

# Minimal bricks have many vertices of small degree

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## Abstract

We prove that every minimal brick on  $n$  vertices has at least  $n/9$  vertices of degree at most 4.

## 1 Introduction

A key element in matching theory is the notion of a brick. We briefly and somewhat informally explain this notion and its role. For a much more detailed treatment we refer to the books of Lovász' and Plummer [4] and Schrijver [7].

A *matching* (a set of independent edges) of a graph is *perfect* if every vertex is incident with a matching edge. Consider a matching covered graph, that is a graph in which every edge lies in some perfect matching. A tight cut of such a graph is a cut that meets every perfect matching in precisely one edge. Contracting one, or the other, side of a tight cut  $F$  we obtain two new graphs (which preserve the perfect matching structure we had in the original graph). This operation is called a 'split along the tight cut  $F$ '.

Clearly, we can go on splitting along tight cuts in the newly obtained graphs until arriving at graphs that contain no tight cuts. It was shown by Lovász [3] that no matter in which order we choose the tight cuts we split along, we will essentially always arrive at the same set of cuts and graphs. The obtained decomposition is generally called a 'brick and brace decomposition' because the set of final graphs (without tight cuts) is divided into those that are bipartite – called *braces* – and those that are not – the *bricks*. This decomposition allows to reduce several problems from matching theory to bricks (e.g. a graph is Pfaffian if and only if its bricks are).

Both bricks and braces have been characterised by Edmonds, Lovász and Pulleyblank [2] in other terms. We omit the characterisation of braces. For the one of bricks, let us first say that a graph  $G$  is *bicritical* if  $G - \{u, v\}$  has a perfect matching for every choice of distinct vertices  $u$  and  $v$ . Now bricks are precisely the bicritical and 3-connected graphs [2]. For practical purposes let us consider a brick to be defined this way.

The focus of this paper lies on *minimal bricks*: Those bricks  $G$  for which  $G - e$  ceases to be a brick for every edge  $e \in E(G)$ . Minimality often leads to sparsity in some respect. Minimal bricks are no exception: It is known [5] that any minimal brick on  $n$  vertices has average degree at most  $5 - 7/n$ , unless it is one of four special bricks (the prism or the wheel  $W_n$  for  $n = 4, 6, 8$ ). While thus minimal bricks do have vertices of degree 3 or 4, they may conceivably be very

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few in number, if the average degree is very close to 5. Of particular interest are vertices that attain the smallest degree possible, which is 3 for a brick.

De Carvalho, Lucchesi and Murty [1] proved that any minimal brick contains a vertex of degree 3, which had been conjectured earlier by Lovász; see [1]. This was extended by Norine and Thomas, who showed the existence of 3 such vertices, and then went on to pose the following stronger conjecture.

**Conjecture 1** (Norine and Thomas [5]). *There is an  $\alpha > 0$  so that every minimal brick  $G$  contains at least  $\alpha|V(G)|$  vertices of degree 3.*

Our main result yields further evidence for this conjecture.

**Theorem 2.** *Every minimal brick  $G$  has at least  $\frac{1}{9}|V(G)|$  vertices of degree at most 4.*

We hope that the methods developed here, if substantially strengthened, will be useful for attacking Norine and Thomas' conjecture.

## 2 Brick generation

For practical purposes, the abstract definition of a brick as a 3-connected and bicritical graph may sometimes be less useful than knowing how to obtain a brick from another brick by a small local operation. De Carvalho, Lucchesi and Murty [1] study such operations, and prove that any brick other than the Petersen graph can be obtained by performing these operations successively, starting with either  $K_4$  or the prism. (In particular, every graph in this sequence is a brick.) Norine and Thomas [6] show a generalisation of this result, which they obtained independently.

In particular, every brick has a generating sequence of ever larger bricks. To be useful in induction proofs about minimal bricks, however, it appears necessary that all intermediate graphs are minimal as well, which is unfortunately not guaranteed by the results above. To mend this situation, Norine and Thomas [5] introduce another family of operations, called *strict extensions*, which we shall describe below. Using strict extensions, they find that each minimal brick has a generating sequence consisting only of minimal bricks:

**Theorem 3** (Norine and Thomas [5]). *Every minimal brick other than the Petersen graph can be obtained by strict extensions starting from  $K_4$  or the prism, where all intermediate graphs are minimal bricks.*

Notice that although a strict extension of a brick is a brick, a strict extension of a minimal brick need not be a minimal brick [5].

Let us now formally define strict extensions, following Norine and Thomas [5]. There are five types of strict extensions: Strict linear, bilinear, pseudolinear, quasiquadratic and quasiquartic extensions. The first three of these are based on an even simpler operation, the bisplitting of a vertex.

For this, consider a graph  $H$  and one of its vertices  $v$  of degree at least 4. Partition the neighbourhood of  $v$  into two sets  $N_1$  and  $N_2$  such that each contains at least two vertices. We now replace  $v$  by two new independent vertices,  $v_1$  and  $v_2$ , where  $v_1$  is incident with the vertices in  $N_1$  and  $v_2$  with the ones in  $N_2$ . Finally, we add a third new vertex  $v_0$  that is adjacent to precisely  $v_1$

and  $v_2$ . We say that any such graph  $H'$  is obtained from  $H$  by *bisplitting*  $v$ . The vertex  $v_0$  is the *inner vertex* of the bisplit, while  $v_1$  and  $v_2$  are the *outer vertices*. Any time we perform a bisplit at a vertex  $v$  we will tacitly assume  $v$  to have degree at least 4.

We will now define turn by turn the strict extensions. At the same time we will specify a small set of vertices, the *fundament* of the strict extension. One should think of the fundament as a minimal set of vertices that needs to be present, should we want to perform the extension in some other, usually smaller, graph.

Let  $v$  be a vertex of a graph  $G$ . We say that  $G'$  is a *strict linear extension* of  $G$  if  $G'$  is obtained by one of the three following operations. (See Figure 1 for an illustration.)

1. We perform a bisplit at  $v$ , denote by  $v_0$  the inner vertex, and by  $v_1$  and  $v_2$  the outer vertices of the bisplit. Choose a vertex  $u_0 \in V(G)$  that is non-adjacent to  $v$ . Add the edge  $u_0v_0$ .
2. We perform bisplits at  $v$  and at a second non-adjacent vertex  $u$ , obtaining outer vertices  $v_1$  and  $v_2$  and inner vertex  $v_0$  from the first bisplit and outer vertices  $u_1$  and  $u_2$  and inner vertex  $u_0$  from the second. Add the edge  $u_0v_0$ .
3. We bisplit  $v$ , obtaining the inner vertex  $u_0$ , and outer vertices  $u_1$  and  $u_2$ . We bisplit  $u_1$ , obtaining an inner vertex  $v_0$  and outer vertices  $v_1$  and  $v_2$ , where  $v_1$  is adjacent to  $u_0$ . Add the edge  $u_0v_0$ .

The *fundament* of the extension depends on the subtype: For 1. the fundament is comprised of  $u_0, v$  plus any choice among the vertices of  $G$  of two neighbours of  $v_1$  and of two neighbours of  $v_2$ ; for 2. it will be  $u, v$  together with any two neighbours for each of  $u_1, u_2, v_1, v_2$  that lie in  $G$ ; and for 3. we choose  $v$ , one neighbour of  $v_1$  and two of each of  $u_2$  and  $v_2$ , all of them vertices of  $G$ .

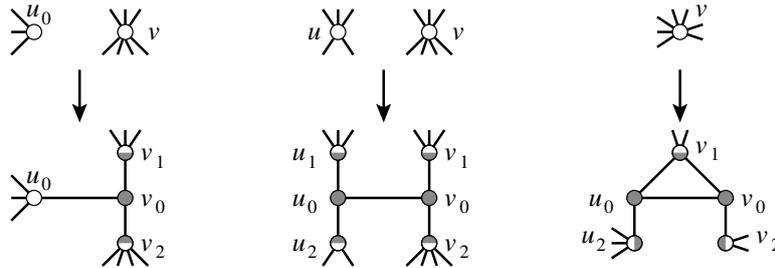


Figure 1: Strict linear extension

Next, assume  $u, v, w$  to be three vertices of  $G$ , so that  $w$  is a neighbour of  $u$  but not of  $v$ . Bisplit  $u$ , and denote by  $u_2$  the new outer vertex that is adjacent to  $w$ , by  $u_1$  the other outer vertex and by  $u_0$  the new inner vertex. Subdivide the edge  $u_2w$  twice, so that it becomes a path  $u_2abw$ , where  $a$  and  $b$  are new vertices. Let  $G'$  be the graph obtained by adding the edges  $bu_0$  and  $av$ ; see Figure 2. We say that  $G'$  is a *bilinear extension* of  $G$ . Its *fundament* consists of  $u, v, w$  together with one neighbour of  $u_2$ , neither  $a$  nor  $u_0$ , and two neighbours of  $u_1$ , none equal to  $u_0$ .

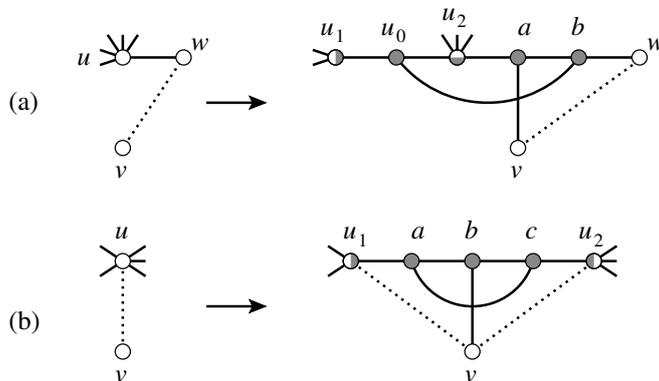


Figure 2: (a) Bilinear extension (b) Pseudolinear extension

A graph  $G'$  is called a *pseudolinear extension* of  $G$  if it may be obtained from  $G$  in the following way. Choose a vertex  $u$  of  $G$  of degree at least 4, and a non-neighbour  $v$  of  $u$ . Partition the neighbours of  $u$  into two sets  $N_1$  and  $N_2$  each of size at least two. Replace the vertex  $u$  by two new ones,  $u_1$  and  $u_2$ , so that  $u_1$  is adjacent to every vertex in  $N_1$  and  $u_2$  to every one in  $N_2$ . Add three new vertices  $a, b, c$  and a path  $u_1abcu_2$ , and let the graph resulting from adding the edges  $ac$  and  $bv$  be  $G'$ ; see Figure 2. We define the *fundament* as  $u, v$  plus two neighbours of each of  $u_1$  and  $u_2$ , all chosen among  $V(G)$ .

The penultimate extension is the *quasiquadratic extension*, shown in Figure 3. Let  $u$  and  $v$  be two distinct vertices of  $G$ , and let  $x$  and  $y$  be not necessarily distinct vertices so that  $x \neq u, y \neq v$  and  $\{u, v\} \neq \{x, y\}$ . If  $u$  and  $v$  are adjacent, delete the edge between them. Add two adjacent new vertices  $u'$  and  $v'$  and join  $u'$  by an edge to  $u$  and  $x$ , and make  $v'$  adjacent to  $v$  and  $y$ . The resulting graph  $G'$  is a quasiquadratic extension of  $G$ .

Norine and Thomas distinguish those quasiquadratic extensions in which the edge  $uv$  was present in  $G$ , calling these extensions *quadratic*. As we will mostly be concerned with non-quadratic quasiquadratic extensions, let us call these extensions *conservative-quadratic*. Thus, in a conservative-quadratic extension the vertices  $u$  and  $v$  are not adjacent in  $G$ , and, in particular,  $G$  is an induced subgraph of  $G'$ . Let us remark rightaway that, as a conservative-quadratic extension is not quadratic its name is ill-chosen. To be more correct, we should call such an extension conservative-quasiquadratic. But life is far too short for such a long name.

The *fundament* of the quasiquadratic extension is simply  $\{u, v, x, y\}$ . For later use, let us call  $\{u, v\}$  the *upper fundament* of the extension.

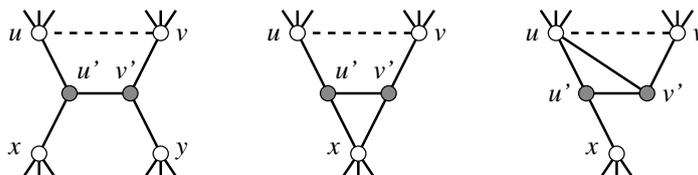


Figure 3: (Quasi-)quadratic extension with different allowed identifications

Finally, consider distinct vertices  $u, v$  and distinct vertices  $x, y$  so that  $u \neq y$ ,  $v \neq x$  and  $\{u, v\} \neq \{x, y\}$ . If present, delete the edges  $uv$  and  $xy$ . We add four new vertices  $u', v', x', y'$  and edges between them so that  $u'v'y'x'u'$  is a 4-cycle. The graph obtained by adding the edges  $uu', vv', xx'$  and  $yy'$  is a *quasiquartic extension* of  $G$ . Its *fundament* consists of  $u, v, x, y$ .

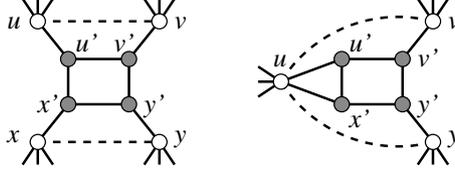


Figure 4: (Quasi-)quartic extension with different allowed identifications

Now, an extension is called *strict* if it is any of the following: quasiquadratic, quasiquartic, bilinear, pseudolinear, and strict linear. We write  $G \rightarrow G'$  if  $G$  is a brick and  $G'$  is obtained from  $G$  by a strict extension.

Let  $F$  be the fundament of the strict extension  $G \rightarrow G'$ . We observe two trivial properties:

$$\text{Any vertex outside } F \text{ has the same degree in } G \text{ as in } G'. \quad (1)$$

$$\text{We have } |F| \leq 3 \cdot (|V(G')| - |V(G)|). \quad (2)$$

We note that the ratio 3 is attained for strict linear extensions of the first type: There the fundament consists of  $u, v$  plus four neighbours of  $v$ , while  $G'$  has only two vertices more than  $G$ .

It is easy to see that a strict extension  $G'$  of a brick  $G$  is 3-connected. Also, it is not difficult to find a perfect matching of  $G' - x - y$  for any pair of vertices  $x, y \in V(G')$ , with exception of the pair  $u_0, v_0$  if  $G \rightarrow G'$  is a strict linear extension, and the pair  $u_0b$ , or  $ac$ , if  $G \rightarrow G'$  is a bi- or pseudolinear extension, respectively. These particular cases can be reduced to the exercise of finding a perfect matching in the graph obtained from  $G$  by bisplitting a vertex, deleting the new inner vertex and another vertex distinct from the new outer vertices. Using Tutte's theorem, and the fact that  $G$  is brick, this is not hard to solve.

This leads to the following lemma, which has also been observed by Norine and Thomas [5]:

**Lemma 4.** *Any strict extension of a brick is a brick.*

We close this section with an example. In Figure 5 we build up a triple ladder by repeatedly alternating between quasiquartic and quasiquadratic extensions, starting from a prism. As by Lemma 4, strict extensions take a brick to a brick, we deduce that the triple ladder is a brick. To see that it is a minimal brick, note that the deletion of any edge results in a graph that fails to be 3-connected.

### 3 Brick on brick

We will call a sequence  $G_0 \rightarrow G_1 \rightarrow \dots \rightarrow G_k$  a *brick-on-brick sequence* if all the  $G_0, \dots, G_k$  are bricks (not necessarily minimal) and if all the  $G_{i-1} \rightarrow G_i$  are strict

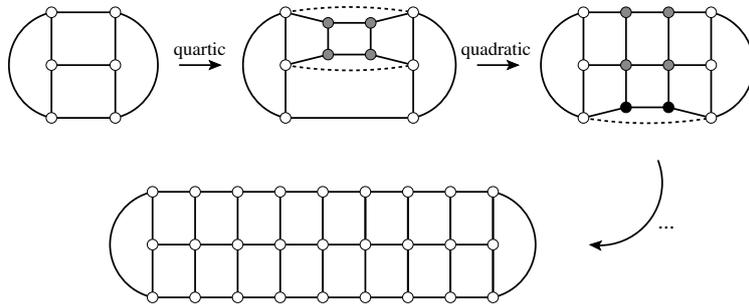


Figure 5: A minimal brick

extensions. Thus, the theorem of Norine and Thomas states that every minimal brick  $G$  has such a brick-on-brick sequence that starts with  $K_4$  or the prism and ends with  $G$ , and in which all intermediate bricks are minimal—unless  $G$  is the Petersen graph.

We formulate a simple lemma that allows us to reorder a brick-on-brick sequence.

**Lemma 5.** *Let  $A \rightarrow B \rightarrow C$  be a brick-on-brick sequence, so that  $A \rightarrow B$  is conservative-quadratic with new vertices  $p, q$  and so that  $p, q$  do not lie in the fundament of  $B \rightarrow C$ . Then there exist a brick  $B'$  so that  $A \rightarrow B' \rightarrow C$  is a brick-on-brick sequence and  $B' \rightarrow C$  is conservative-quadratic with new vertices  $p, q$ .*

*Proof.* Since  $A \rightarrow B$  is conservative-quadratic, we have that  $B - \{p, q\} = A$ . It is easy to verify that thus  $A \rightarrow C - \{p, q\}$  is a strict extension (of the same type as  $B \rightarrow C$ ). For this, it is important to note that by assumption,  $p$  and  $q$  are not in the fundament of  $B \rightarrow C$ . In particular, any bisplittings of  $B \rightarrow C$  can also be performed in  $A$  at vertices of degree  $\geq 4$ . Using Lemma 4, we see that  $B' := C - \{p, q\}$  is a brick.

It remains to show that  $B' \rightarrow C$  is a conservative-quadratic extension. This is easy to check if none of the vertices of the fundament  $F$  of  $A \rightarrow B$  has suffered a bisplit during the operation  $A \rightarrow B'$ . So assume there is a vertex  $s \in F$  which is bisplit in  $A \rightarrow B'$ , and say  $s$  is adjacent to  $p$  in  $B$ . Then, however,  $s$  is also bisplit in  $B \rightarrow C$ , and in  $C$ , one of the new outer vertices, say  $s_1$ , is adjacent to  $p$ . So  $B' \rightarrow C$  is a quasiquadratic extension. Note that the number of edges gained in  $A \rightarrow B'$  and in  $B \rightarrow C$  is the same, and so, with  $A \rightarrow B$ , also  $B' \rightarrow C$  is conservative-quadratic.  $\square$

Let us now examine how the edge density changes in a brick-on-brick sequence. Suppose  $G = (V, E)$  is a minimal brick other than the Petersen graph, and let  $G_0 \rightarrow \dots \rightarrow G_k$  be a brick-on-brick sequence for  $G$  as given by Theorem 3, that is,  $G = G_k$  and  $G_0$  is either the  $K_4$  or the prism. For a set of indices  $I \subseteq \{1, \dots, k\}$  we define  $\nu(I)$  to be the total number of vertices added in extensions corresponding to  $I$ :

$$\nu(I) := \sum_{i \in I} (|V(G_i)| - |V(G_{i-1})|).$$

Similarly, we define

$$\epsilon(I) := \sum_{i \in I} (|E(G_i)| - |E(G_{i-1})|).$$

Now, let  $I_1$  be the set of indices  $i \in \{1, \dots, k\}$  for which  $G_{i-1} \rightarrow G_i$  is a strict linear, bilinear or pseudolinear extension, and set  $\nu_1 = \nu(I_1)$  and  $\epsilon_1 = \epsilon(I_1)$ . We define analogously  $I_2, \nu_2$  and  $\epsilon_2$  (resp.  $I_2^c, \nu_2^c$  and  $\epsilon_2^c$ ) for quasiquadratic (resp. conservative-quadratic) extensions and  $I_3, \nu_3$  and  $\epsilon_3$  for quasiquartic extensions.

Finally, let  $\nu_0 := |V(G_0)|$  and  $\epsilon_0 := |E(G_0)|$ . As  $G_0$  is either  $K_4$  or the prism it follows that  $(\nu_0, \epsilon_0) \in \{(4, 6), (6, 9)\}$ . Moreover, we clearly have that

$$|V(G)| = \nu_0 + \nu_1 + \nu_2 + \nu_3 \text{ and } |E(G)| = \epsilon_0 + \epsilon_1 + \epsilon_2 + \epsilon_3. \quad (3)$$

It is easy to calculate that

$$\epsilon_0 = \frac{3}{2}\nu_0, \epsilon_1 \leq \frac{3}{2}\nu_1, (\epsilon_2 - \epsilon_2^c) = \frac{4}{2}(\nu_2 - \nu_2^c), \epsilon_2^c = \frac{5}{2}\nu_2^c \text{ and } \epsilon_3 \leq \frac{8}{4}\nu_3. \quad (4)$$

From (4), we see that the ‘edge density gain’ is largest when performing conservative-quadratic extensions. In fact, the greater the average degree of a minimal brick, the more conservative-quadratic extensions must have been used in any of its brick-on-brick sequences:

**Lemma 6.** *Let  $\delta > 0$ , and let  $G$  be a minimal brick with average degree  $d(G) \geq 4 + \delta$ . For any brick-on-brick sequence  $G_0 \rightarrow \dots \rightarrow G_k$  with  $G = G_k$  and  $G_0 \in \{K_4, \text{Prism}\}$  it holds that  $\nu_2^c \geq \delta|V(G)|$ .*

*Proof.* Let  $G = (V, E)$ . Using (3) and (4), we find that

$$\begin{aligned} \frac{4 + \delta}{2} &\leq \frac{|E|}{|V|} \\ &= \frac{1}{|V|} (\epsilon_0 + \epsilon_1 + \epsilon_2 + \epsilon_3) \\ &\leq \frac{1}{|V|} \left( \frac{3}{2}\nu_0 + \frac{3}{2}\nu_1 + 2(\nu_2 - \nu_2^c) + \frac{5}{2}\nu_2^c + 2\nu_3 \right) \\ &\leq \frac{1}{|V|} \left( 2|V| + \frac{1}{2}\nu_2^c \right), \end{aligned}$$

and consequently,  $\nu_2^c \geq \delta|V|$ .  $\square$

We postpone the proof of the following lemma to the next section.

**Lemma 7.** *Let  $G$  be a brick, and let  $G''$  be a conservative-quadratic extension of a conservative-quadratic extension  $G'$  of  $G$ . Let  $u'$  and  $v'$  be the new vertices of  $G'$ . If one of  $u', v'$  is used for the fundament of  $G' \rightarrow G''$  then  $G''$  is not a minimal brick.*

**Lemma 8.** *Every minimal brick  $G$  of average degree  $d(G) \geq 4 + \delta$  with  $\delta > 0$  has at least  $(4\delta - 3)|V(G)|$  vertices of degree 3.*

*Proof.* By Theorem 3, there is a brick-on-brick sequence  $\mathcal{B} := G_0 \rightarrow \dots \rightarrow G_k$  for  $G$ , where all intermediate graphs are minimal bricks. With Lemma 6 we find that

$$\nu_2^c \geq \delta|V(G)|. \quad (5)$$

This means that there is a set  $Q$  of at least  $\delta|V(G)|$  vertices that arise as new vertices in some conservative-quadratic extension of  $\mathcal{B}$ . Denote by  $Q_1$  the set of those vertices in  $Q$  that are used in the fundament of any later extension of  $\mathcal{B}$ , and let  $Q_2 := Q \setminus Q_1$ . Then  $Q_2 \subseteq V(G)$  and the vertices of  $Q_2$  have degree 3 in  $G$  by (1).

Hence if  $|Q_2| \geq (4\delta - 3)|V(G)|$ , then we are done. So assume otherwise. Then

$$|Q_1| = |Q| - |Q_2| > \delta|V(G)| - (4\delta - 3)|V(G)| = 3(1 - \delta)|V(G)|. \quad (6)$$

Let  $I$  be the set of indices of extensions of  $\mathcal{B}$  that use some vertex of  $Q_1$  in their fundament which has not been used in the fundament of earlier extensions of  $\mathcal{B}$ . Then (2) together with (6) implies that  $\nu(I) > (1 - \delta)|V(G)|$ .

This means that by (3) and by (5), there is an index  $j \in I$  that corresponds to a conservative-quadratic extension  $G_{j-1} \rightarrow G_j$  of  $\mathcal{B}$ . Let  $q \in Q_1$  lie in the fundament of this extension.

We apply Lemma 5 repeatedly in order to finally obtain a brick  $G'_{j-2}$  so that

$$G'_{j-2} \rightarrow G_{j-1} \rightarrow G_j$$

is a brick-on-brick sequence, with  $q$  being one of the new vertices in the conservative-quadratic extension  $G'_{j-2} \rightarrow G_{j-1}$ . This contradicts Lemma 7.  $\square$

We are now ready to prove our main theorem.

*Proof of Theorem 2.* Given a minimal brick  $G$  we distinguish two cases. If the average degree of  $G$  is at least  $4 + \frac{7}{9}$ , then we apply Lemma 8 to see that at  $\frac{1}{9}|V(G)|$  of the vertices have degree 3.

So, we may assume that  $G$  has average degree at most  $5 - \frac{2}{9}$ . Denote by  $V_{\leq 4}$  the set of all vertices of degree at most 4, and by  $V_{\geq 5}$  the set of all vertices of degree at least 5. Then

$$\left(5 - \frac{2}{9}\right)|V(G)| \geq \sum_{v \in V(G)} d(v) \geq 3|V_{\leq 4}| + 5|V_{\geq 5}| = 5|V(G)| - 2|V_{\leq 4}|,$$

which leads to  $|V_{\leq 4}| \geq \frac{1}{9}|V(G)|$ .

In either case we find that at least a ninth of the vertices of  $G$  have degree at most 4.  $\square$

## 4 Proof of Lemma 7

We dedicate this section entirely to the proof of Lemma 7. We shall use the notation from Figure 6, that is,  $\{x, u, v, y\}$  is the fundament of the conservative-quadratic extension  $G \rightarrow G'$ , and  $\{u', r, s, t\}$  is the fundament of the conservative-quadratic extension  $G' \rightarrow G''$ , with new vertices  $r'$  and  $s'$ , where  $r'$  is adjacent

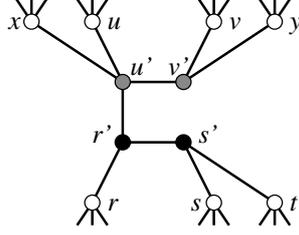


Figure 6: Applying two quadratic extension on top of each other. Note that several of the vertices in the figure might be identified.

to  $u'$  and  $r$ , and  $s'$  is adjacent to  $s$  and  $t$ . Several of these vertices may be identified, some of them are by definition distinct:

$$u \neq x, v \neq y, u' \neq r, s \neq t, \text{ and } u', v', r', s' \text{ are pairwise distinct.}$$

Assume for contradiction that  $G''$  is a minimal brick. From this we will deduce that  $G'' - uu'$  is bicritical and 3-connected, which is clearly impossible.

We start by proving that

$$v' \notin \{s, t\}. \quad (7)$$

Indeed, suppose otherwise. Then, as  $G''$  is a conservative-quadratic extension of  $G'$ , the graph  $G'' - v'u'$  is a quadratic extension of  $G'$  (with  $v'$  and  $u'$  constituting the upper fundament of  $G' \rightarrow G'' - v'u'$ ). Hence  $G'' - v'u'$  is a brick. Thus  $G''$  is not minimal, against our assumption.

Next, we show that

$$G'' - uu' \text{ is bicritical.} \quad (8)$$

For this, let  $a, b$  be two vertices of  $G''$ . Our aim is to find a perfect matching of  $G'' - a - b$  that avoids  $uu'$ . We may assume that neither of  $a, b$  is incident with  $uu'$  as otherwise any perfect matching of  $G'' - a - b$  serves for our purpose (and exists since  $G''$  is bicritical). Therefore, the following three cases cover all possible cases (after possibly swapping  $a$  and  $b$ ).

**Case 1:**  $a, b \in V(G)$ .

Since  $G$  is bicritical there is a perfect matching  $M$  of  $G - a - b$ . This matching together with the edges  $u'v'$  and  $r's'$  yields a perfect matching of  $G'' - a - b$  that avoids  $uu'$ .

**Case 2:**  $a \in V(G)$  but  $b \notin V(G)$ .

Our aim is to find a substitute  $b' \in V(G)$  different from  $a$ , so that a perfect matching  $M$  of  $G - a - b'$  together with two edges  $e, f$  form a perfect matching  $M'$  of  $G'' - a - b$  that avoids  $uu'$ .

*Subcase:*  $b = s'$ . As  $v \neq y$ , we may choose  $b' \in \{v, y\}$  distinct from  $a$ , and let  $M' := M + u'r' + v'b'$ .

*Subcase:*  $b \in \{v', r'\}$  and  $\{s, t\} \neq \{u', a\}$ . Choose  $b' \in \{s, t\}$  distinct from  $u'$  and  $a$ , and note that (7) implies that  $b' \neq v'$ . Then the matching  $M' := M + u'v' + s'b'$  if  $b = r'$  or the matching  $M' := M + u'r' + s'b'$  if  $b = v'$  is as desired.

*Subcase:*  $b = v'$  and  $\{s, t\} = \{u', a\}$ . Note that then  $r \neq a$ . Choose  $b' := r$ . Then the matching  $M' := M + u's' + r'b'$  is as desired.

*Subcase:*  $b = r'$  and  $\{s, t\} = \{u', a\}$ . Choose  $b' \in \{v, y\}$  distinct from  $a$ . Then the matching  $M' := M + u's' + v'b'$  is as desired.

**Case 3:**  $a, b \notin V(G)$ .

Then  $a, b \in \{v', r', s'\}$ . If  $\{a, b\} = \{r', s'\}$  we take a perfect matching  $M$  of  $G$  plus  $u'v'$ . If  $\{a, b\} = \{v', s'\}$  we choose a perfect matching of  $G$  together with  $u'r'$ .

It remains the case when  $\{a, b\} = \{r', v'\}$ . We choose  $b' \in \{s, t\}$  distinct from  $x$ . By (7) we have  $b' \neq v'$ . If  $b' = u'$  then a perfect matching of  $G$  together with  $u's'$  is as desired. Otherwise, we can use a perfect matching of  $G - x - b'$  together with  $u'x$  and  $s'b'$ .

We have thus proved (8).

We finish the proof of the lemma by showing that

$$G'' - uu' \text{ is 3-connected.} \quad (9)$$

Suppose otherwise. Then there are vertices  $w, z$  such that  $G'' - uu' - w - z$  is disconnected. In other words,  $G'' - uu'$  is the union of two subgraphs  $A, B$  with  $\{w, z\} = V(A \cap B)$  and  $A \setminus B \neq \emptyset \neq B \setminus A$ . Since  $G''$  is 3-connected we know that  $w \neq z$  and that  $uu'$  goes from  $A \setminus B$  to  $B \setminus A$ . Say  $u \in A \setminus B$  and  $u' \in B \setminus A$ . Since  $G''$  is 3-connected we know that  $w \neq z$  and that  $uu'$  goes from  $A - B$  to  $B - A$ . Say  $u \in V(A - B)$  and  $u' \in V(B - A)$ .

Since  $G$  is 3-connected, all of  $G$  is contained in either  $A$  or in  $B$ . As  $u \notin V(B)$ , it must be that  $G \subseteq A$ . Thus  $x$ , as a neighbour of  $u'$ , lies in  $A \cap B = \{w, z\}$ . Say  $x = w$ .

Now, either  $v' = z$  or  $v' \in B \setminus A$  and  $\{v, y\} = \{x, z\}$  (then, in particular, one of  $v, y$  is equal to  $z$ ). In either case, we find that the the new vertices  $r', s'$  of  $G''$  lie in  $B - A$ . For the neighbours  $r, s, t$  of  $r'$  and  $s'$  we deduce that

$$\{r, s, t\} \subseteq \{u', v', x, z\}. \quad (10)$$

We claim that  $x \notin \{s, t\}$ . Otherwise, say if  $x = s$ , we perform a quadratic extension in  $G'$  with fundament  $\{u', x, r, t\}$  and upper fundament  $\{u', x\}$ . In this way, the edge  $u'x$  vanishes in the quadratic extension  $G' \rightarrow G'' - xu'$ . Thus,  $G'' - xu'$  is a brick, which contradicts the minimality of  $G''$ . With a similar reasoning we see  $v' \notin \{s, t\}$ .

Thus, one of  $s, t$  must be equal to  $u'$ , and the other equal to  $z$  (recall that  $u' \neq z$ ). As the fundament of the quasiquadratic extension  $G' \rightarrow G''$ , the set  $\{r, s, t, u'\}$  contains at least three vertices. By (10), this implies that  $r$  is either  $x$  or  $v'$ . But then, either choosing  $u'$  and  $x$  as the upper fundament of the quadratic extension  $G' \rightarrow G'' - xu'$ , or choosing  $u'$  and  $v'$  as the upper fundament of the quadratic extension  $G' \rightarrow G'' - u'v'$ , we find a contradiction to the fact that  $G''$  is a minimal brick. This concludes the proof of (9).

## 5 Discussion

In this work, we proved that in a minimal brick the number of vertices of degree  $\leq 4$  is a positive fraction of the total number of vertices. On the other hand, if we look for large degree vertices in a minimal brick, it is not difficult to find examples with a few vertices of arbitrary large degree (for instance even

wheels). It seems less evident that one can also construct minimal bricks with many vertices of degree  $\geq 5$ . We provide an example in Figure 7, where about a seventh of the vertices have degree 6. This graph is indeed a brick, since it can be built from the triple ladder of Figure 5 by performing two quadratic extensions at triples like  $r, s, t$ . It is a minimal brick as clearly every edge is necessary for 3-connectivity.

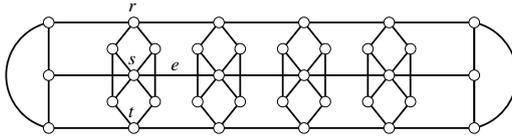


Figure 7: A minimal brick

Vertices of degree  $\leq 4$  and even cubic vertices seem to be abundant in all examples. In the example with fewest proportion of degree 3 vertices we know, the triple ladder in Figure 5, they still make up two thirds of the vertices. In that respect, our result with a fraction of  $\geq \frac{1}{9}$  of the vertices seems quite low.

The main aim of this paper was to develop ideas and techniques that ultimately should serve to settle the Norine-Thomas conjecture. While we believe to have done a substantial step in that direction, there are still serious obstacles lying on that route. Let us briefly outline some of them.

Clearly, an average degree of at most  $4 - \gamma$  (for some small constant  $\gamma > 0$ ) yields a positive fraction of degree 3 vertices. We may therefore assume that our minimal bricks have average degree of about 4 and higher. While an average degree of about 5 and higher leads to a brick-on-brick sequence with many conservative-quadratic extensions (cf. Lemma 6), the now lower bound on the average degree will give us less information on the kind of extensions our brick-on-brick sequence is composed of. In particular, quadratic and conservative-quartic (those that do not involve edge deletions) might appear, as they push the average degree towards 4. Even worse, because conservative-quadratic extensions yield a relatively large edge-density increase, we may also have lots of strict linear, bilinear or pseudolinear extensions.

To handle this, we would seem to need a much stronger version of Lemma 7, that also forbids two chained quadratic extensions, say, that increase the degree of a fundament vertex. Unfortunately, two such extension might actually occur while still yielding a minimal brick: This is exactly what happened to produce the degree 6 vertices in Figure 7.

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