yields in this case

$$\{c_F = 0\} = \bigcap_{\varepsilon > 0} \{c_F \leqslant \varepsilon\} = \bigcap_{\varepsilon > 0} \{c_F < \varepsilon\}.$$

It suffices then to apply (4.1). □

COROLLARY 4.2. *For any $F \in \mathcal{M}(H)$, one has*

$$\mathrm{dist}[\lambda; W(F)] \leqslant c_F(\lambda) \leqslant \mathrm{dist}[\lambda; \Lambda_+(F)] \quad \forall \lambda \in \mathbb{R}.$$

Proof. The first inequality is a consequence of Propositions 3.2 and 4.2. The second inequality follows from Proposition 4.1(d) and Corollary 4.1. □

COROLLARY 4.3. *For $F \in \mathcal{M}(H)$, the following statements are equivalent*:

(a) $\forall \varepsilon \in \mathbb{R}_+$, $\Lambda_\varepsilon(F)$ *is bounded*;
(b) c_F *is* coercive *in the sense that* $c_F(\lambda) \to \infty$ *as* $|\lambda| \to \infty$.

Proof. The coercivity of c_F amounts to saying that each sublevel set $\{c_F \leqslant \varepsilon\}$ is bounded. The implication (b) $\Rightarrow$ (a) follows from the second inclusion in (4.1). Conversely, if (a) holds, then each sublevel set $\{c_F \leqslant \varepsilon\}$ is bounded. Indeed,

$$\{c_F \leqslant \varepsilon\} \subset \{c_F < \varepsilon + 1\} \subset \Lambda_{\varepsilon+1}(F).$$

The proof is thus complete. □

Straightforwardly from Proposition 4.1(d), one sees that

$$\limsup_{|\lambda|\to\infty} \frac{c_F(\lambda)}{|\lambda|} \leqslant 1. \tag{4.3}$$

The behavior at infinity of c_F can be described, however, in a more accurate way. To do this, it is helpful to introduce the expression

$$\delta_{\max}(F) := \left[\sup\left\{\frac{\|y\|^2\|x\|^2 - \langle y, x\rangle^2}{\|x\|^4} \mid (x, y) \in \operatorname{Gr} F,\ x \neq 0\right\}\right]^{\frac{1}{2}},$$

which we call the *maximal deviation* of F. Similarly, one says that

$$\delta_{\min}(F) := \left[\inf\left\{\frac{\|y\|^2\|x\|^2 - \langle y, x\rangle^2}{\|x\|^4} \mid (x, y) \in \operatorname{Gr} F,\ x \neq 0\right\}\right]^{\frac{1}{2}}$$

is the *minimal deviation* of F. It is immediate to see that

$$0 \leqslant \delta_{\min}(F) \leqslant \delta_{\max}(F) \leqslant \|F\|_+ \quad \forall F \in \mathcal{M}(H).$$

PROPOSITION 4.3. *Assume that $F \in \mathcal{M}(H)$ has a finite maximal deviation. Then, the following three statements are equivalent*:

(a) $\forall \varepsilon \in \mathbb{R}_+$, $\Lambda_\varepsilon(F)$ *is bounded*;
(b) $\lim_{|\lambda|\to\infty} \frac{c_F(\lambda)}{|\lambda|} = 1$;
(c) *the numerical range $W(F)$ is bounded.*

If one drops the assumption $\delta_{\max}(F) < \infty$, then one still has the chain of implications (c) $\Rightarrow$ (b) $\Rightarrow$ (a).

Proof. According to Proposition 4.1, one has

$$\frac{c_F^2(\lambda)}{\lambda^2} \geqslant 1 - \frac{2}{\lambda^2}\Psi^*_{W(F)}(\lambda) + \frac{c_F^2(0)}{\lambda^2} \quad \forall \lambda \in \mathbb{R} \setminus \{0\}.$$

If $W(F)$ is bounded, then

$$\lim_{|\lambda|\to\infty} \frac{2}{\lambda^2}\Psi^*_{W(F)}(\lambda) = 0.$$

Consequently,

$$\liminf_{|\lambda|\to\infty} \frac{c_F^2(\lambda)}{\lambda^2} \geqslant 1.$$

The above inequality, combined with (4.3), yields condition (b). Of course, (b) forces c_F to be coercive. By Corollary 4.3, the coercivity of c_F is equivalent to condition (a). Assume now that F has finite maximal deviation. To prove that (a) implies (c), it suffices to show that

$$W(F) \subset \Lambda_\alpha(F) \quad \text{with } \alpha = \delta_{\max}(F). \tag{4.4}$$

If $\lambda \in W(F)$, then one can write

$$\lambda = \frac{\langle y, x\rangle}{\langle x, x\rangle} \quad \text{with } (x, y) \in \operatorname{Gr} F,\ x \neq 0.$$

Consequently,

$$\frac{\|\lambda x - y\|^2}{\|x\|^2} = \lambda^2 - 2\lambda \frac{\langle y, x\rangle}{\|x\|^2} + \frac{\|y\|^2}{\|x\|^2} = \frac{\|y\|^2}{\|x\|^2} - \frac{\langle y, x\rangle^2}{\|x\|^4} \leqslant \alpha^2.$$

This shows that $\lambda \in \Lambda_\alpha(F)$. □

The next proposition gives a necessary and sufficient condition for $\Lambda_\varepsilon(F)$ to coincide with the sublevel set $\{c_F \leqslant \varepsilon\}$. Recall that a set-valued mapping $\Phi\colon X \rightrightarrows Y$ between two Hausdorff topological spaces is said to be *graph-closed* at $u_0 \in X$ if

$$\Phi(u_0) = \bigcap_{V \in \mathcal{N}(u_0)} \operatorname{cl}\left[\bigcup_{u \in V} \Phi(u)\right],$$

where $\mathcal{N}(u_0)$ stands for the filter of all neighborhoods of u_0. If X and Y are metrizable spaces, then graph-closedness of Φ at u_0 is equivalent to the condition

$$\{(u_n, v_n)\}_{n\in\mathbb{N}} \subset \operatorname{Gr}\Phi \quad \text{and} \quad \{(u_n, v_n)\}_{n\in\mathbb{N}} \to (u_0, v_0)$$
$$\text{imply } (u_0, v_0) \in \operatorname{Gr}\Phi.$$

PROPOSITION 4.4. *Let $F \in \mathcal{M}(H)$. For any $\varepsilon_0 \in \mathbb{R}_+$, the following statements are equivalent*:

(a) $\Lambda_{\varepsilon_0}(F) = \{c_F \leqslant \varepsilon_0\}$;
(b) $\varepsilon \in \mathbb{R}_+ \rightrightarrows \Lambda_\varepsilon(F)$ *is graph-closed at* ε_0.

Proof. Assume (a). Take $\{(\varepsilon_n, \lambda_n)\}_{n\in\mathbb{N}} \to (\varepsilon_0, \lambda_0)$ with $\lambda_n \in \Lambda_{\varepsilon_n}(F), \forall n \in \mathbb{N}$. By Proposition 4.2, one can write

$$c_F(\lambda_n) \leqslant \varepsilon_n \quad \forall n \in \mathbb{N}.$$

By passing to the limit, one arrives at the desired conclusion. Conversely, let (b) be true. The graph-closedness of $\varepsilon \in \mathbb{R}_+ \rightrightarrows \Lambda_\varepsilon(F)$ at ε_0 can be expressed in the form

$$\Lambda_{\varepsilon_0}(F) = \bigcap_{\delta>0} \mathrm{cl}\Big[\bigcup_{\varepsilon\in[\varepsilon_0-\delta,\varepsilon_0+\delta]} \Lambda_\varepsilon(F)\Big].$$

By monotonicity, the above equality reduces to

$$\Lambda_{\varepsilon_0}(F) = \bigcap_{\delta>0} \mathrm{cl}[\Lambda_{\varepsilon_0+\delta}(F)].$$

But,

$$\mathrm{cl}\{c_F < \varepsilon_0 + \delta\} \subset \mathrm{cl}[\Lambda_{\varepsilon_0+\delta}(F)] \subset \{c_F \leqslant \varepsilon_o + \delta\},$$

and consequently

$$\bigcap_{\delta>0} \mathrm{cl}\{c_F < \varepsilon_0 + \delta\} \subset \bigcap_{\delta>0} \mathrm{cl}[\Lambda_{\varepsilon_0+\delta}(F)] \subset \bigcap_{\delta>0} \{c_F \leqslant \varepsilon_0 + \delta\}.$$

The set on the left coincides with $\{c_F \leqslant \varepsilon_0\}$, and so does the set on the right. The conclusion is that $\Lambda_{\varepsilon_0}(F)$ must also be equal to $\{c_F \leqslant \varepsilon_0\}$. □

Unless stated otherwise, the Hilbert space H is equipped with the strong (or norm) topology. The notation H_ω is used to indicate that the topological properties on H must be understood in the weak sense.

PROPOSITION 4.5. *Suppose that the graph of $F \in \mathcal{M}(H)$ is a closed set in the product space $H \times H_\omega$. Assume also that* cl$[D(F)]$ *admits a compact base, that is to say,*

$$\mathrm{cl}[D(F)] = \mathbb{R}_+ B := \{\alpha\, u \mid \alpha \in \mathbb{R}_+\,,\ u \in B\}$$

for some compact set $B \subset H$ which does not contain the origin. Then,

$$\Lambda_\varepsilon(F) = \{c_F \leqslant \varepsilon\} \quad \forall \varepsilon \in \mathbb{R}_+.$$

In particular, $\Lambda(F) = \Lambda_+(F)$.

Proof. By Proposition 4.4, what must be proven is that

$$\{(\varepsilon, \lambda) \in \mathbb{R}_+ \times \mathbb{R} \mid \lambda \in \Lambda_\varepsilon(F)\}$$

is a closed set. Take a sequence $\{(\varepsilon_n, \lambda_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}_+ \times \mathbb{R}$ converging to $(\varepsilon_0, \lambda_0) \in \mathbb{R}_+ \times \mathbb{R}$, and such that

$$\lambda_n \in \Lambda_{\varepsilon_n}(F) \quad \forall n \in \mathbb{N}.$$

So, one can write

$$\|\lambda_n x_n - y_n\| \leqslant \varepsilon_n \|x_n\|$$

with $(x_n, y_n) \in \operatorname{Gr} F$ and $x_n \neq 0$. But $x_n = \alpha_n u_n$, with $\alpha_n \in \mathbb{R}_+ \setminus \{0\}$ and $u_n \in B$. By taking a subsequence if necessary, one may assume that $\{u_n\}_{n\in\mathbb{N}}$ converges to some vector $u \in B$. If one defines $v_n := y_n/\alpha_n$, then one gets

$$\|\lambda_n u_n - v_n\| \leqslant \varepsilon_n \|u_n\|, \quad \text{with } (u_n, v_n) \in \operatorname{Gr} F. \tag{4.5}$$

The sequence $\{v_n\}_{n\in\mathbb{N}}$ is clearly bounded because

$$\|v_n\| \leqslant (|\lambda_n| + \varepsilon_n)\|u_n\|.$$

Therefore, $\{v_n\}_{n\in\mathbb{N}}$ admits a subsequence that converges weakly to some vector $v \in H$. By working with this subsequence, and passing to the limit in (4.5), one arrives at

$$\|\lambda_0 u - v\| \leqslant \varepsilon_0 \|u\| \quad \text{with } (u, v) \in \operatorname{Gr} F.$$

Since $0 \notin B$, the case $u = 0$ must be ruled out. Hence, $\lambda_0 \in \Lambda_{\varepsilon_0}(F)$. □

Some remarks are now in order:

(i) The equality $\Lambda(F) = \Lambda_+(F)$ is obtained by choosing $\varepsilon_0 = 0$ in the above proof. With such a choice, the sequence $\{(u_n, v_n)\}_{n\in\mathbb{N}} \subset \operatorname{Gr} F$ converges strongly to $(u, \lambda u)$. So, it is enough to request the closedness of $\operatorname{Gr} F$ only with respect to the strong topology of $H \times H$.

(ii) That $\operatorname{cl}[D(F)]$ admits a compact base is a crucial hypothesis in Proposition 4.5. This hypothesis holds trivially if H is finite dimensional; indeed, $\operatorname{cl}[D(F)]$ admits $B = S_H \cap \operatorname{cl}[D(F)]$ as compact base.

While dealing with operators defined on infinite dimensional Hilbert spaces, one should not expect to have always the representation formula

$$\Lambda_\varepsilon(F) = \{c_F \leqslant \varepsilon\}.$$

Example 1.1 displays a case in which $\Lambda_\varepsilon(F)$ is not even closed. A more realistic requirement on F is that $\Lambda_\varepsilon(F)$ coincides with $\{c_F \leqslant \varepsilon\}$ up to a closure operation.

DEFINITION 4.2. One says that $F \in \mathcal{M}(H)$ is *amenable* at the level ε if

$$\operatorname{cl}[\Lambda_\varepsilon(F)] = \{c_F \leqslant \varepsilon\}. \tag{4.6}$$

Essential amenability of F simply means that the above equality holds at each $\varepsilon \in \mathbb{R}_+$ for which $\Lambda_\varepsilon(F)$ is nonempty.

The operator introduced in Example 1.1 is essentially amenable, and so are most operators appearing in practice. Sufficient conditions for amenability are given in the next proposition. Recall that $\Phi\colon X \rightrightarrows Y$ is called *upper-semicontinuous* at $u_0 \in X$ if

$$\left.\begin{array}{l}\Phi(u_0) \subset M,\\ M \text{ is open in } Y\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\text{there is a neighborhood } V \text{ of } u_0\\ \text{such that } \Phi(u) \subset M\ \forall u \in V.\end{array}\right.$$

PROPOSITION 4.6. *Any of the following three assumptions is sufficient for $F \in \mathcal{M}(H)$ to be amenable at the level $\varepsilon_0 > 0$:*

(a) $\mathrm{cl}\{c_F < \varepsilon_0\} = \{c_F \leqslant \varepsilon_0\}$;
(b) $\Lambda_{\varepsilon_0}(F) = \{c_F \leqslant \varepsilon_0\}$;
(c) $\varepsilon \in \mathbb{R}_+ \rightrightarrows \Lambda_\varepsilon(F)$ *is upper-semicontinuous at* ε_0.

Proof. The case (b) is trivial, and the case (a) is a direct consequence of Proposition 4.2. Assume that $\varepsilon \in \mathbb{R}_+ \rightrightarrows \Lambda_\varepsilon(F)$ is upper-semicontinuous at ε_0. For an arbitrary $n \in \mathbb{N}$, the set on the right-hand side of

$$\Lambda_{\varepsilon_0}(F) \subset \Lambda_{\varepsilon_0}(F) + \frac{1}{n}\,]-1, 1[$$

is open. Therefore, there is a strictly increasing function $\varphi\colon \mathbb{N} \to \mathbb{N}$ such that

$$\Lambda_{\varepsilon_0+\frac{1}{m}}(F) \subset \Lambda_{\varepsilon_0}(F) + \frac{1}{n}\,]-1, 1[\quad \forall n \in \mathbb{N},\ \forall m \geqslant \varphi(n).$$

By monotonicity of $\varepsilon \in \mathbb{R}_+ \rightrightarrows \Lambda_\varepsilon(F)$, it follows that

$$K := \bigcap_{m\in\mathbb{N}} \Lambda_{\varepsilon_0+\frac{1}{m}}(F) \subset \Lambda_{\varepsilon_0}(F) + \frac{1}{n}\,]-1, 1[\quad \forall n \in \mathbb{N}.$$

Hence,

$$K \subset \bigcap_{n\in\mathbb{N}} \left\{ \Lambda_{\varepsilon_0}(F) + \frac{1}{n}\,]-1, 1[\right\} = \mathrm{cl}[\Lambda_{\varepsilon_0}(F)].$$

Proposition 4.2 yields the inclusion $\{c_F \leqslant \varepsilon_0\} \subset K$, from where one obtains

$$\{c_F \leqslant \varepsilon_0\} \subset \mathrm{cl}[\Lambda_{\varepsilon_0}(F)].$$

This takes care of the amenability of F at the level ε_0. □

5. The Resolvent

The *resolvent* of $F \in \mathcal{M}(H)$ at $\lambda \in \mathbb{R}$ is the multivalued mapping $(F - \lambda I)^{-1}$: $H \rightrightarrows H$ given by

$$(F - \lambda I)^{-1}(v) := \{x \in H \mid v \in (F - \lambda I)(x)\}.$$

Notice that $(F - \lambda I)^{-1}$ corresponds to the inverse of $F - \lambda I$, with the inversion operation being understood in the multivalued sense. Since $(F - \lambda I)^{-1}$ is a positively homogeneous operator, it makes sense to evaluate its magnitude

$$r_F(\lambda) := \|(F - \lambda I)^{-1}\|_+.$$

A surprising fact is that $r_F\colon \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ turns out to be reciprocal to the companion function c_F. This and other related results are stated below:

THEOREM 5.1. *Let $F \in \mathcal{M}(H)$. For each $\lambda \in \mathbb{R}$, one has*

$$r_F(\lambda) > 0 \quad \text{and} \quad c_F(\lambda) = 1/r_F(\lambda). \tag{5.1}$$

In particular, the set

$$\operatorname{dom} r_F := \{\lambda \in \mathbb{R} \mid r_F(\lambda) < \infty\}$$

is open, and

$$\Lambda_+(F) = \{\lambda \in \mathbb{R} \mid r_F(\lambda) = \infty\}.$$

Proof. From the expression

$$r_F(\lambda) = \sup\{\|x\| \mid x \in (F - \lambda I)^{-1}(v), v \in B_H\},$$

one arrives at

$$r_F(\lambda) = \sup\{\|x\| \mid (x, y) \in \operatorname{Gr} F, \|\lambda x - y\| \leqslant 1\}.$$

If r_F vanished at some point $\lambda_0 \in \mathbb{R}$, then one should have

$$\left.\begin{array}{l}(x, y) \in \operatorname{Gr} F \\ \|\lambda_0 x - y\| \leqslant 1\end{array}\right\} \Rightarrow x = 0. \tag{5.2}$$

But, if one takes $(x_0, y_0) \in \operatorname{Gr} F$, with $x_0 \neq 0$, and chooses

$$(x, y) = \alpha(x_0, y_0), \quad \text{with } 0 < \alpha \leqslant \frac{1}{\|\lambda_0 x_0 - y_0\|},$$

then one sees that (5.2) does not hold. As a consequence, r_F takes only strictly positive values. To prove the second formula in (5.1), fix $\lambda \in \mathbb{R}$ and pick up any sequence

$$\{(x_n, y_n)\}_{n \in \mathbb{N}} \subset \operatorname{Gr} F \quad \text{with } \|\lambda x_n - y_n\| \leqslant 1 \text{ and } \|x_n\| \to r_F(\lambda).$$

Since $r_F(\lambda) > 0$, one can assume that $x_n \neq 0$ for all $n \in \mathbb{N}$. Hence,

$$c_F(\lambda) \leqslant \frac{\|\lambda x_n - y_n\|}{\|x_n\|} \leqslant \frac{1}{\|x_n\|} \quad \forall n \in \mathbb{N}.$$

By passing to the limit, one obtains $c_F(\lambda) \leqslant 1/r_F(\lambda)$. Of course, the above argument applies also when $r_F(\lambda) = \infty$. To prove the reverse inequality, we distinguish between two cases:

I. If λ is an eigenvalue of F, then the eigenset

$$(F - \lambda I)^{-1}(0) = \{x \in H : \lambda x \in F(x)\}$$

is unbounded, and therefore

$$r_F(\lambda) \geqslant \sup\{\|x\| \mid x \in (F - \lambda I)^{-1}(0)\} = \infty.$$

II. If λ is not an eigenvalue of F, then we take a sequence

$$\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset \operatorname{Gr} F \quad \text{with } x_n \neq 0 \quad \text{and} \quad \frac{\|\lambda x_n - y_n\|}{\|x_n\|} \to c_F(\lambda).$$

One has necessarily $\lambda x_n \neq y_n$. Observe that

$$(\tilde{x}_n, \tilde{y}_n) := \frac{1}{\|\lambda x_n - y_n\|}(x_n, y_n) \in \operatorname{Gr} F \quad \text{and} \quad \|\lambda \tilde{x}_n - \tilde{y}_n\| = 1.$$

As a consequence, $\|\tilde{x}_n\| \leqslant r_F(\lambda)$, or equivalently,

$$\frac{1}{r_F(\lambda)} \leqslant \frac{\|\lambda x_n - y_n\|}{\|x_n\|}.$$

By passing to the limit in the above inequality, one completes the proof of (5.1). The last part of the theorem follows from (5.1) and Corollary 4.1. □

6. The Spectral Threshold

The term ε in Definition 2.1 can be understood as a level of tolerance for $\lambda \in \mathbb{R}$ to be admitted as an 'eigenvalue' of $F \in \mathcal{M}(H)$. By increasing the tolerance level if necessary, one can always arrive to a situation in which $\Lambda_\varepsilon(F)$ is nonempty. Of course, one would like to take ε as small as possible.

DEFINITION 6.1. Let $F \in \mathcal{M}(H)$. The real number $\varepsilon_F := \inf\{\varepsilon \in \mathbb{R}_+ \mid \Lambda_\varepsilon(F) \neq \emptyset\}$ is called the *spectral threshold* of F.

The spectral threshold is a concept that admits several interesting characterizations. In the proposition stated below, the notation

$$\operatorname{gap}[A, B] := \inf_{a\in A,\ b\in B} \|a - b\|$$

refers to the *gap* between the sets $A, B \subset H$.

PROPOSITION 6.1. *Let $F \in \mathcal{M}(H)$. Then,*

(a) $\varepsilon_F = \sup\{\varepsilon \in \mathbb{R}_+ \mid \Lambda_\varepsilon(F) = \emptyset\}$;
(b) *ε_F is equal to the infimal value of c_F;*
(c) *ε_F is equal to the minimal deviation of F;*
(d) $\varepsilon_F = \inf_{x\neq 0} \frac{\operatorname{gap}[\mathbb{R}x, F(x)]}{\|x\|}$.

Proof. The formula given in (a) follows from the monotonicity of $\varepsilon \in \mathbb{R}_+ \rightrightarrows \Lambda_\varepsilon(F)$. To prove (b), let us write

$$\gamma := \inf\{c_F(\lambda) \mid \lambda \in \mathbb{R}\}.$$

For every $\varepsilon > 0$, the sublevel set $\{c_F < \gamma + \varepsilon\}$ is nonempty. By applying Proposition 4.2 one gets

$$\Lambda_{\gamma+\varepsilon}(F) \neq \emptyset \quad \forall \varepsilon > 0.$$

This shows that $\varepsilon_F \leqslant \gamma$. If ε_F was strictly smaller than γ, then one could pick up a number λ in $\Lambda_\alpha(F)$, with $\alpha \in\ ]\varepsilon_F, \gamma[$ chosen arbitrarily. In such a case, one would obtain $c_F(\lambda) \leqslant \alpha < \gamma$, contradicting in this way the definition of γ. To prove (c) it suffices to write

$$\begin{aligned} \inf_{\lambda\in\mathbb{R}} c_F(\lambda) &= \inf_{\lambda\in\mathbb{R}} \inf\left\{\frac{\|\lambda x - y\|}{\|x\|} \mid (x, y) \in \operatorname{Gr} F,\ x \neq 0\right\} \\ &= \inf\left\{\inf_{\lambda\in\mathbb{R}} \frac{\|\lambda x - y\|}{\|x\|} \mid (x, y) \in \operatorname{Gr} F,\ x \neq 0\right\}, \end{aligned}$$

and observe hat

$$\left[\inf_{\lambda\in\mathbb{R}} \frac{\|\lambda x - y\|}{\|x\|}\right]^2 = \inf_{\lambda\in\mathbb{R}} \frac{\|\lambda x - y\|^2}{\|x\|^2} = \frac{\|y\|^2\|x\|^2 - \langle y, x\rangle^2}{\|x\|^4}.$$

Part (d) follows from

$$\begin{aligned} \inf_{\lambda\in\mathbb{R}} c_F(\lambda) &= \inf_{\lambda\in\mathbb{R}} \inf_{x\neq 0} \inf_{y\in F(x)} \frac{\|\lambda x - y\|}{\|x\|} \\ &= \inf_{x\neq 0} \inf\left\{\frac{\|\lambda x - y\|}{\|x\|} \mid \lambda \in \mathbb{R},\ y \in F(x)\right\}. \end{aligned}$$

□

7. Some Calculus Rules

In this section we compare the approximate point spectra of two arbitrary operators $F, G \in \mathcal{M}(H)$ that are linked through a simple relation. To keep the lenght of the exposition at bay, only the following three situations will be considered:

$$\begin{aligned} G &= F + \mu I; \\ G &= S^{-1} \circ F \circ S; \\ G &= F \quad \text{on a dense subset of } H. \end{aligned}$$

The first case corresponds to a translation, and it comes without surprise that:

PROPOSITION 7.1. *For any $F \in \mathcal{M}(H)$ and $\mu \in \mathbb{R}$, one has*:

(a) $\Lambda_\varepsilon(F + \mu I) = \Lambda_\varepsilon(F) + \mu, \quad \forall \varepsilon \in \mathbb{R}_+;$
(b) $\Lambda_+(F + \mu I) = \Lambda_+(F) + \mu;$
(c) $c_{F+\mu I}(\lambda) = c_F(\lambda - \mu), \quad \forall \lambda \in \mathbb{R}.$

Proof. It is immediate. □

The second case involves a *similarity transformation* $S\colon\ H \to H$. The relation

$$G(u) := [S^{-1} \circ F \circ S](u) = S^{-1}[F(Su)], \quad \forall u \in H$$

amounts to saying that

$$\operatorname{Gr} G = \{(u, v) \in H \times H \mid (Su, Sv) \in \operatorname{Gr} F\}.$$

PROPOSITION 7.2. *Let $G := S^{-1} \circ F \circ S$, where F belongs to $\mathcal{M}(H)$ and $S \in \mathcal{L}(H)$ is assumed to be a bijection. Then,*

(a) $\forall \varepsilon \in \mathbb{R}_+$, *one has* $\Lambda_\varepsilon(G) \subset \Lambda_{\varepsilon\|S\|\,\|S^{-1}\|}(F)$ *and* $\Lambda_\varepsilon(F) \subset \Lambda_{\varepsilon\|S\|\,\|S^{-1}\|}(G)$;
(b) $\Lambda_+(F) = \Lambda_+(G)$.

Moreover, if S is an isometry (i.e. $\|Su\| = \|u\|\ \forall u \in H$), then

$$\forall \varepsilon \in \mathbb{R}_+ \quad \Lambda_\varepsilon(G) = \Lambda_\varepsilon(F), \quad \textit{and} \quad c_G = c_F.$$

Proof. Let $\lambda \in \Lambda_\varepsilon(G)$. For some $(u, v) \in \operatorname{Gr} G$, with $u \neq 0$, one has

$$\|\lambda u - v\| \leqslant \|u\|.$$

If one writes $x := Su$ and $y := Sv$, then $(x, y) \in \operatorname{Gr} F$ and $x \neq 0$. Moreover,

$$\|\lambda x - y\| \leqslant \|S\|\|\lambda u - v\| \leqslant \varepsilon\|S\|\|S^{-1}x\| \leqslant \varepsilon\|S\|\|S^{-1}\|\|x\|.$$

This proves that $\lambda \in \Lambda_{\varepsilon\|S\|\,\|S^{-1}\|}(F)$. To obtain the second inclusion in part (a), it suffices to apply the same argument to the relation $F = S \circ G \circ S^{-1}$. The remaining part of the proposition can be proven without difficulty. □

Next on our agenda is the case of two operators $F, G \in \mathcal{M}(H)$ that coincide on a dense subset of H. Recall that $\Phi\colon\ X \rightrightarrows Y$ is called *lower-semicontinuous* at $u_0 \in X$ if

$$\left.\begin{array}{l}\Phi(u_0) \cap M \neq \emptyset,\\ M \text{ is open in } Y\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\text{there is a neighborhood } V \text{ of } u_0\\ \text{such that } \Phi(u) \cap M \neq \emptyset \quad \forall u \in V.\end{array}\right.$$

Lower-semicontinuity of Φ simply means that the above property holds for each $u_0 \in X$.

PROPOSITION 7.3. *Let $F, G \in \mathcal{M}(H)$ be two lower-semicontinuous operators. Assume that $G = F$ on some dense subset of H. Then $c_G = c_F$ and $\Lambda_+(F) = \Lambda_+(G)$.*

Proof. Let us show that $c_G \leqslant c_F$. Take any $\lambda \in \mathbb{R}$ and set $\alpha = c_G(\lambda)$. It follows from the definition of c_G that

$$\alpha\|x\| \leqslant \text{dist}[\lambda x; G(x)] \quad \forall x \in H.$$

In particular,

$$\alpha\|x\| \leqslant \text{dist}[\lambda x; F(x)] \quad \forall x \in D,$$

where $D \subset H$ is the dense set over which F and G coincide. Consider now an arbitrary $(\overline{x}, \overline{y}) \in \text{Gr}\, F$, with $\overline{x} \neq 0$. By density of D, one can construct a sequence $\{x_n\}_{n\in\mathbb{N}} \subset D$ converging to $\overline{x}$. The lower-semicontinuity of F at $\overline{x}$ guarantees the existence of a sequence $\{y_n\}_{n\in\mathbb{N}} \to \overline{y}$ such that $y_n \in F(x_n)$ for all n sufficiently large (see [1]). Passing to the limit in

$$\alpha\|x_n\| \leqslant \text{dist}[\lambda x_n; F(x_n)] \leqslant \|\lambda x_n - y_n\|,$$

one obtains

$$\alpha\|\overline{x}\| \leqslant \|\lambda\overline{x} - \overline{y}\|.$$

This shows that $\alpha \leqslant c_F(\lambda)$. The inequality $c_F \leqslant c_G$ is obtained by exchanging the roles of F and G. □

8. The Linear Case

This section focusses the attention on the special case

$$F(x) = \{Ax\} \quad \text{with } A \in \mathcal{L}(H).$$

Much has been said on the ε-eigenvalue analysis of linear continuous operators, so we mention here just one original result. It concerns the behavior of the companion function

$$\lambda \in \mathbb{R} \to c_A(\lambda) = \inf_{x\neq 0} \frac{\|\lambda x - Ax\|}{\|x\|}.$$

As usual, $A^* \in \mathcal{L}(H)$ denotes the adjoint of $A \in \mathcal{L}(H)$, and

$$\mu_{\min}(E) := \inf_{\|x\|=1} \langle x, Ex\rangle, \qquad \mu_{\max}(E) := \sup_{\|x\|=1} \langle x, Ex\rangle \quad \forall E \in \mathcal{L}(H).$$

THEOREM 8.1. *The companion function of $A \in \mathcal{L}(H)$ admits the representation*

$$c_A(\lambda) = [\lambda^2 - \lambda f_A(\lambda) + \mu_{\min}(A^*A)]^{\frac{1}{2}} \quad \forall \lambda \in \mathbb{R}, \tag{8.1}$$

where f_A: $\mathbb{R} \to \mathbb{R}$ is a nondecreasing function such that

$$\lim_{\lambda\to-\infty} f_A(\lambda) = \mu_{\min}(A + A^*) \quad \textit{and} \quad \lim_{\lambda\to\infty} f_A(\lambda) = \mu_{\max}(A + A^*). \tag{8.2}$$

Proof. As a matter of computation, one obtains

$$c_A^2(\lambda) = \inf_{\|x\|=1} \|\lambda x - Ax\|^2 = \lambda^2 + \inf_{\|x\|=1} \{\langle x, A^*Ax\rangle - 2\lambda\langle x, Ax\rangle\}.$$

Thus,

$$c_A^2(\lambda) = \lambda^2 + \mu_{\min}(A^*A - 2\lambda A_S) = \lambda^2 - \mu_{\max}(-A^*A + 2\lambda A_S),$$

where $A_S = (A + A^*)/2$ corresponds to the self-adjoint part of A. To obtain (8.1) it suffices to define f_A by means of the relation

$$\mu_{\max}(-A^*A + 2\lambda A_S) = \mu_{\max}(-A^*A) + \lambda f_A(\lambda).$$

The value of f_A at 0 is irrelevant in (8.1). The first-order differential quotient

$$f_A(\lambda) = \frac{\mu_{\max}(-A^*A + 2\lambda A_S) - \mu_{\max}(-A^*A)}{\lambda}$$

defines f_A over $\mathbb{R} \setminus \{0\}$. The monotonicity of f_A and the limiting behavior (8.2) follow from standard arguments of convex analysis (cf. [14]). Indeed, $\mu_{\max}(\cdot)$ is a positively homogeneous convex function over the space

$$\mathcal{S}(H) := \{E \in \mathcal{L}(H) \mid E \text{ is self-adjoint}\}.$$

To make sure that f_A is nondecreasing over the whole $\mathbb{R}$, the value of f_A at 0 must be chosen between the left derivative

$$\sup_{\lambda<0} f_A(\lambda) = \lim_{\lambda\to 0^-} \frac{\mu_{\max}(-A^*A + 2\lambda A_S) - \mu_{\max}(-A^*A)}{\lambda}$$

and the right derivative

$$\inf_{\lambda>0} f_A(\lambda) = \lim_{\lambda\to 0^+} \frac{\mu_{\max}(-A^*A + 2\lambda A_S) - \mu_{\max}(-A^*A)}{\lambda}.$$

This completes the proof. □

9. An Application to the Resonance Phenomenon

This section illustrates how the concept of approximate eigenvalue has a bearing in the analysis of the first-order differential system

$$-\dot{\Psi}(t) + F(\Psi(t)) \ni 0. \tag{9.1}$$

Solutions to (9.1) are sought, for instance, in the class of continuously differentiable functions $\Psi\colon \mathbb{R}_+ \to H$.

We want to know how does the system (9.1) respond to a forcing term of exponential type:

$$-\dot{\Psi}(t) + F(\Psi(t)) \ni \mathrm{e}^{\lambda t} z. \tag{9.2}$$

The next theorem shows that the forced system (9.2) may exhibit resonance when the parameter λ is chosen in the approximate point spectrum of F.

THEOREM 9.1. *Let $F \in \mathcal{M}(H)$ be nonempty-valued.*

(a) *If $\lambda \in \Lambda(F)$, then strong resonance occurs*:

$$\begin{cases} \text{there are a unit vector } z \in H \text{ and a solution } \Psi \\ \text{to (9.2) such that } \lim_{t\to\infty} \mathrm{e}^{-\lambda t}\|\Psi(t)\| = \infty. \end{cases}$$

(b) *If $\lambda \in \Lambda_+(F) \setminus \Lambda(F)$, then mild resonance occurs*:

$$\begin{cases} \text{for any constant } M > 0, \text{ there are a unit vector } z \in H \\ \text{and a solution } \Psi \text{ to (9.2) such that } \|\Psi(t)\| \geqslant M\mathrm{e}^{\lambda t}\ \forall t \in \mathbb{R}_+. \end{cases}$$

Proof. Let x be a unit vector in $(F - \lambda I)^{-1}(0)$. If one chooses $z = -x$ and Ψ given by

$$\Psi(t) := t\mathrm{e}^{\lambda t}x \quad \forall t \in \mathbb{R}_+,$$

then Ψ solves (9.2), and $\mathrm{e}^{-\lambda t}\|\Psi(t)\| \to \infty$ as $t \to \infty$. Take now λ in $\Lambda_+(F) \setminus \Lambda(F)$, and consider an arbitrary constant $M > 0$. According to Theorem 5.1, one can write

$$\|(F - \lambda I)^{-1}\|_+ = \sup_{z' \in B_H} \sup_{x' \in (F-\lambda I)^{-1}(z')} \|x'\| = \infty.$$

Hence, there are $z \in B_H$ and $x \in (F - \lambda I)^{-1}(z)$ such that $\|x\| \geqslant M$. Since λ is not an eigenvalue of F, the set $(F - \lambda I)^{-1}(0)$ reduces to the singleton $\{0\}$. Thus, z must be a nonzero vector. By positive homogeneity of $(F - \lambda I)^{-1}$, one can take z in the unit sphere S_H. Finally, one can easily check that the trajectory

$$t \in \mathbb{R}_+ \to \Psi(t) = \mathrm{e}^{\lambda t}x$$

solves (9.2) and satisfies the required growth condition. □

10. Conclusions

The ε-eigenvalue analysis of linear continuous operators has been the object of a number of publications. Some authors have dealt with this topic because they had very specific applications at hand. Some others were more concerned with the theoretical implications of the concept of ε-eigenvalue.

The purpose of our work has been to lay out the basic ingredients for the building of a general ε-eigenvalue theory for positively homogeneous multivalued operators. Concepts like companion function, amenability, spectral threshold, ... have been identified as important tools for the understanding of the spectral behavior of such multivalued operators.

The lenght of the paper has exceeded by far our original expectation. Further applications of the general theory are still under investigation and will be reported in due course.

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