On the Ramsey number of the double star

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Abstract

The double star $S(m_1, m_2)$ is obtained from joining the centres of a star with m_1 leaves and a star with m_2 leaves. We give a short proof of a new upper bound on the two-colour Ramsey number of $S(m_1, m_2)$ which holds for all m_1, m_2 with $\frac{\sqrt{5}+1}{2}m_2 < m_1 < 3m_2$. Our result implies that for all positive m, the Ramsey number of the double star $S(2m, m)$ is at most $\lceil 4.275m \rceil + 1$.

1 Introduction

The much studied Ramsey number $R(H)$ of a graph H is defined as the smallest integer n such that every 2-colouring of the edges of K_n contains a monochromatic copy of H. The case when H is a complete graph is the subject of Ramsey's famous theorem from the 1930's, and determining Ramsey numbers of complete graphs is notoriously difficult. For a recent breakthrough, see [3].

Among the earliest non-complete graphs H to be studied were different kinds of trees. In 1967, Gerencsér and Gyárfás [4] showed that $R(P_k) = k + \lfloor \frac{k+1}{2} \rfloor$, where P_k is the k-edge path. For k-edge stars $K_{1,k}$, the Ramsey number is larger: Harary [6] observed in 1972 that $R(K_{1,k}, K_{1,k}) = 2k$ if k is odd, and $R(K_{1,k}, K_{1,k}) = 2k - 1$ if k is even.

Burr and Erdős [2] conjectured in 1976 that $R(T_k) \leq R(K_{1,k}, K_{1,k})$, for any tree T_k with k edges. For large k, it is known that $R(T_k) \leq 2k$, by the results of [9]. However, this bound far from best possible for paths, which motivated the search for a more fine-tuned conjecture. Note that paths are (almost) completely balanced trees, while stars are the most unbalanced trees. So, it seems natural to suspect that the Ramsey

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number of a tree might be related to its unbalancedness, i.e. the difference in size between the two bipartition classes.

It is easy to see that

$$R_B(T) := \max\{2t_1, t_1 + 2t_2\} - 1$$

is a lower bound for the Ramsey number of any tree T with bipartition classes of sizes $t_1 \geq t_2 \geq 2$. This can be seen by considering the *canonical colourings*, which are defined as follows. Take a complete graph G on $R_B(T) - 1$ vertices. If $t_1 > 2t_2$, partition $V(G)$ into two sets of equal size, colour all edges inside each set red and colour all remaining edges blue. If $t_1 \leq 2t_2$, take a set of $t_1 + t_2 - 1$ vertices, colour all edges inside this set red, and colour all remaining edges blue. It is straightforward to see that no monochromatic copy of T is present in this colouring.

Note that if T is a path then $R_B(T) = R(T)$, and the same holds if T is a star with an even number of edges. In [1], Burr discusses the canonical colourings and expresses his belief that $R(T)$ may be equal to $R_B(T)$ unless T is an odd star. In 2002, Haxell, Łuczak, and Tingley [7] confirmed this suspicion asymptotically for all trees with linearly bounded maximum degree. Namely, they proved that for every $\eta > 0$, there exist t_0 and δ such that $R(T) \leq (1+\eta)R_B(T)$ for each tree T with $\Delta(T) \leq \delta t_1$ and $t_1 > t_0$, where $t_1 \geq t_2$ are, as before, the sizes of the bipartition classes of the tree T.

But already in 1979, Grossman, Harary and Klawe [5] found that, contrary to Burr's suspicion, there are values of m_1, m_2 such that $R(S(m_1, m_2)) > R_B(S(m_1, m_2))$ (where $S(m_1, m_2)$ is the double star with m_i leaves in partition class i). However, the examples from [5] still allowed for the possibility that for every tree T we would have that $R(T) \leq R_B(T) + 1$. The authors of [5] conjectured this to be the truth for all double stars, which they confirmed for a range of values of m_1, m_2. Currently, it is known that this holds if $m_1 \geq 3m_2$ [5] or if $m_1 \leq 1.699(m_2 + 1)$ [8]. In other words, for $m_1, m_2 \in \mathbb{N}^+$ it holds that

$$R(S(m_1, m_2)) \leq \max\{2m_1, m_1 + 2m_2\} + 2 = R_B(S(m_1, m_2)) + 1 \tag{1}$$

unless

$$1.699(m_2 + 1) < m_1 < 3m_2. \tag{2}$$

But in general, inequality (1) is not true. Norin, Sun and Zhao [8] showed that $R(S(m_1, m_2)) \geq 5m_1/3 + 5m_2/6 + o(m_2)$ for all $m_1 \geq m_2 \geq 0$ and $R(S(m_1, m_2)) \geq 189m_1/115 + 21m_2/23 + o(m_2)$ for all $m_1 \geq 2m_2 \geq 0$. In particular, their results imply that $R(S(m_1, m_2)) > R_B(S(m_1, m_2)) + 1$ if m_1, m_2 fulfill

$$\frac{7}{4}m_2 + o(m_2) \leq m_1 \leq \frac{105}{41}m_2 + o(m_2).$$

This range covers the special case that $m_1 = 2m_2$. For this case, the results from [8] yield that $R(S(2m, m)) \geq 4.2m + o(m)$ while $R_B(S(2m, m)) = 4m+2$. This discovery lead the authors of [8] to pose the following question.

Question 1 (Norin, Sun and Zhao [8])**.** *Is it true that $R(S(2m, m)) = 4.2m + o(m)$?*

There are few results giving upper bounds on the Ramsey number of the double star for the range of m_1, m_2 where (1) does not hold. The inequality $R(S(m_1, m_2)) \leq 2m_1 + m_2 + 2$ for all $m_1 \geq m_2 \geq 0$ was established in [5], where it is described as a 'weak upper bound'. In the preprint [8], very good asymptotic bounds for $R(S(m_1, m_2))$ are obtained from a computer-assisted proof using the flag algebra method, but as these are not quick to state, we refer the reader to [8]. We remark that Theorem 4.5 from [8], used with the invalid pair number 5 from Table 1 of [8], implies that $\lim_{m\to\infty} R(S(2m, m))/m$ is bounded from above by 4.21526.

Our main result is a short elementary proof of a new upper bound on $R(S(m_1, m_2))$ which holds for all values of $m_1, m_2 \in \mathbb{N}^+$ fulfilling $\frac{\sqrt{5}+1}{2} m_2 < m_1 < 3m_2$. Observe that $\frac{\sqrt{5}+1}{2} > 1.618$, and thus our result covers the whole range of values of m_1, m_2 from (2).

Theorem 2. *Let $m_1, m_2 \in \mathbb{N}^+$, with $\frac{\sqrt{5}+1}{2} m_2 < m_1 < 3m_2$. Then*

$$R(S(m_1, m_2)) \leq \left\lceil \sqrt{2m_1^2 + (m_1 + \frac{m_2}{2})^2} + \frac{m_2}{2} \right\rceil + 1.$$

As an immediate corollary of our theorem, we obtain for the double star $S(2m, m)$ the following bound.

Corollary 3. $R(S(2m, m)) \leq \lceil 4.27492m \rceil + 1$ *for all $m \in \mathbb{N}^+$.*

2 Preliminaries

In this section we prepare the proof of the main result, Theorem 2, by proving some auxiliary results. We start with a very simple lemma for recurrent later use. A similar lemma appears in [8].

Lemma 4. *Let $m_1, m_2 \in \mathbb{N}$, let G be a graph and let $vw \in E(G)$ such that $d(v) > m_1$, $d(w) > m_2$, and $|N(v) \cup N(w)| \geq m_1 + m_2 + 2$. Then $S(m_1, m_2) \subseteq G$.*

Proof. To form the double star with central edge vw, first choose m_1 neighbours of v, as many as possible outside $N(w) \cup \{w\}$, the others in $N(w)$. Then, choose m_2 neighbours of w in $N(w)$, different from v and from the previously chosen neighbours of v. This concludes the proof. $\square$

Next we show a useful statement about vertex degrees when no double star is present.

Lemma 5. *Let $m_1, m_2 \in \mathbb{N}$, and let G be a graph on $n \geq m_1 + m_2 + 2$ vertices such that $S(m_1, m_2) \not\subseteq G$. Let $v \in V(G)$, let $A \subseteq N(v)$ with $|A| > m_1$ and $d(u) > m_2$ for each $u \in A$. Let $w \in A$. Then w has at most $m_1 + m_2 - |A|$ neighbours in $V(G) \setminus (A \cup \{v\})$. Furthermore, there is a vertex $z \in V(G) \setminus (A \cup \{v\})$ having at most*

$$\frac{m_1 + m_2 - |A|}{n - |A| - 1} \cdot |A|$$

neighbours in A.

Proof. Set $D := V(G) \setminus (A \cup \{v\})$. If w has $m_1 + m_2 - |A| + 1$ or more neighbours in D, then $|N(v) \cup N(w)| \geq |A| + (m_1 + m_2 - |A| + 1) + |\{v\}| = m_1 + m_2 + 2$ (we count v as a neighbour of w), and we can apply Lemma 4 to see that $S(m_1, m_2) \subseteq G$, which is a contradiction.

So w has at most $m_1 + m_2 - |A|$ neighbours in D, which is as desired. Further, as this holds for every $u \in A$, the average number of neighbours in A of a vertex from D is at most

$$\frac{(m_1 + m_2 - |A|) \cdot |A|}{|D|} = \frac{m_1 + m_2 - |A|}{n - |A| - 1} \cdot |A|.$$

So any vertex $z \in D$ having at most the average number of neighbours in A is as desired. $\square$

We will also need a lemma from [8], whose elementary proof can be found there.

Lemma 6 (Lemma 2.3 in [8])**.** *Let $n \geq \max\{2m_1, m_1 + 2m_2\} + 2$, and let the edges of K_n be coloured with red and blue such that there is no monochromatic $S(m_1, m_2)$. Then there is a colour $C \in \{red, blue\}$ such that each vertex of K_n has degree at most m_1 in colour C.*

3 Proof of Theorem 2.

The whole section is devoted to the proof of Theorem 2. Let $m_1, m_2 \in \mathbb{N}^+$ be given, fulfilling

$$\frac{\sqrt{5}+1}{2} m_2 < m_1 < 3m_2. \tag{3}$$

Set

$$m_3 := \left\lceil \sqrt{2m_1^2 + (m_1 + \frac{m_2}{2})^2} - (m_1 + \frac{m_2}{2}) \right\rceil. \tag{4}$$

Using (3) and (4), it is easy to calculate that

$$m_3 > \max\{m_2, m_1 - m_2\}, \tag{5}$$

and in particular, we have that $m_3 \geq 1$. Set $n := m_1 + m_2 + m_3 + 1$, and let a red and blue colouring of the edges of K_n be given. Let G_r be the subgraph of K_n induced by the red edges, and G_b be the subgraph of K_n induced by the blue edges. For any $u \in V(K_n)$, let $N_r(u)$ be the set of all neighbours of u in G_r, and let $N_b(u)$ be the set of all neighbours of u in G_b. Set $d_r(u) := |N_r(u)|$ and $d_b(u) := |N_b(u)|$.

For contradiction assume that there is no monochromatic $S(m_1, m_2)$. Note that $n \geq \max\{2m_1, m_1 + 2m_2\} + 2$ because of (5) and since n is an integer. So, we can use Lemma 6 to see that there is a colour, which we may assume to be blue, such that every vertex has degree at most m_1 in that colour. That is, $d_b(u) \leq m_1$ for all $u \in V(G)$, and thus,

$$\delta(G_r) \geq m_2 + m_3. \tag{6}$$

Now choose any vertex v and a subset A of $N_r(v)$ with

$$|A| = m_2 + m_3. \tag{7}$$

By (6), and since $m_2 + m_3 > m_1$ by (5), we know that $|A| > m_1$ and $\delta(G_r) > m_2$. So, we can use Lemma 5 in G_r to see that for any $w \in A$, we have

$$|N_r(w) \setminus (A \cup \{v\})| \leq m_1 + m_2 - (m_2 + m_3) = m_1 - m_3.$$

and therefore,

$$\begin{aligned} |N_r(w) \cap (A \cup \{v\})| &= d_r(w) - |N_r(w) \setminus (A \cup \{v\})| \\ &\geq m_2 + m_3 - (m_1 - m_3) \\ &= m_2 + 2m_3 - m_1. \end{aligned} \tag{8}$$

We employ Lemma 5 once more, this time to find a vertex $z \notin A \cup \{v\}$ such that

$$|N_r(z) \cap A| \leq \frac{m_1 + m_2 - |A|}{n - |A| - 1} \cdot |A| = \frac{m_1 - m_3}{m_1} \cdot (m_2 + m_3),$$

where we use (7) for the equality. We deduce that

$$\begin{aligned}|N_r(z) \setminus A| &= d_r(z) - |N_r(z) \cap A| \\ &\geq (m_2 + m_3) - \frac{m_1 - m_3}{m_1} \cdot (m_2 + m_3) \\ &= (m_2 + m_3)\frac{m_3}{m_1}. \end{aligned} \tag{9}$$

Further, note that $d_b(z) \leq m_1 < m_2 + m_3 = |A|$ because of (6), (5) and (7). Therefore, we know that vertex z sends at least one red edge to A. Consider any red edge uz with $u \in A$. Using (8) and (9), we get

$$\begin{aligned}|N_r(u) \cup N_r(z)| &\geq |N_r(u) \cap (A \cup \{v\})| + |N_r(z) \setminus A| + |\{u, z\}| \\ &\geq m_2 + 2m_3 - m_1 + (m_2 + m_3)\frac{m_3}{m_1} + 2 \\ &\geq m_1 + m_2 + 2,\end{aligned}$$

where for the last inequality we use the fact that $2m_1m_3 + m_2m_3 + m_3^2 \geq 2m_1^2$ which can be calculated from (4). So, we can apply Lemma 4 to find a red double star with central edge uz, and are done.

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References

[1] Burr, S. A. Generalized Ramsey theory for graphs—a survey. In *Graphs and combinatorics (Proc. Capital Conf., George Washington Univ., Washington, D.C., 1973)*. Springer, Berlin, 1974, pp. 52–75. Lecture Notes in Mat., Vol. 406.

[2] Burr, S. A., and Erdős, P. Extremal Ramsey theory for graphs. *Utilitas Mathematica 9* (1976), 247–258.

[3] Campos, M., Griffiths, S., Morris, R., and Sahasrabudhe, J. An exponential improvement for diagonal ramsey. Preprint 2023, arXiv:2303.09521.

[4] Gerencsér, L., and Gyárfás, A. On Ramsey-type problems. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 10* (1967), 167–170.

[5] Grossman, J. W., Harary, F., and Klawe, M. Generalized Ramsey theory for graphs. X. Double stars. *Discrete Math. 28*, 3 (1979), 247–254.

[6] Harary, F. Recent results on generalized Ramsey theory for graphs. In *Graph theory and applications (Proc. Conf., Western Michigan Univ., Kalamazoo, Mich., 1972; dedicated to the memory of J. W. T. Youngs)*. Springer, Berlin, 1972, pp. 125–138. Lecture Notes in Math., Vol. 303.

[7] Haxell, P. E., Łuczak, T., and Tingley, P. W. Ramsey numbers for trees of small maximum degree. *Combinatorica 22*, 2 (2002), 287–320. Special issue: Paul Erdős and his mathematics.

[8] Norin, S., Sun, Y. R., and Zhao, Y. Asymptotics of Ramsey numbers of double stars. Preprint 2016, arXiv:1605.03612.

[9] Zhao, Y. Proof of the $(n/2-n/2-n/2)$ conjecture for large n. *Electr. J. Comb. 18* (2011).