

Almost partitioning a 3-edge-coloured $K_{n,n}$ into 5 monochromatic cycles

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Abstract

We show that for any colouring of the edges of the complete bipartite graph $K_{n,n}$ with 3 colours there are 5 disjoint monochromatic cycles which together cover all but $o(n)$ of the vertices. In the same situation, 18 disjoint monochromatic cycles together cover all vertices.

Keywords: Monochromatic cycle partition, Ramsey-type problem, complete bipartite graph

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1 Introduction

The monochromatic cycle partition problem is a Ramsey-type problem that originated in work of Gerencsér and Gyárfás [7] and Gyárfás [8], and lately received a considerable amount of attention from the community. Given a graph G , and a (not necessarily proper) colouring of its edges with r colours, we are interested in covering $V(G)$ with mutually disjoint monochromatic cycles, using as few cycles as possible. (For technical reasons, single vertices, single edges and the empty set count as cycles as well.) To state the problem more precisely, the aim is to determine the smallest number $m = m(r, G)$ such that for any r -edge colouring of G , there are m disjoint monochromatic cycles that cover $V(G)$.

The case $G = K_n$ received the most attention so far. An easy construction shows that at least r cycles are necessary to cover all the vertices, and Erdős, Gyárfás and Pyber [6] showed that the number of cycles needed is a function of r (independent of n). The currently best known upper bound of $100r \log r$ (for large n) for this function is due to Gyárfás, Ruszinkó, Sárközy and Szemerédi [9]. For $r = 2$, Bessy and Thomassé [4] showed that a partition into 2 cycles (even of different colours) always exists, thus proving a conjecture of Lehel [2] and extending earlier work of [21, 1]. (See also [24] for an alternative proof.) Motivated by ideas of Schelp, Balogh et al. [3] suggested a strengthening of Lehel's conjecture: Every 2-coloured n -vertex graph of minimum degree at least $3n/4$ can be partitioned into a red and a blue cycle. As evidence for

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their conjecture, Balogh et al. [3] proved an asymptotic version: All but $o(n)$ vertices of any 2-coloured n -vertex graph of minimum degree $(3/4 + o(1))n$ can be partitioned into a red and a blue cycle. DeBiasio and Nelsen [5] adapted the absorbing method of [25], to show that under the same conditions, *all* vertices of the graph can be partitioned into a red and a blue cycle. Extending this technique, Letzter [17] proved the conjecture of Balogh et al. for large n .

The conjecture [6] that r monochromatic cycles suffice to partition any r -coloured complete graph for all $r \geq 3$, was disproved by Pokrovskiy [23]. However, his examples allow partitions of all but one vertex. In light of this, it has been proposed to tone down the conjecture, allowing for a constant number of uncovered vertices [3, 23]. On the positive side, for $r = 3$, three monochromatic cycles suffice to partition all but $o(n)$ vertices of K_n , and, for large enough n , 17 monochromatic cycles partition all of $V(K_n)$; this was shown by Gyárfás, Ruszinkó, Sárközy, and Szemerédi [12]. (Actually, by a slight modification of their method, one can replace the number 17 with 10, see Section 5.3). Very recently, Pokrovskiy [24] showed that it is indeed possible to partition all but a constant number of vertices of a 3-coloured complete graph into at most 3 cycles [24]. This was independently confirmed by Letzter with a better constant [18].

For G being the balanced complete bipartite graph $K_{n,n}$, first upper bounds for monochromatic cycle partitions were given by Haxell [14] and by Peng, Rödl and Rućinski [22]. The current best known result is that $4r^2$ monochromatic cycles suffice to partition all vertices of $K_{n,n}$, if n is large [16].

For a lower bound, an easy construction shows we need at least $2r - 1$ cycles to cover all the vertices. For instance, starting out with a properly r -edge-coloured $K_{r,r}$, blow up each vertex in one partition class to a set of size r , while in the other partition class only blow up one vertex to a set of size $r(r - 1) + 1$. A similar construction is given in [23].

We believe that the lower bound of $2r - 1$ might be the correct answer to the monochromatic cycle partition problem in balanced complete bipartite graphs. This suspicion has recently been confirmed for $r = 2$ by Letzter [18], after preliminary work of Schaudt and Stein [26]. (See also [19] for a short proof for a partition into 4 cycles. Our contribution here is that the lower bound of $2r - 1$ is asymptotically correct also for $r = 3$.)

Theorem 1.1. *For any 3-edge-colouring of $K_{n,n}$,*

- (a) *there is a partition of all but $o(n)$ vertices of $K_{n,n}$ into five monochromatic cycles, and*
- (b) *if n is large enough, then the vertices of $K_{n,n}$ can be partitioned into 18 monochromatic cycles.*

The second part of our theorem improves the formerly best bound of 1695 disjoint monochromatic cycles for covering any 3-edge coloured $K_{n,n}$ [14]. We remark that in [26] it is shown that 12 monochromatic cycles suffice to partition all the vertices of any two-coloured $K_{n,n}$.

A related result for $r = 2$ and for partitions into *paths*, is due to Pokrovskiy [23]. He showed that a 2-edge-coloured $K_{n,n}$ can be partitioned into two monochromatic paths, unless the colouring is a *split colouring*, that is, an edge-colouring that has a colour-preserving homomorphism to a properly

edge-coloured $K_{2,2}$. In a split colouring, three disjoint monochromatic cycles (or paths) are always enough to cover all vertices. Pokrovskiy [23] conjectures $2r - 1$ disjoint monochromatic paths suffice for arbitrary r .

We now briefly sketch the proof of our main result, Theorem 1.1, thereby explaining the structure of the paper. The proof of Theorem 1.1(a) involves the construction of large monochromatic connected matchings (see below) and an application of the Regularity Lemma [27]. This method has been introduced by Łuczak [20] and became a standard approach.

A *monochromatic connected matching* is a matching in a connected component of the graph spanned by the edges of a single colour, and such a component is called a *monochromatic component*. Slightly abusing notation, we treat matchings as both edge subsets and 1-regular subgraphs. The following is our key lemma. Its proof is given in Section 2.

Lemma 1.2. *Let the edges of $K_{n,n}$ be coloured with three colours. Then there is a partition of the vertices of $K_{n,n}$ into five or less monochromatic connected matchings.*

Now for the proof of Theorem 1.1(a), apply the Regularity Lemma to the given 3-edge-coloured $K_{n,n}$. The reduced graph Γ is almost complete bipartite and inherits a 3-colouring (via majority density of the pairs). A robust version of Lemma 1.2, namely Lemma 3.1 (see Section 3), permits us to partition almost all of R into five monochromatic connected matchings. In the subsequent step, presented in Section 4, we apply a specific case of the Blow-up Lemma [11, 15, 20] to get from our matchings to five monochromatic cycles which together partition almost all vertices of $K_{n,n}$.

The proof of Theorem 1.1(b) is given in Section 5.2. It combines ideas of Haxell [14] and Gyárfás et al. [12] with Theorem 1.1(a). First, we fix a large monochromatic subgraph H , which is Hamiltonian and remains so even if some of the vertices are deleted from it. Then, using Theorem 1.1(a), we cover almost all vertices of $K_{n,n} - V(H)$ with five vertex-disjoint monochromatic cycles. The amount of still uncovered vertices being much smaller than the order of H , we can apply a Lemma from [9] in order to absorb these vertices using vertices from H , and producing only a few more cycles. We finish by taking one more monochromatic cycle, which covers the remainder of H .

2 Covering with connected matchings

In this section we give the proof of the exact version of Lemma 1.2. Its proof has been written with the proof of the more technical robust counterpart (Lemma 3.1 in Section 3) in mind, in order to ease the transition between the two proofs. It may therefore appear to be a bit overly lengthy in some of its parts.

2.1 Preliminaries

This subsection contains some preliminary results for the proof of our key lemma, Lemma 1.2, which is given in the subsequent subsection. We start with some definitions. The *biparts* of a bipartite graph H are its partition classes, which we denote by \overline{H} and \underline{H} . If $X \subseteq \overline{H}$ and $Y \subseteq \underline{H}$, or if $X \subseteq \underline{H}$ and $Y \subseteq \overline{H}$,

we write $[X, Y]$ for the bipartite subgraph induced by the edges between X and Y .

Definition 2.1 (empty graph, trivial graph). *A bipartite graph is empty if it has no vertices and trivial if one of its biparts has no vertices.*

For a colouring of the edges of H with colours red, green and blue, a *red component* R is a connected component in the subgraph obtained by deleting the non-red edges and a *red matching* is a matching whose edges are red. The same terms are defined for colours green and blue. We now introduce two types of colourings for 2-coloured bipartite graphs. We call an edge colouring of a bipartite graph H in red and blue a *V-colouring* if there are monochromatic components R and B of distinct colours such that

1. each of R and B is non-trivial;
2. $R \cup B$ is spanning in H ;
3. $|V(\overline{R \cap B})| = |V(\overline{H})|$ or $|V(R \cap B)| = |V(\underline{H})|$.

A colouring of $E(H)$ in red and blue is *split*, if

1. all monochromatic components are non-trivial;
2. each colour has exactly two monochromatic components.

The following lemma classifies the component structure of a 2-coloured bipartite graph.

Lemma 2.2. *If the bipartite 2-edge-coloured graph H is complete, then one of the following holds:*

- (a) *There is a spanning monochromatic component,*
- (b) *H has a V-colouring, or*
- (c) *the edge-colouring is split.*

Proof. Let R be a non-trivial component in colour red, say. Set $X := H - R$ and note that all edges in $[\overline{R}, \underline{X}]$ and $[\underline{R}, \overline{X}]$ are blue.

We first assume that $|\overline{X}| = 0$. If also $|\underline{X}| = 0$, we are done, since then R is spanning. Otherwise, $|\underline{X}| > 0$, and thus the colouring is a V-colouring.

So by symmetry we can assume that both $|\overline{X}| > 0$ and $|\underline{X}| > 0$. If there is a blue edge in R or in X , then H is spanned by one blue component. Hence, all edges inside R and X are red and the colouring is split. \square

Corollary 2.3. *If a bipartite 2-edge-coloured graph H is complete, then*

- (a) *there are one or two non-trivial monochromatic components that together span H , and*
- (b) *if the colouring is not split, then there is a colour with exactly one non-trivial component.*

Let us now turn to monochromatic matchings.

Lemma 2.4. *Let H be a balanced bipartite complete graph whose edges are coloured red and blue. Then either*

- (a) *H is spanned by two vertex disjoint monochromatic connected matchings, one of each colour, or*
- (b) *the colouring is split and*
 - *H is spanned by one red and two blue vertex disjoint connected monochromatic matchings and*
 - *H is spanned by one blue and two red vertex disjoint connected monochromatic matchings.*

Proof. First assume that the colouring is split. We take one red maximum matching in each of the two red components. This leaves at least one of the blue components with no vertices on each side. We extract a third maximum matching from the leftover of the other blue component, thus leaving one of its sides with no vertices. Thus the three matchings together span H . Note that we could have switched the roles of red and blue in order to obtain two blue and one red matching that span H .

So by Lemma 2.2, we may assume that either there is a colour, say red, with a spanning component R , or H has a V -colouring, with components R in red and B in blue, say. In either case, we take a maximum red matching M in R . Then there is an induced balanced bipartite subgraph of H , whose edges are all blue, which contains all uncovered vertices of each bipart of H . If this subgraph is trivial, we are done. Otherwise, we finish by extracting from it a maximum blue matching $M' \subseteq B$. As H is complete and there are no leftover edges in said subgraph, we obtain that $M \cup M'$ spans H , and we are done. \square

We continue with a lemma about the component structure of 3-edge-coloured bipartite graphs.

Lemma 2.5. *Let the edges of the complete bipartite graph H be coloured in red, green and blue, such that each colour has at least four non-trivial components; then there are three monochromatic components that together span H .*

Proof. Let R be a red non-trivial component. Since there are three more red non-trivial components, the three graphs $X := H - R$, $[\overline{R}, \underline{X}]$ and $[\underline{R}, \overline{X}]$ are each non-trivial. Moreover, the edges of the latter two graphs are green and blue. By Corollary 2.3(a) there are one or two non-trivial monochromatic components that together span $[\underline{R}, \overline{X}]$. So, if $[\overline{R}, \underline{X}]$ has a spanning monochromatic component, then we can span H with at most three components, which is as desired. Therefore and by symmetry we may assume from now on that none of $[\overline{R}, \underline{X}]$ and $[\underline{R}, \overline{X}]$ has a spanning monochromatic component. Suppose $[\overline{R}, \underline{X}]$ has a split-colouring. By Lemma 2.2, either $[\underline{R}, \overline{X}]$ is split or one of \underline{R} and \overline{X} is contained in the intersection of a blue and a green monochromatic component. In the latter case the union of three monochromatic components of the same colour contains one of the biparts of H . But this is impossible as each colour has at least four non-trivial components. On the other hand, if both $[\overline{R}, \underline{X}]$ and $[\underline{R}, \overline{X}]$ have a split colouring, then each bipart of H is contained in the union of four green components as well as in the union of four blue components, and thus

all edges in X are red. But then there are only two non-trivial red components, R and X , a contradiction.

So by Lemma 2.2, and by symmetry, we know that $[\overline{R}, \underline{X}]$ and $[\underline{R}, \overline{X}]$ both have green/blue V -edge-colourings. Thus each of $[\overline{R}, \underline{X}]$ and $[\underline{R}, \overline{X}]$ has a non-trivial blue component and a non-trivial green component, say these are B_1, G_1 and B_2, G_2 respectively. Furthermore, \underline{X} or \overline{R} is contained in the intersection $B_1 \cap G_1$, and \overline{X} or \underline{R} is spanned by the intersection $B_2 \cap G_2$.

We first look at the case where \underline{X} is contained in $B_1 \cap G_1$. If \underline{R} is contained in $B_2 \cap G_2$, then both green and blue have at most two spanning components, which is a contradiction. On the other hand, if \overline{X} is contained in $B_2 \cap G_2$, then H is spanned by the union of R and the blue components in H that contain B_1 and B_2 , and we are done.

Consequently we can assume by symmetry and by Lemma 2.2 that \overline{R} is spanned by $B_1 \cap G_1$ and \underline{R} is spanned by $B_2 \cap G_2$. Observe that $[\underline{G}_1, \overline{G}_2]$ is coloured red and blue and $[\underline{B}_1, \overline{B}_2]$ is coloured red and green, since otherwise, we obtain the desired cover. Suppose there is a red component of $[\underline{G}_1, \overline{G}_2]$ that is spanning in $[\underline{G}_1, \overline{G}_2]$. Such a component, together with B_1 and B_2 , spans H . So, we can assume $[\underline{G}_1, \overline{G}_2]$ has no red spanning red component. Moreover, since there are at least four non-trivial blue components, $[\underline{G}_1, \overline{G}_2]$ contains two blue components, which are non-trivial each.

Since these blue components are non-trivial in H , $[\underline{G}_1, \overline{G}_2]$ does not have a V -colouring (in itself). Thus, by Lemma 2.2, $[\underline{G}_1, \overline{G}_2]$ is split coloured in red and blue. Similarly we see that $[\underline{B}_1, \overline{B}_2]$ is split coloured in red and green.

Consider the edges in $[\underline{G}_1, \overline{B}_2]$ and $[\underline{B}_1, \overline{G}_2]$. If any of these edges is green or blue, then our graph is spanned by three green or by three blue components. On the other hand, if all edges in $[\underline{G}_1, \overline{B}_2]$ and $[\underline{B}_1, \overline{G}_2]$ are red, then H has only three non-trivial red components, a contradiction. \square

2.2 Proof of Lemma 1.2

We are now ready to prove Lemma 1.2. Let H be a balanced bipartite complete graph of order $2n$. Our aim is to show that H can be spanned with five vertex disjoint monochromatic connected matchings. We suppose that this is wrong in order to obtain a contradiction. We prove a series of claims in order to reduce the problem to a specific colouring, which then receives a distinct treatment.

Claim 2.6. *Each colour has at least three non-trivial components.*

Proof. Suppose the claim is wrong for colour red, say. By assumption, there are two (possibly trivial) red components R_1 and R_2 in H , such that all other red components are trivial. Let M be a maximum red matching in $R_1 \cup R_2$. Then every edge in the balanced bipartite subgraph $X := H - M$ is green or blue. By Lemma 2.4, H can be spanned with three vertex-disjoint monochromatic connected matchings. So in total we found at most five vertex-disjoint monochromatic connected matchings that together span H . \square

Claim 2.7. *There are no two monochromatic components that together span H .*

Proof. Suppose the claim is wrong and there are monochromatic components R and B that together span H . By Claim 2.6 we can assume that they have distinct

colours, say R is red and B is blue. Take a red matching M^{red} of maximum size in R and a blue matching M^{blue} of maximum size in $B - V(M^{\text{red}})$. Set $R' := R - V(M^{\text{red}} \cup M^{\text{blue}})$ and $B' := B - V(M^{\text{red}} \cup M^{\text{blue}})$. By maximality, any edge between $\overline{B'}$ and $\underline{R'}$ is green. The same holds for the edges between $\underline{B'}$ and $\overline{R'}$.

If $[\underline{B'}, \overline{R'}]$ is empty, we finish by picking a maximum matching in $[\underline{R'}, \overline{B'}]$. We proceed analogously if $[\underline{R'}, \overline{B'}]$ is empty. Assuming that both are non-empty we now pick a maximum matching in each of the green components of $H - V(M^{\text{red}} \cup M^{\text{blue}})$ that contain $[\overline{B'}, \underline{R'}]$, $[\underline{B'}, \overline{R'}]$. (If this is the same component, we only pick one matching. If R' or B' is empty, we let the matchings be empty.) Call these green matchings M_1^{green} resp. M_2^{green} . Let $B'' := B' - V(M_1^{\text{green}} \cup M_2^{\text{green}})$ and $R'' := R' - V(M_1^{\text{green}} \cup M_2^{\text{green}})$.

Observe that by the maximality of M_1^{green} and M_2^{green} , if one of $\underline{R''}$, $\overline{B''}$ is non-empty, then the other one is empty. The same holds for the sets $\underline{B''}$, $\overline{R''}$. Thus one of the two graphs R'' , B'' is empty, say this is B'' .

The edges in R'' are green and blue. If R'' contains no green edges, we can pick another blue matching of maximum size and are done. Then again, if R'' contains a green edge, it follows by maximality of M_1^{green} and M_2^{green} that both of them are empty, which implies that there are no green edges in $R' \cup B'$. In this case we ignore M_1^{green} and M_2^{green} and finish as follows: By Lemma 2.4, R' can be spanned by at most 3 vertex disjoint monochromatic connected matchings. This proves the claim. \square

Claim 2.8. *Let Y and Z be monochromatic components of distinct colours such that $Y \cap Z$ is non-trivial. Then $Y - Z$ is not empty.*

Proof. Let Y be a red component, Z be a blue component, and let $X := H - (Y \cup Z)$. Suppose that $Y - Z$ is empty. We first note that all edges in $[\overline{Y \cap Z}, \underline{X}]$ and $[\underline{Y \cap Z}, \overline{X}]$ are green. Moreover, by Claim 2.6, there is another non-trivial blue component in H , which implies that X is non-trivial.

The subgraphs $[\overline{Y \cap Z}, \underline{X}]$ and $[\underline{Y \cap Z}, \overline{X}]$ cannot belong to the same green component, since otherwise H is spanned by the union of said green component and Z , which is not possible by Claim 2.7. Consequently, X has no green edges. By Claim 2.6 there is a green non-trivial component $G \subseteq Y \cup Z$. As $H = Z \cup (Y - Z) \cup X$ and $Y - Z$ is empty, we obtain that $G \cap Z$ is non-trivial in H and $G - Z \subseteq Y - Z$ is empty. Thus G has the same properties as Y with respect to Z and we can repeat the same arguments as above to obtain that all edges in X are blue. But this is a contradiction to Claim 2.7, as X and Z together span H . \square

Claim 2.9. *There is a colour that has exactly three non-trivial components.*

Proof. We show that there is a colour with at most three non-trivial components. This together with Claim 2.6 yields the desired result. So suppose otherwise. Then each colour has at least four non-trivial components. By Lemma 2.5, there are components X , Y and Z that together span H .

By assumption, not all of X , Y and Z have the same colour. If two of these components, say X and Y , have the same colour, say red, then $H - (X \cup Y)$ contains a red component that is non-trivial, by the assumption that our claim is false. The intersection of this red component with Z is non-trivial. Hence we get a contradiction to Claim 2.8.

So assume X is red, Y is blue and Z is green. We claim that (after possibly swapping top and bottom parts)

$$\overline{(Y \cap Z) - X} \text{ is empty.} \quad (1)$$

Indeed, otherwise $(Y \cap Z) - X$ is non-trivial. Then, as $[\underline{X}, \overline{(Y \cap Z) - X}]$ is non-trivial and its edges are green and blue, we get $\underline{X} \subseteq Y \cup Z$ since every vertex in \underline{X} sees a vertex in $\overline{Y \cap Z}$. In the same way we obtain $\overline{X} \subseteq Y \cup Z$. Thus $Z \cup Y$ is spanning, which is not possible by Claim 2.7. This proves (1).

By assumption, $H - X$ contains three non-trivial red components R_1 , R_2 and R_3 , say. For $i \neq j$, $[R_i \cap (Y - Z), R_j \cap (Z - Y)]$ has no red, blue or green edges and thus is trivial. So for at most one $i \in \{1, 2, 3\}$ the subgraph $R_i \cap [\overline{Y - Z}, \underline{Z - Y}]$ is non-trivial. The same holds for $[R_i \cap (Y - Z), R_j \cap (Z - Y)]$. Consequently, and by the pigeonhole principle, we can assume that,

$$R_1 \cap [\overline{Y - Z}, \underline{Z - Y}] \text{ and } R_1 \cap [\underline{Y - Z}, \overline{Z - Y}] \text{ are both trivial.} \quad (2)$$

As R_1 is non-trivial, we can suppose that without loss of generality $R_1 \cap Y$ is non-trivial. Thus, by (1) $R_1 \cap (Y - Z)$ is non-empty. Hence, by (2) we get:

$$|\overline{R_1} \cap \overline{Z - Y}| = 0. \quad (3)$$

Moreover, Claim 2.8 (applied to R_1 and Y) implies that R_1 has at least one vertex in $\overline{Z - Y}$ or $\underline{Z - Y}$. By (3) we have the latter case and hence

$$R_1 \cap (Z - Y) \text{ and } R_1 \cap (Y - Z) \text{ are each non-empty.} \quad (4)$$

The fact that $[\overline{Y - (X \cup Z)}, R_1 \cap (Z - Y)]$ and $[\overline{Z - (X \cup Y)}, R_1 \cap (Y - Z)]$ only have red edges, together with (2) and (4), yields that

$$\overline{Y - (X \cup Z)} \text{ and } \overline{Z - (X \cup Y)} \text{ are each empty} \quad (5)$$

Now by (5) (and by the existence of R_1, R_2, R_3), we know that $\overline{(Y \cap Z) - X}$ is non-empty. So each vertex of \underline{X} has a neighbour in $\overline{(Y \cap Z) - X}$ and hence $\underline{X} \subseteq Y \cup Z$. Since, by Claim 2.7, H is not spanned by $Y \cup Z$, we have that $\overline{X - (Y \cup Z)}$ is non-empty. This and (4) imply that $[\overline{X - (Y \cup Z)}, \underline{Y - (X \cup Z)}]$ and $[\overline{X - (Y \cup Z)}, \underline{Z - (X \cup Y)}]$ are non-trivial each. As the edges of these subgraphs are green and blue respectively, there are green and blue components G and B such that $\overline{H - X - [(G \cap Y) \cup (B \cap Z)]}$ is empty.

Now let G' be another non-trivial green component. Then $\overline{G' - X}$ is empty, while $\overline{G' \cap X}$ is non-empty. By (5) it follows that $\overline{G' - X}$ is empty, while $\overline{G' \cap X}$ is non-empty. This is not possible by Claim 2.8 and completes the proof. \square

Using Claim 2.9 we assume from now on that without loss of generality, colour red has exactly three non-trivial components R_1, R_2 and R_3 . For $i = 1, 2, 3$, let M_i be a red matching of maximum size in R_i .

The remaining graph $Y := H - M_1 - M_2 - M_3$ has no red edges. If Y is trivial, then as $|\overline{Y}| = |\underline{Y}|$, the graph Y is empty, and so we are done. If Y can be spanned by two disjoint monochromatic connected matchings, we are also done, since in that case, we found five matchings which together span H . So we can assume that the colouring of Y is split, by Lemma 2.4 and as the edges of

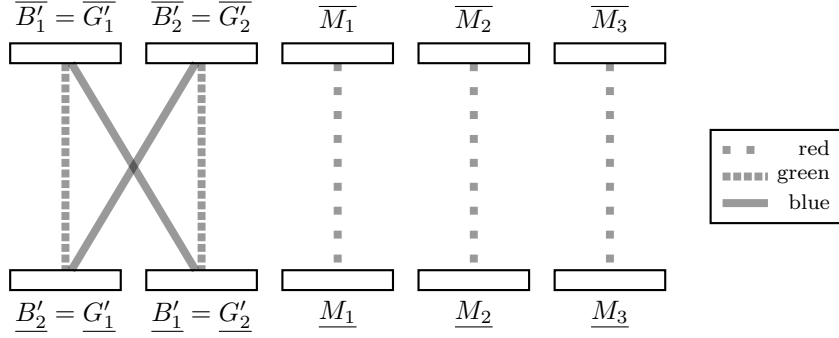


Figure 1: The structure of the colouring before Claim 2.10

Y are green and blue. We denote the blue and green components of Y by B'_1 , B'_2 , respectively G'_1 , G'_2 , where $\overline{B'_1} = \overline{G'_1}$, $\overline{B'_2} = \overline{G'_2}$, $\underline{B'_1} = \underline{G'_2}$, and $\underline{B'_2} = \underline{G'_1}$. Note that the subgraph

$$B'_1 \cup B'_2 \cup M_1 \cup M_2 \cup M_3 \text{ is spanning in } H. \quad (6)$$

By Lemma 2.4, Y can be spanned by two blue matchings $M_4 \subseteq B'_1$, $M_5 \subseteq B'_2$ and an additional green matching. If any of the matchings M_i is trivial, we can ignore it and still have a sufficiently large cover of H . Thus we get that

$$B'_1, B'_2, G'_1, G'_2, M_1, M_2, \text{ and } M_3 \text{ are non-trivial.} \quad (7)$$

Moreover, let B_1 and B_2 be the blue components in H that contain B'_1 and B'_2 , respectively. We define G_1 and G_2 analogously. If $B_1 = B_2$, we are done as $M_4 \cup M_5$ is a connected matching. This and symmetry imply

$$B_1 \neq B_2 \text{ and } G_1 \neq G_2. \quad (8)$$

The colouring so far is shown in Figure 1.

Claim 2.10. *For each $i = 1, 2, 3$ we have that*

- (a)
 - if $|\overline{M_i} \setminus \overline{G_1 \cup G_2}| > 0$, then $\underline{B'_1} \subseteq \underline{R_i}$ or $\underline{B'_2} \subseteq \underline{R_i}$;
 - if $|\overline{M_i} \setminus \overline{B_1 \cup B_2}| > 0$, then $\underline{G'_1} \subseteq \underline{R_i}$ or $\underline{G'_2} \subseteq \underline{R_i}$;
- (b)
 - if $|\underline{M_i} \setminus \underline{G_1 \cup G_2}| > 0$, then $\overline{B'_1} \subseteq \overline{R_i}$ or $\overline{B'_2} \subseteq \overline{R_i}$;
 - if $|\underline{M_i} \setminus \underline{B_1 \cup B_2}| > 0$, then $\overline{G'_1} \subseteq \overline{R_i}$ or $\overline{G'_2} \subseteq \overline{R_i}$;
- (c)
 - if $|\overline{M_i} \setminus \overline{G_1 \cup G_2 \cup B_1 \cup B_2}| > 0$, then $\underline{B'_1 \cup B'_2} = \underline{G'_1 \cup G'_2} \subseteq \underline{R_i}$;
 - if $|\underline{M_i} \setminus \underline{G_1 \cup G_2 \cup B_1 \cup B_2}| > 0$, then $\overline{B'_1 \cup B'_2} = \overline{G'_1 \cup G'_2} \subseteq \overline{R_i}$.

Proof. For the first part of (a), assume $|\overline{M_1} \setminus \overline{G_1 \cup G_2}| > 0$. Note that there is no green edge between $\overline{M_1} \setminus \overline{G_1 \cup G_2}$ and $\underline{G'_1}$. First assume that $\overline{M_1} \cap \overline{B_1} \setminus \overline{G_1 \cup G_2}$ is non-empty. Then, by (8), any edge between $\overline{M_1} \cap \overline{B_1} \setminus \overline{G_1 \cup G_2}$ and $\underline{B'_2} = \underline{G'_1}$ is red. So, by (7) the result follows. So we can assume that this is not true. Similarly the result holds if $|\overline{M_1} \cap \overline{B_2} \setminus \overline{G_1 \cup G_2}| > 0$. Therefore we can

assume that $\overline{M_1 \setminus B_1 \cup B_2 \cup G_1 \cup G_2}$ is non-empty. In this case, since all edges between $\overline{M_1 \setminus G_1 \cup G_2 \cup B_1 \cup B_2}$ and B'_1 are red, the result follows again by (7). Statement (b) and the second part of (a) follow similarly.

For the first part of (c), note that any edge between $\overline{M_i \setminus G_1 \cup G_2 \cup B_1 \cup B_2}$ and $B'_1 \cup B'_2 = \overline{G'_1 \cup G'_2}$ has to be red and use (7). The second part of (c) is analogous. \square

By Claim 2.6 there are green and blue non-trivial components $G_3 \neq G_1, G_2$ and $B_3 \neq B_1, B_2$ in H .

Claim 2.11. *It holds that $|V(G_3 \cap B_3 \cap (M_1 \cup M_2 \cup M_3))| > 0$.*

Proof. Assume otherwise. That is, assume

$$|V(G_3 \cap B_3 \cap (M_1 \cup M_2 \cup M_3))| = 0.$$

The components B_3 and G_3 do not meet with $B'_1 \cup B'_2 = G'_1 \cup G'_2$ and by (6), there are no vertices outside of $B'_1 \cup B'_2 \cup M_1 \cup M_2 \cup M_3$. We conclude that $B_3 \cap (M_1 \cup M_2 \cup M_3)$ and $G_3 \cap (M_1 \cup M_2 \cup M_3)$ are each non-trivial. Hence there are indices i, i', j, j' such that there is a blue non-trivial subgraph $B'_3 \subseteq B_3$ and a green non-trivial subgraph $G'_3 \subseteq G_3$ such that $\overline{B'_3} \subseteq \overline{M_i}$ and $\overline{B'_3} \subseteq \overline{M_{i'}}$, and $\overline{G'_3} \subseteq \overline{M_j}$ and $\overline{G'_3} \subseteq \overline{M_{j'}}$. Actually, we can choose these indices such that $i \neq i'$ and $j \neq j'$. Since if $i = i'$, say, Claim 2.8 yields that $(B_3 \cap H) \setminus M_i$ is not empty and therefore, by (6), there is some index $k \neq i$ such that $B_3 \cap M_k$ is not empty, which allows us to swap i' for k .

For an index $k \neq i$, the edges between $\overline{B'_3} \subseteq \overline{R_1 \cap M_i}$ and $G'_3 \cap M_k$ are blue and green. As by our initial assumption $|V(G_3 \cap B_3 \cap (M_1 \cup M_2 \cup M_3))| = 0$, this implies that $|G_3 \cap M_k| = 0$. In the same way we obtain that $|G_3 \cap M_k| = 0$ for $k \neq i'$ or $|\overline{B'_3} \cap M_i| = 0$, but the latter cannot happen by the choice of B'_3 . Hence we have $i = j'$ and $i' = j$; in other words,

$$|\overline{M_i} \cap G_3| > 0, |\overline{M_j} \cap G_3| > 0, |\overline{M_i} \cap B_3| > 0 \text{ and } |\overline{M_j} \cap B_3| > 0.$$

So by Claim 2.10 (a) and (b), either we have $B'_1 \subseteq R_i$ and $B'_2 \subseteq R_j$, or we have $G'_1 \subseteq R_i$ and $G'_2 \subseteq R_j$. Indeed, the fact that $|\overline{M_i} \cap G_3| > 0$ together with Claim 2.10 (b) implies that one of $\overline{B'_1} = \overline{G'_1} \subseteq \overline{R_i}$, $\overline{B'_2} = \overline{G'_2} \subseteq \overline{R_i}$ holds. Without loss of generality, we assume the latter. Next, as $|\overline{M_i} \cap B_3| > 0$, Claim 2.10 (a) implies that $\overline{G'_1} = \overline{B'_2} \subseteq \overline{R_i}$ or $\overline{G'_2} = \overline{B'_1} \subseteq \overline{R_i}$. Without loss of generality, we assume the former. We repeat the same with index j , and since we already know that $B'_2 \subseteq R_i$, the output of Claim 2.10 has to be $\overline{B'_1} = \overline{G'_2} \subseteq \overline{R_j}$ for $|\overline{M_j} \cap G_3| > 0$ and $\overline{B'_1} = \overline{G'_1} \subseteq \overline{R_j}$ for $|\overline{M_j} \cap B_3| > 0$. For the remainder, let us assume that $B'_1 \subseteq R_i$ and $B'_2 \subseteq R_j$.

Then $G'_1 \cap R_k = \emptyset = G'_2 \cap R_k$, where k is the third index, which together with Claim 2.10 (a) and (b), gives that $|R_k \cap (G_3 \cup B_3)| = 0$. The edges between $B'_2 = \overline{G'_1} \subseteq \overline{G_1 \cap R_j}$ and $B'_3 \cap R_i$ have to be green, which implies $\overline{B'_3} \subseteq \overline{G_1}$. As any edge between $\overline{B'_3}$ and $R_k - B_3$ has to be green this implies $|R_k \cap G_1| > 0$ since R_k is non-trivial and $|R_k \cap B_3| = 0$. This also implies that $|R_k - G_1| = 0$.

By repeating the same argument with $\overline{B'_1} = \overline{G'_1} \subseteq \overline{G_1}$ and $\overline{B'_3}$, it follows that $|R_k \cap G_1| > 0$ and $|\overline{R_k} - \overline{G_1}| = 0$. So $R_k \cap G_1$ is non-trivial and $R_k - G_1$ is empty, a contradiction to Claim 2.8. \square

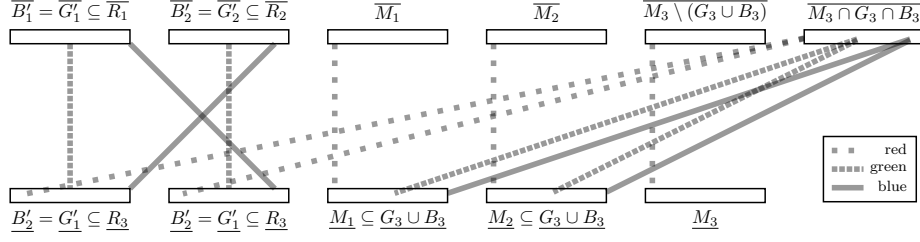


Figure 2: The structure of the colouring after (13).

Claim 2.11 and the symmetry between the M_i in both biparts allow us to assume that without loss of generality

$$|\overline{M_3} \cap \overline{G_3} \cap \overline{B_3}| > 0. \quad (9)$$

This implies $|\overline{M_3} \setminus \overline{G_1} \cup \overline{G_2} \cup \overline{B_1} \cup \overline{B_2}| > 0$ and thus by Claim 2.10(c) with $i = 3$ we obtain

$$\underline{B'_1} \cup \underline{B'_2} = \underline{G'_1} \cup \underline{G'_2} \subseteq \underline{R_3}. \quad (10)$$

This implies that $(\underline{R_1} \cup \underline{R_2}) \cap (\underline{G'_1} \cup \underline{G'_2}) = \emptyset$. Since the edges between $\overline{M_3} \cap \overline{G_3} \cap \overline{B_3}$ and $\underline{R_1} \cup \underline{R_2}$ are coloured green and blue, we have by (9) that

$$\underline{M_1} \cup \underline{M_2} \subseteq \underline{R_1} \cup \underline{R_2} \subseteq \underline{G_3} \cup \underline{B_3}. \quad (11)$$

So, by (7) and Claim 2.10(b) with $i = 1$, we can assume that without loss of generality

$$\overline{B'_1} = \overline{G'_1} \subseteq \overline{R_1} \quad (12)$$

and hence by (7) and Claim 2.10(b) with $i = 2$ it follows that

$$\overline{B'_2} = \overline{G'_2} \subseteq \overline{R_2}. \quad (13)$$

The structure of the colouring so far is sketched in Figure 2. The assertions (12) and (13) imply that $\overline{R_3} \cap \overline{G'_1} \cup \overline{G'_2} = \emptyset$. Suppose that there is an $x \in \underline{R_1} \cup \underline{R_2} \setminus \underline{G_1} \cup \underline{G_2} \cup \underline{B_1} \cup \underline{B_2}$. By (10), the edges between x and $\underline{G'_1} \cup \underline{G'_2} = \underline{B'_1} \cup \underline{B'_2}$ are not red, and neither green or blue by choice of x . As $\underline{G'_1}$ and $\underline{G'_2}$ are both non-trivial in H by (7) and H is complete, we obtain a contradiction. Hence

$$\overline{M_1} \cup \overline{M_2} \setminus \overline{G_1} \cup \overline{G_2} \cup \overline{B_1} \cup \overline{B_2} = \emptyset. \quad (14)$$

In the same fashion, suppose there is an $x \in (\underline{M_3} \setminus \underline{G_1} \cup \underline{G_2}) \cup (\underline{M_3} \setminus \underline{B_1} \cup \underline{B_2})$. By (12) and (13), the edges between x and $\overline{B'_1} = \overline{G'_1}$ respectively $\overline{B'_2} = \overline{G'_2}$ are neither green nor blue by choice of x . Again, using (7) and the completeness of H , we obtain a contradiction as

$$\underline{M_3} \setminus \underline{G_1} \cup \underline{G_2} = \underline{M_3} \setminus \underline{B_1} \cup \underline{B_2} = \emptyset. \quad (15)$$

Finally, suppose there is an $x \in \overline{B_3} \cup \overline{G_3} \cap \overline{M_1} \cup \overline{M_2}$. By (7), x sees vertices in $\underline{M_3}$. This, however, contradicts (15) and thus

$$\overline{B_3} \cup \overline{G_3} \cap \overline{M_1} \cup \overline{M_2} = \emptyset. \quad (16)$$

Next, we restore the symmetry between the colours.

Claim 2.12. *Each colour has exactly three components.*

Proof. We already know that R_1, R_2 and R_3 are the only red components in H . Suppose there is a (possibly trivial) green component G_4 distinct from G_1, G_2 and G_3 . Assume first that $\overline{G_4} \neq \emptyset$. Note that any edge between $\overline{G_4}$ and $\overline{G'_1 \cup G'_2}$ is red or blue. By (8), no vertex of $\overline{G_4}$ can send blue edges to both $\overline{G'_1}$ and $\overline{G'_2}$. Moreover, by (12) and (13), no vertex of $\overline{G_4}$ can send red edges to both $\overline{G'_1}$ and $\overline{G'_2}$. Since H is complete and $\overline{G'_1} = \overline{B'_1}$ and $\overline{G'_2} = \overline{B'_2}$ are non-trivial, we derive $\overline{G_4} \subseteq \overline{R_1 \cup R_2} \cap \overline{B_1 \cup B_2}$. But this contradicts (9), because H is complete.

Now let us assume that $\overline{G_4} = \emptyset$, and so, $\overline{G_4} \neq \emptyset$. In other words, G_4 consists of a single vertex with no incident green edges. Suppose that $\overline{G_4} \cap \overline{M_3} = \emptyset$. So by (7) and (10), the edges between $\overline{G_4}$ and $\overline{G'_1 \cup G'_2}$ are blue, which contradicts that $\overline{B'_1}$ and $\overline{B'_2}$ lie in distinct blue components, as asserted by (8). Therefore $\overline{G_4} \subseteq \overline{M_3}$. As $\overline{G_4} = \emptyset$, all edges between $\overline{G_4}$ and $\overline{M_1 \cup M_2}$ are blue. By (15) and (16), $\overline{B_3} \subseteq \overline{[M_1 \cup M_2, M_3]}$. Since H is complete and $\overline{B_3}$ is non-trivial, we obtain that $\overline{G_4} \subseteq \overline{B_3}$. We also have that $\overline{G_3} \subseteq \overline{[M_1 \cup M_2, M_3]}$ by (15) and (16). Since G_3 is non-trivial it follows that, $\overline{G_3} \cap \overline{M_1 \cup M_2}$ is non-empty. Since the edges between $\overline{G_4}$ and $\overline{G_3}$ are blue, we obtain that $\overline{M_1 \cup M_2} \cap \overline{G_3} \cap \overline{B_3} \neq \emptyset$. But this represents a contradiction to (12) or (13), since there is no colour left for the edges between $\overline{G_3} \cap \overline{B_3}$ and $\overline{B'_1 \cup B'_2}$. Since a fourth blue component would behave the same way as G_4 , this finishes the proof of the claim. \square

By (10) it follows that $\overline{R_i} = \overline{M_i}$ for $i = 1, 2$. In the same way (12) and (13) imply that

$$\overline{R_3} = \overline{M_3}. \quad (17)$$

For $1 \leq i, j, k \leq 3$ we denote $\overline{i|j|k} := \overline{R_i \cap G_j \cap B_k}$ and $\underline{i|j|k} := \underline{R_i \cap G_j \cap B_k}$. From (7), (9), (12) and (13) we obtain that

$$|\overline{1|1|1}|, |\overline{2|2|2}|, |\overline{3|3|3}| > 0. \quad (18)$$

Note that by definition and completeness it follows that for all i, i', j, j', k, k' with $i \neq i', j \neq j'$ and $k \neq k'$ we have (modulo switching biparts)

$$\text{if } |\overline{i|j|k}| > 0, \text{ then } |\underline{i'|j'|k'}| = 0. \quad (19)$$

Let us show that $\underline{i|j|k} = \emptyset$, unless i, j, k are pairwise different. Indeed, otherwise, if say $\underline{1|1|k} \neq \emptyset$ for $k = 1, 2$ or 3 , we obtain a contradiction to (19) as $|\overline{2|2|2}|, |\overline{3|3|3}| > 0$ by (18).

Hence \underline{H} can be decomposed into sets $\underline{i|j|k}$, where $1 \leq i, j, k \leq 3$ are pairwise different. So we have:

$$\underline{1|3|2} \cup \underline{1|2|3} \cup \underline{2|3|1} \cup \underline{2|1|3} \cup \underline{3|2|1} \cup \underline{3|1|2} = \underline{H}. \quad (20)$$

Claim 2.13. *We have $\overline{H} = \overline{1|1|1} \cup \overline{2|2|2} \cup \overline{3|3|3} \cup \overline{3|1|2} \cup \overline{3|2|1}$.*

Proof. First, we show there is no $\overline{i|j|k} \neq \emptyset$ such that exactly two of i, j, k are equal. If $\overline{3|1|1} \neq \emptyset$, say, then $|\underline{1|2|3}|, |\underline{1|3|2}| = 0$ by (19). Together with (20), this implies that $\underline{R_1}$ is trivial, a contradiction. Second, note that (10) implies that $\overline{3|1|2}$ and $\overline{3|2|1}$ are non-empty. Again, by (19), it follows that $\underline{i|j|k} = \emptyset$, if $i \neq 3$ and $3 \in \{j, k\}$. This proves the claim. \square

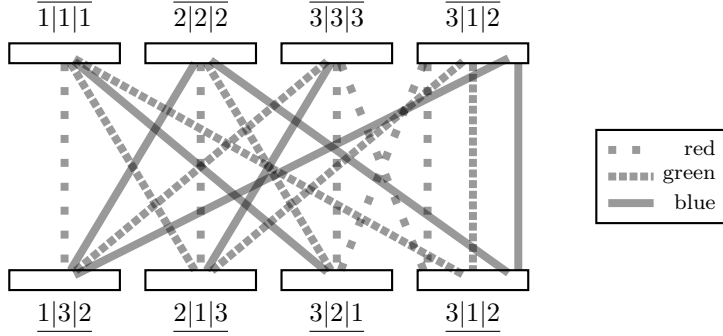


Figure 3: The colouring from the proof of Claim 2.14

Claim 2.14. *We have $\overline{H} = \overline{1|1|1} \cup \overline{2|2|2} \cup \overline{3|3|3}$.*

Proof. By the previous claim it remains to show that $\overline{3|1|2} = \overline{3|2|1} = \emptyset$. To this end, suppose that $\overline{3|1|2} \neq \emptyset$ and thus $|\underline{1|2|3}|, |\underline{2|3|1}| = 0$ by (19). If $\overline{3|2|1} \neq \emptyset$ as well, then by (19) also $|\underline{1|3|2}| = 0$ which, by Claim 2.13 and (20) gives the contradiction that $R_1 \subseteq \overline{1|1|1}, \underline{1|2|3} \cup \underline{1|3|2}$ is trivial. So we have

$$\overline{H} = \overline{1|1|1} \cup \overline{2|2|2} \cup \overline{3|3|3} \cup \overline{3|1|2},$$

with $\overline{3|1|2} \neq \emptyset$. This partition is shown in Figure 3.

Ignoring from now on the matchings M_1 and M_2 , we aim at covering H with M_3 and four other matchings. To this end take a green matching M_1^{green} of maximum size in $G_1 - M_3$ and next a blue matching M_2^{blue} of maximum size in $B_2 - M_3 - M_1^{\text{green}}$. Denote

- $\overline{i|j|k}' := \overline{i|j|k} \setminus \overline{M_3 \cup M_1^{\text{green}} \cup M_2^{\text{blue}}}$ and
- $\underline{i|j|k}' := \underline{i|j|k} \setminus \underline{M_3 \cup M_1^{\text{green}} \cup M_2^{\text{blue}}}$.

We can assume that $M_3 \cup M_1^{\text{green}} \cup M_2^{\text{blue}}$ is not spanning. Thus, as H is complete, the maximality of the matchings M_3 , M_1^{green} and M_2^{blue} implies that $\overline{3|1|2}', \underline{3|1|2}' = \emptyset$.

Moreover it follows that

- $|\overline{1|1|1}'| = 0$ or $|\underline{2|1|3}'| = 0$ by maximality of $M_1^{\text{green}} \subseteq G_1$,
- $|\overline{2|2|2}'| = 0$ or $|\underline{1|3|2}'| = 0$ by maximality of $M_2^{\text{blue}} \subseteq B_2$,
- $\overline{3|3|3}' = \emptyset$ as $\overline{R_3} = \overline{M_3}$ by (17).

If $|\overline{1|1|1}'|, |\underline{2|2|2}'| = 0$, then we have found three disjoint connected matchings that span H , contradicting our assumption. If $|\underline{2|1|3}'|, |\underline{1|3|2}'| = 0$, we take a green matching in G_2 and a blue maximum matching in B_1 , among the yet unmatched vertices. After this step, there are no vertices of $\underline{3|2|1}'$ left uncovered and therefore all vertices of \underline{H} are covered. Thus, as H is balanced, we have found five disjoint monochromatic connected matchings which together span H .

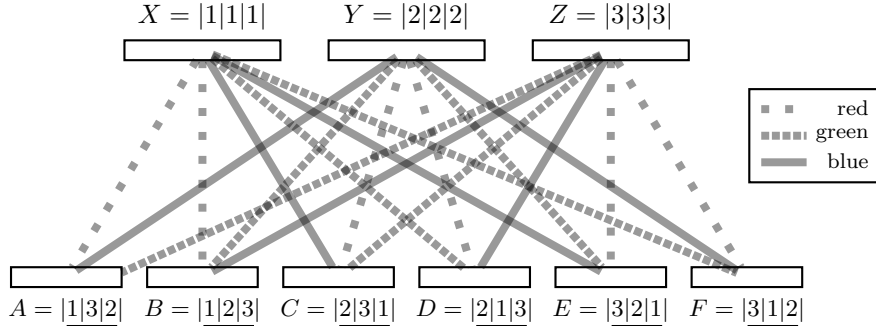


Figure 4: The partition of $K_{n,n}$.

So, either $|\overline{2|2|2'}|, |2|1|3'| = 0$, or $|\overline{1|1|1'}|, |1|3|2'| = 0$. In either case we can find two disjoint monochromatic connected matchings that cover all vertices of the two other sets from the previous sentence and all vertices of $3|2|1'$. So we have five disjoint monochromatic connected matchings spanning \overline{H} , a contradiction. \square

For ease of notation we set

$$X := |\overline{1|1|1}|, Y := |\overline{2|2|2}|, Z := |\overline{3|3|3}| \text{ and}$$

$$A := |1|3|2|, B := |1|2|3|, C := |2|3|1|, D := |2|1|3|, E := |3|2|1|, F := |3|1|2|.$$

By Claim 2.14 and (20) we have $|\overline{H}| = X+Y+Z$ and $|\underline{H}| = A+B+C+D+E+F$. Note that the edges between any upper and lower part are monochromatic (see Figure 4).

Also note that we reached complete symmetry between the colours and the indices of the components, so we will from now on again treat them as interchangeable.

Observe that for (at least) one index $i \in \{1, 2, 3\}$ it holds that $|\overline{R}_i| \leq |\underline{R}_i|$. We shall call such an index i a *weak* index for the colour red. If furthermore $|\overline{R}_i| < |\underline{R}_i \cap B_j| = |\underline{R}_i \cap G_k|$ and $|\overline{R}_i| < |\underline{R}_i \cap B_k| = |\underline{R}_i \cap G_j|$, where j, k are the other two indices from $\{1, 2, 3\}$, then we call i *very weak* for colour red. Analogously define (*very*) *weak* indices for colours blue and red.

Claim 2.15. *If index i is weak for colour c , then*

- (a) *the indices in $\{1, 2, 3\} - \{i\}$ are not weak for colour c , and*
- (b) *index i is very weak for colour c .*

Proof. Let us show this for $i = 2$ and colour red (the other cases are analogous). By assumption, $Y \leq C+D$. Since $X < A+B$ and $Z < E+F$ cannot both hold, we can assume without loss of generality that $Z \geq E+F$. Now if $X \leq A+B$, then we pick maximal red matchings in $|\overline{1|1|1}|, |1|3|2| \cup |1|2|3|$, $|\overline{2|2|2}|, |2|3|1| \cup |2|1|3|$ and $|\overline{3|3|3}|, |3|2|1| \cup |3|1|2|$, thus covering all vertices of $|\overline{1|1|1}| \cup |\overline{2|2|2}| \cup |\overline{3|3|3}|$. To finish we cover all of the remaining vertices in $|\overline{3|3|3}| \cup (H \setminus R_3)$ with a blue

and a green matching, a contradiction. Hence $X > A + B$. Using this fact, $Z > E + F$ follows by symmetry. This proves (a).

In order to show (b), let us first prove that $Y < C$. We pick a maximal red matching in each of R_1 and R_3 , thus covering all vertices of $\overline{R_1 \cup R_3}$. Now if $Y \geq C$, then all vertices of $\overline{2|3|1}$ are contained in a maximal red matching that also contains all vertices of $\overline{2|2|2}$. We cover all of the remaining vertices in $\overline{R_1 \cup R_3}$ with a blue and a green matching, a contradiction. The fact that $Y < D$ follows analogously. \square

Suppose two of the three indices 1, 2, 3 are weak for different colours, say 1 is weak for red and 2 is weak for green. Then Claim 2.15(b) gives that $X < A$ and $Y < E$. Thus we can match all vertices of $\overline{1|1|1}$ into $\overline{1|3|2}$ and all vertices of $\overline{2|2|2}$ into $\overline{3|2|1}$ with two matchings, one red and one green, and cover all of the remaining vertices with three disjoint matchings, one from each of R_3, G_3, B_3 , a contradiction.

Hence, since each colour has a weak index, there is an index i that is weak for all three colours, $i = 2$ say. We match all vertices of $\overline{2|2|2}$ into $\overline{3|1|2}$ with a blue matching M . Let us from now work with the remaining set $\overline{3|1|2'} = \overline{3|1|2} \setminus V(M)$ of cardinality $F' = F - Y$. Set $n' = n - Y$. (So instead of five we will have to find four monochromatic connected matchings covering all vertices of $H - M$.) Without loss of generality assume $Z \geq X$. Claim 2.15(a) gives that

$$X > A + B, C + E, D + F' \text{ and } Z > A + C, B + D, E + F'. \quad (21)$$

Hence $X > n'/3$. So, one of the three sums $A + C, B + D, E + F'$ has to be strictly smaller than X , say $A + C < X$. Consequently, $Z = n' - X < B + D + E + F'$.

If $Z \geq D + E + F'$, then we cover all vertices of $\overline{R_3 - M}$ with a red matching, and cover all vertices of the remains of $\overline{3|3|3}$ with a blue matching that also covers all vertices of $\overline{2|1|3}$. Now all that is left on the top is $\overline{1|1|1}$, which we can match with a red and a blue matching into the remains of $\overline{1|3|2} \cup \overline{1|2|3} \cup \overline{2|3|1}$. Thus we found four connected matchings that cover all vertices of $\overline{H - V(M)}$, and are done.

So we may assume $Z < D + E + F'$ and thus $X > A + B + C$. If $X \leq A + B + C + E$, then we can proceed similarly as in the previous paragraph to find four matchings covering all vertices of H . Hence $X > A + B + C + E$, implying that $Z < D + F'$. But by (21) we have $D + F' < X$ a contradiction to our assumption that $X \leq Z$. This finishes the proof of Lemma 1.2.

3 Covering almost all vertices with connected matchings

3.1 Preliminaries

The goal of this section is to prove a version of Lemma 1.2 for almost complete graphs. This result is given in Lemma 3.1.

Let G be a graph with biparts A and B and let H be a subgraph of G . We call H γ -dense in G if it has at least $\gamma|A||B|$ edges. If $H = G$, we often simply say G is γ -dense. Let H be a subgraph of G . If H has biparts $X \subseteq A$ and $Y \subseteq B$ such that $|X| \geq \gamma|A|$ and $|Y| \geq \gamma|B|$, then we call H γ -non-trivial (in

G), or we say G is γ -spanned by H . Usually, we use the term γ -non-trivial when $\gamma \approx 0$ and we use the term γ -spanned when $\gamma \approx 1$.

Lemma 3.1. *There is an $\epsilon_0 > 0$ such that for each $0 < \epsilon \leq \epsilon_0$ there are n_0 and $\rho = \rho(\epsilon)$ such that for all $n \geq n_0$ the following holds.*

Every 3-edge-coloured balanced bipartite $(1 - \epsilon)$ -dense graph of size $2n$ is $(1 - \rho)$ -spanned by at most five disjoint monochromatic connected matchings.

For the proof of Lemma 3.1 we need some more notation. Again, let G be a graph with biparts A and B and let H be a subgraph of G . We say H has γ -complete degree in G if $\deg_H(y) > \gamma|A|$ for $y \in B \cap V(H)$ and $\deg_H(x) > \gamma|B|$ for $x \in A \cap V(H)$. Clearly, if H has γ -complete degree in G , then in particular, H is γ -dense in G .

The following lemmas are well-known and follow from standard averaging arguments.

Lemma 3.2. *For $\epsilon > 0$ let H be a $(1 - \epsilon)$ -dense bipartite graph. Then H has a $(1 - \sqrt{\epsilon})$ -spanning subgraph H' with $(1 - 2\sqrt{\epsilon})$ -complete degree (in H).*

Lemma 3.3. *For $1/4 > \epsilon > 0$ let H be a bipartite graph with biparts A, B , having $(1 - \epsilon)$ -complete degree. Then any 2ϵ -non-trivial subgraph of H is connected.*

We omit the easy proofs of the next two lemmas.

Lemma 3.4. *For $\delta, \epsilon > 0$ let H be a $(1 - \epsilon)$ -dense bipartite graph with a δ -subgraph H' . Then H' is $(1 - \epsilon/\delta^2)$ -dense in H' .*

Lemma 3.5. *For $\delta, \epsilon > 0$ let H be a bipartite graph of $(1 - \epsilon)$ -complete degree and H' be a δ -non-trivial subgraph. Then H' has $(1 - \epsilon/\delta)$ -complete degree in itself.*

The proof of the next lemma is given as a warm-up. In the remainder of this section H is a bipartite graph with biparts \overline{H} and \underline{H} .

Lemma 3.6. *For $1/5 \geq \epsilon > 0$ let H be a 2-edge-coloured bipartite graph of $(1 - \epsilon)$ -complete degree, with bipartition A, B . Then H has a $((1 - \epsilon)/2)$ -spanning monochromatic component.*

Proof. Having $(1 - \epsilon)$ -complete degree, H has a monochromatic component R with $|\overline{R}| \geq (1 - \epsilon)|\overline{H}|/2$. If R is $((1 - \epsilon)/2)$ -spanning we are done. Otherwise the monochromatic subgraph $[\overline{R}, \underline{H} - R]$ is $((1 - \epsilon)/2)$ -spanning, and it is connected by Lemma 3.3. \square

In order to formulate a dense version of Lemma 2.2 we need to define dense variants of V -colourings and split colourings. We say a colouring of $E(H)$ in red and blue is an ϵ - V -colouring if there are monochromatic components R and B of distinct colours such that

1. each of R and B is ϵ -non-trivial in H ;
2. $R \cup B$ is $(1 - \epsilon)$ -spanning in H ;
3. $|V(\overline{R \cap B})| \geq (1 - \epsilon)|V(\overline{H})|$ or $|V(\underline{R \cap B})| \geq (1 - \epsilon)|V(\underline{H})|$.

A colouring of $E(H)$ in red and blue is ϵ -split, if

1. all monochromatic components are ϵ -non-trivial;
2. each colour has exactly two monochromatic components.

The following is a robust analogue of Lemma 2.2.

Lemma 3.7. *Let $\epsilon < 1/6$. If the bipartite 2-edge-coloured graph H has $(1 - \epsilon)$ -complete degree, then one of the following holds:*

- (a) *There is a $(1 - 3\epsilon)$ -spanning monochromatic component,*
- (b) *H has a 3ϵ - V -colouring, or*
- (c) *the edge-colouring is 2ϵ -split.*

Proof. Let R be an $((1 - \epsilon)/2)$ -spanning component in colour red, say. Such a component exists by Lemma 3.6. Set $X := H - R$ and note that all edges in $[\underline{R}, \underline{X}]$ and $[\underline{R}, \overline{X}]$ are blue.

We first assume that $|\overline{X}| < 3\epsilon|V(\overline{H})|$. If also $|\underline{X}| < 3\epsilon|V(\underline{H})|$, we are done, since then R is $(1 - 3\epsilon)$ -spanning. Otherwise, $|\underline{X}| \geq 3\epsilon|V(\underline{H})|$, and thus the blue subgraph $[\underline{X}, \overline{R}]$ is connected by Lemma 3.3 and the colouring is a 3ϵ - V -colouring.

So by symmetry we can assume that both $|\overline{X}| \geq 3\epsilon|V(\overline{H})|$ and $|\underline{X}| \geq 3\epsilon|V(\underline{H})|$. If there is a blue edge in R or in X , then H is spanned by one blue component by Lemma 3.3. Hence, all edges inside R and X are red and the colouring is 2ϵ -split, again using Lemma 3.3. \square

Corollary 3.8. *Let $\epsilon < 1/6$. If a bipartite 2-edge-coloured graph H has $(1 - \epsilon)$ -complete degree, then*

- (a) *there are one or two 2ϵ -non-trivial monochromatic components that together $(1 - 3\epsilon)$ -span H , and*
- (b) *if the colouring is not 2ϵ -split, then there is a colour with exactly one 3ϵ -non-trivial component.*

Now we prove an analogue of Lemma 2.4.

Lemma 3.9. *Let $\epsilon < 1/6$, and let H be a balanced bipartite graph of $(1 - \epsilon)$ -complete degree whose edges are coloured red and blue. Then either*

- (a) *H is $(1 - 5\epsilon)$ -spanned by two vertex disjoint monochromatic connected matchings, one of each colour, or*
- (b) *the colouring is 2ϵ -split and*
 - *H is $(1 - 2\epsilon)$ is spanned by one red and two blue vertex disjoint monochromatic connected matchings and*
 - *H is $(1 - 2\epsilon)$ is spanned by one blue and two red vertex disjoint monochromatic connected matchings.*

Proof. First assume that the colouring is 2ϵ -split. We take one red maximum matching in each of the two red components. This leaves at least one of the blue components with less than $\epsilon|\overline{H}|$ vertices on each side. We extract a third maximum matching from the leftover of the other blue component, thus leaving

one of its sides with less than $\epsilon|\overline{H}|$ vertices. All three matchings are clearly connected (or possibly empty, in case of the third matching) Thus the three matchings together $(1 - 2\epsilon)$ -span H . Note that we could have switched the roles of red and blue in order to obtain two blue and one red matching that $(1 - 2\epsilon)$ -span H .

So by Lemma 3.7, we may assume that either there is a colour, say red, with an $(1 - 3\epsilon)$ -spanning component R , or H has a 3ϵ - V -colouring, with components R in red and B in blue, say. In either case, we take a maximum red matching M in R . Then there is an induced balanced bipartite subgraph of H , whose edges are all blue, which contains all but at most $3\epsilon|V(H)|$ of the uncovered vertices of each bipart of H . If this subgraph is not 2ϵ -non-trivial, we are done. Otherwise, we finish by extracting from it a maximum blue matching $M' \subseteq B$, note that M' is connected by Lemma 3.3. As H has $(1 - \epsilon)$ -complete degree and there are no leftover edges in said subgraph, we obtain that $M \cup M'$ $(1 - 4\epsilon)$ -span H , and we are done. \square

We now prove a robust analogue of Lemma 2.5.

Lemma 3.10. *Let $1/6^6 > \epsilon > 0$. Let the edges of the bipartite graph H of $(1 - \epsilon)$ -complete degree be coloured in red, green and blue, such that each colour has at least four $\epsilon^{1/6}$ -non-trivial components; then there are three monochromatic components that together $(1 - \epsilon^{1/6})$ -span H .*

Proof. Set $\gamma := \epsilon^{1/6}$ and let R be a red γ -non-trivial component. Throughout the proof we shall make use of Lemma 3.3 without mentioning it explicitly. Since there are three more red γ -non-trivial components, the three graphs $X := H - R$, $[\overline{R}, \underline{X}]$ and $[\underline{R}, \overline{X}]$ are each γ -non-trivial and by Lemma 3.5, each of them has $(1 - \gamma^2)$ -complete degree (in themselves). Moreover, the edges of the latter two graphs are green and blue. By Corollary 3.8(a) there are one or two $2\gamma^2$ -non-trivial monochromatic components that together $(1 - 3\gamma^2)$ -span $[\underline{R}, \overline{X}]$. So, if $[\overline{R}, \underline{X}]$ has a $(1 - 3\gamma^2)$ -spanning monochromatic component, then we can $(1 - 3\gamma^2)$ -span H with at most three components, which is as desired. Therefore and by symmetry we may assume from now on that none of $[\overline{R}, \underline{X}]$ and $[\underline{R}, \overline{X}]$ has a $(1 - 3\gamma^2)$ -spanning monochromatic component. Suppose $[\overline{R}, \underline{X}]$ has a $2\gamma^2$ -split-colouring. By Lemma 3.7, either $[\underline{R}, \overline{X}]$ is $2\gamma^2$ -split or a fraction of $(1 - 3\gamma^2)$ of one of \underline{R} and \overline{X} is contained in the intersection of a blue and a green monochromatic component. In the latter case the union of three monochromatic components of the same colour contains a fraction of $(1 - 3\gamma^2)$ vertices of one of the biparts of H . But this is impossible as each colour has at least four γ -non-trivial components, and $\gamma > 3\gamma^2$. On the other hand, if both $[\overline{R}, \underline{X}]$ and $[\underline{R}, \overline{X}]$ have a $2\gamma^2$ -split colouring, then each bipart of H is contained in the union of four green components as well as in the union of four blue components, and thus all edges in X are red. But then there are only two γ -non-trivial red components, R and X , a contradiction.

So by Lemma 3.7, and by symmetry, we know that $[\overline{R}, \underline{X}]$ and $[\underline{R}, \overline{X}]$ both have green/blue $3\gamma^2$ - V -edge-colourings. Thus each of $[\overline{R}, \underline{X}]$ and $[\underline{R}, \overline{X}]$ has a $3\gamma^2$ -non-trivial blue component and a $3\gamma^2$ -non-trivial green component, say these are B_1, G_1 and B_2, G_2 respectively. Furthermore, a fraction of $(1 - 3\gamma^2)$ of \underline{X} or \overline{R} is contained in the intersection $B_1 \cap G_1$, and a fraction of \overline{X} or \underline{R} is contained in the intersection $B_2 \cap G_2$.

We first look at the case where a fraction of $(1 - 3\gamma^2)$ of \underline{X} is contained in $\overline{B_1 \cap G_1}$. If a fraction of $(1 - 3\gamma^2)$ of \underline{R} is contained in $\overline{B_2 \cap G_2}$, then, as $\gamma > 6\gamma^2$, both green and blue have at most two γ -non-trivial components, which is a contradiction. On the other hand, if a fraction of $(1 - 3\gamma^2)$ of \overline{X} is contained in $\overline{B_2 \cap G_2}$, then H is $(1 - 3\gamma^2)$ -spanned by the union of R and the blue components in H that contain B_1 and B_2 , and we are done.

Consequently we can assume by symmetry and by Lemma 3.7 that a fraction of $(1 - 3\gamma^2)$ of \overline{R} is contained in the intersection $\overline{B_1 \cap G_1}$ and a fraction of $(1 - 3\gamma^2)$ of \underline{R} is contained in the intersection $\overline{B_2 \cap G_2}$. Observe that $[G_1, \overline{G_2}]$ is coloured red and blue and $[B_1, \overline{B_2}]$ is coloured red and green, since otherwise, we obtain the desired cover. As these two graphs are each $3\gamma^3$ -non-trivial subgraphs of H , and as $\epsilon/(3\gamma^3) = \gamma^3/3$, Lemma 3.5 implies they have $(1 - \gamma^3/3)$ -complete degree (in themselves). Suppose there is a red component of $[G_1, \overline{G_2}]$ that is $(1 - \gamma)$ -spanning in $[G_1, \overline{G_2}]$. Such a component, together with B_1 and B_2 , $(1 - 2\gamma)$ -spans H as $\gamma < 1/3$. So, we can assume $[G_1, \overline{G_2}]$ has no red $(1 - \gamma)$ -spanning red component. Moreover, since there are at least four γ -non-trivial blue components, $[G_1, \overline{G_2}]$ contains two blue components, which are $\gamma/2$ -non-trivial each as $\gamma/2 > 3\gamma^2$.

Since these blue components are γ -non-trivial in H , $[G_1, \overline{G_2}]$ does not have a γ^3 - V -colouring (in itself). Thus, by Lemma 3.7 with input $\epsilon_{Lem3.7} = \gamma^3/3$, $[G_1, \overline{G_2}]$ is $2\gamma^3/3$ -split coloured in red and blue. Similarly we see that $[B_1, \overline{B_2}]$ is $2\gamma^3/3$ -split coloured in red and green.

Consider the edges in $[G_1, \overline{B_2}]$ and $[B_1, \overline{G_2}]$. If any of these edges is green or blue, then our graph is $(1 - 2\gamma^3/3)$ -spanned by three green or by three blue components. On the other hand, if all edges in $[G_1, \overline{B_2}]$ and $[B_1, \overline{G_2}]$ are red, then $[G_1 \cup B_1, \overline{B_2 \cup G_2}]$ is connected in red by Lemma 3.3, and thus, H has only three γ -non-trivial red components, a contradiction. \square

3.2 Proof of Lemma 3.1

We are now ready to prove Lemma 3.1. We will not give specific bounds for $\epsilon_0 > 0$ and n_0 but assume that they are sufficiently small respectively large as we go through the proof. For $0 < \epsilon \leq \epsilon_0$ let $n \geq n_0$ and H be a balanced bipartite $(1 - \epsilon)$ -dense graph which has $(1 - \epsilon)$ -complete degree and order $2n$, where $n \geq n_0$.

We choose numbers δ, γ, ρ such that

$$\epsilon \ll \delta \ll \gamma \ll \rho < 1. \quad (22)$$

Although these numbers could in principle be specified, we refrain from doing so in order to not spoil the neatness of the argumentation. Our aim is to show that H can be $(1 - \rho)$ -spanned with five vertex disjoint monochromatic connected matchings. We suppose that this is wrong in order to obtain a contradiction. Lemma 3.1 then follows by Lemma 3.2.

The next claim is the robust analogue of Claim 2.6.

Claim 3.11. *Each colour has at least three γ -non-trivial components.*

Proof. Suppose the claim is wrong for colour red, say. Let \mathcal{Y} be the set of all red components with top bipart smaller than γn and let \mathcal{Z} be the set of all red

components with bottom bipart smaller than γn . The total number of edges in red components that are not γ -non-trivial is less than

$$\sum_{Y \in \mathcal{Y}} \gamma n |\underline{Y}| + \sum_{Z \in \mathcal{Z}} \gamma n |\overline{Z}| < 2\gamma n^2.$$

Thus, deleting the (red) edges of all $Y \in \mathcal{Y} \cup \mathcal{Z}$, we obtain a spanning subgraph H' of H that is $(1 - 3\gamma)$ -dense in itself and in which each red component is either γ -non-trivial or trivial.

By assumption, there are two (possibly trivial) red components R_1 and R_2 in H' , such that all other red components are trivial. Let M be a maximum red matching in $R_1 \cup R_2$. Then every edge in the balanced bipartite subgraph $X := H' - M$ is green or blue.

If the (at most) two connected matchings in M together $(1 - \rho)$ -span H , we are done. Otherwise X is ρ -non-trivial in H' , and thus $(1 - (\rho/20)^2)$ -dense, by Lemma 3.4 and since we assume $3\gamma \leq (\rho^2/20)^2$.

We apply Lemma 3.2 to obtain a subgraph $H'' \subseteq X$ that $(1 - \rho/20)$ -spans X and has $(1 - \rho/10)$ -complete degree. By Lemma 3.9, H'' can be $(1 - \rho/2)$ -spanned with three vertex-disjoint monochromatic connected matchings. So in total we found at most five vertex-disjoint monochromatic connected matchings that together $(1 - \rho)$ -span H . \square

A subgraph $X \subseteq H$ is called ϵ -empty, if both $|\underline{X}| < \epsilon|\underline{H}|$ and $|\overline{X}| < \epsilon|\overline{H}|$ hold. The next claim is a robust version of Claim 2.7.

Claim 3.12. *There are no two monochromatic components that together $(1 - \gamma/2)$ -span H .*

Proof. Suppose the claim is wrong and there are monochromatic components R and B that together $(1 - \gamma/2)$ -span H . By Claim 3.11 we can assume that they have distinct colours, say R is red and B is blue. Take a red matching M^{red} of maximum size in R and a blue matching M^{blue} of maximum size in $B - V(M^{\text{red}})$. Set $R' := R - V(M^{\text{red}} \cup M^{\text{blue}})$ and $B' := B - V(M^{\text{red}} \cup M^{\text{blue}})$. By maximality, any edge between $\overline{B'}$ and $\underline{R'}$ is green. The same holds for the edges between $\underline{B'}$ and $\overline{R'}$.

If $[\underline{B'}, \overline{R'}]$ is γ -empty, we finish by picking a maximum matching in $[\underline{R'}, \overline{B'}]$. We proceed analogously if $[\underline{R'}, \overline{B'}]$ is γ -empty. So at least one R' or B' is γ -non-trivial. Thus, since H has $(1 - \epsilon)$ -complete degree, all edges of $[\underline{B'}, \overline{R'}]$ lie in the same green component. The same holds for $[\underline{R'}, \overline{B'}]$.

Assuming that both are non-empty we now pick now pick a maximum matching in each of the green components of $H - V(M^{\text{red}} \cup M^{\text{blue}})$ that contain $[\overline{B'}, \underline{R'}]$, $[\underline{B'}, \overline{R'}]$. (If this is the same component, we only pick one matching. If R' or B' is γ -empty, we let the matchings be empty.) Call these green matchings M_1^{green} resp. M_2^{green} . Let $B'' := B' - V(M_1^{\text{green}} \cup M_2^{\text{green}})$ and $R'' := R' - V(M_1^{\text{green}} \cup M_2^{\text{green}})$.

Observe that by the maximality of M_1^{green} and M_2^{green} , if one of $\underline{R''}$, $\overline{B''}$ has size at least ϵn , then the other one is empty. The same holds for the sets $\underline{B''}$, $\overline{R''}$. Thus one of the two graphs R'' , B'' is ϵ -empty, say this is B'' . If R'' is 2γ -empty, we are done, so we can assume that R'' is γ -non-trivial.

The edges in R'' are green and blue. If R'' contains no green edges, we can pick another blue matching of maximum size and are done. Then again, if R''

contains a green edge, it follows by maximality of M_1^{green} and M_2^{green} that both of them have a size of less than $2\epsilon n$. In this case we ignore M_1^{green} and M_2^{green} and finish as follows: By Lemma 3.5, R'' has $(1 - \epsilon/\gamma)$ -complete degree in itself. So, by Lemma 3.9, R'' can be $(1 - 5\epsilon/\gamma)$ -spanned by at most 3 vertex disjoint monochromatic connected matchings. This proves the claim. \square

Claim 3.13. *Let Y and Z be monochromatic components of distinct colours such that $Y \cap Z$ is 2ϵ -non-trivial. Then $Y - Z$ is not $\gamma/4$ -empty.*

Proof. Let Y be a red component, Z be a blue component, and let $X := H - (Y \cup Z)$. Suppose that $Y - Z$ is $\gamma/4$ -empty. We first note that all edges in $[\overline{Y \cap Z}, \underline{X}]$ and $[\underline{Y \cap Z}, \overline{X}]$ are green. Moreover, by Claim 3.11, there is another γ -non-trivial blue component in H and hence, X is 2ϵ -non-trivial in H , since $\gamma - \gamma/4 > 2\epsilon$ by (22).

Thus the subgraphs $[\overline{Y \cap Z}, \underline{X}]$ and $[\underline{Y \cap Z}, \overline{X}]$ are connected in green by Lemma 3.3. But they cannot belong to the same green component, since otherwise H is $(1 - \gamma/4)$ -spanned by the union of said green component and Z , which is not possible by Claim 3.12. Consequently, X has no green edges. By Claim 3.11 there is a green γ -non-trivial component $G \subseteq Y \cup Z$. As $H = Z \cup (Y - Z) \cup X$ and $Y - Z$ is $(\gamma/4)$ -empty, we obtain that $G \cap Z$ is $(3\gamma/4)$ -non-trivial in H and $G - Z \subseteq Y - Z$ is $(\gamma/4)$ -empty. Thus G has the same properties as Y with respect to Z and we can repeat the same arguments as above to obtain that all edges in X are blue. Hence X is connected in blue by Lemma 3.3. But this is a contradiction to Claim 3.12, as X and Z together $(1 - \gamma/4)$ -span H . \square

Claim 3.14. *There is a colour that has exactly three δ -non-trivial components.*

Proof. We show that there is a colour with at most three δ -non-trivial components. This together with Claim 3.11 yields the desired result. So suppose otherwise. Then each colour has at least four δ -non-trivial components. By Lemma 3.10, there are components X, Y and Z that together $(1 - \epsilon^{1/6})$ -span H .

By assumption, and as $\delta > \epsilon^{1/6}$ by (22), not all of X, Y and Z have the same colour. If two of these components, say X and Y , have the same colour, say red, then $H - (X \cup Y)$ contains a red component that is δ -non-trivial in H , by the assumption that our claim is false. As $\delta \geq \epsilon^{1/6} + 2\epsilon$ by (22), we have that the intersection of this red component with Z is 2ϵ -non-trivial in H . Hence we get a contradiction to Claim 3.13 as $\gamma/4 > \epsilon^{1/6}$ by (22).

So assume X is red, Y is blue and Z is green. We claim that (after possibly swapping top and bottom parts)

$$\underline{(Y \cap Z) - X} \text{ has less than } \epsilon n \text{ vertices.} \quad (23)$$

Indeed, otherwise $(Y \cap Z) - X$ is ϵ -non-trivial. Then, as $[\underline{X}, \overline{(Y \cap Z) - X}]$ is ϵ -non-trivial and its edges are green and blue, we get $\underline{X} \subseteq Y \cup Z$ since every vertex in \underline{X} sees a vertex in $\overline{Y \cap Z}$. In the same way we obtain $\overline{X} \subseteq Y \cup Z$. Thus $Z \cup Y$ is $(1 - \epsilon^{1/6})$ -non-trivial, which is not possible by Claim 3.12. This proves (23).

By assumption, $H - X$ contains three δ -non-trivial red components R_1, R_2 and R_3 , say. For $i \neq j$, $[\overline{R_i \cap (Y - Z)}, \overline{R_j \cap (Z - Y)}]$ has no red, blue or green edges and thus cannot be ϵ -non-trivial. So for at most one $i \in \{1, 2, 3\}$ the subgraph $R_i \cap [\overline{Y - Z}, \underline{Z - Y}]$ is ϵ -non-trivial. The same holds for

$[R_i \cap (Y - Z), \overline{R_j \cap (Z - Y)}]$. Consequently, and by the pigeonhole principle we can assume that

$$\text{none of } R_1 \cap [\overline{Y - Z}, Z - Y] \text{ and } R_1 \cap [Y - Z, \overline{Z - Y}] \text{ is } \epsilon\text{-non-trivial.} \quad (24)$$

By (24) and as R_1 is δ -non-trivial, at least one of $R_1 \cap Z$, $R_1 \cap Y$ is 3ϵ -non-trivial. We will assume the former. Thus, by (23) $\overline{R_1 \cap (Y - Z)}$ has a size of at least $2\epsilon n$. Hence, by (24) we get:

$$|\overline{R_1 \cap Z - Y}| < \epsilon n. \quad (25)$$

Moreover, Claim 3.13 (applied to R_1 and Y implies that R_1 has at least $\gamma n/4 - \epsilon^{1/6}n > 2\epsilon n$ vertices in $\overline{Z - Y}$ or $\underline{Z - Y}$. By (25) we have the latter case and hence

$$\underline{R_1 \cap (Z - Y)} \text{ and } \underline{R_1 \cap (Y - Z)} \text{ each have a size of at least } 2\epsilon n. \quad (26)$$

The fact that $[\overline{Y - (X \cup Z)}, R_1 \cap (Z - Y)]$ and $[\overline{Z - (X \cup Y)}, R_1 \cap (Y - Z)]$ only have red edges, together with (24) and (26), yields that

$$\overline{Y - (X \cup Z)} \text{ and } \overline{Z - (X \cup Y)} \text{ each have less than } \epsilon n \text{ vertices.} \quad (27)$$

Now by (27) (and by the existence of R_1, R_2, R_3), we know that $\overline{(Y \cap Z) - X}$ has at least ϵn vertices. So each vertex of \underline{X} has a neighbour in $(Y \cap Z) - \underline{X}$ and hence $\underline{X} \subseteq \underline{Y \cup Z}$. Since, by Claim 3.12, H is not $(1 - \epsilon^{1/6} - 2\epsilon)$ -spanned by $Y \cup Z$, we have that $\overline{X - (Y \cup Z)}$ has a size of at least $2\epsilon n$. This and (26) imply that $[X - (Y \cup Z), \underline{Y - (X \cup Z)}]$ and $[X - (Y \cup Z), \underline{Z - (X \cup Y)}]$ are 2ϵ -non-trivial each. As the edges of these subgraphs are green and blue respectively and as Lemma 3.3 applies, there are green and blue components G and B such that $\overline{H - X - [(G \cap Y) \cup (B \cap Z)]}$ has a size of less than $\epsilon n + \epsilon^{1/6}n$ by (23).

Now let G' be another δ -non-trivial green component. Then $\overline{G' - X}$ has at most $\epsilon^{1/6}n$ vertices, while $\underline{G' \cap X}$ has at least $2\epsilon n$ vertices. By (27) it follows that $\overline{G' - X}$ has at most $\epsilon n + \epsilon^{1/3}n$ vertices, while $\overline{G' \cap X}$ has at least $2\epsilon n$ vertices. This is not possible by Claim 3.13 and completes the proof. \square

Using Claim 3.14 we assume from now on that without loss of generality the colour red has exactly three δ -non-trivial components R_1, R_2 and R_3 . For $i = 1, 2, 3$ let M_i be a red matching of maximum size in R_i .

None of the red edges in $Y := H - M_1 - M_2 - M_3$ is in a red δ -non-trivial component. As seen in the proof of Claim 3.11, the number of red edges which are not in δ -non-trivial red components sums up to at most $2\delta n^2$. Therefore the number of red edges in Y is at most $2\delta n^2$. Let Y' be the subgraph of Y where these edges have been deleted. Note that the edges of Y' are coloured in blue and green. Moreover, H is still $(1 - 3\delta)$ -dense after the removal of the red edges of Y .

If Y' is not $(3\delta)^{1/3}$ -non-trivial, then we are done as Y' is balanced. Otherwise Y' is $(1 - (3\delta)^{1/3})$ -dense by Lemma 3.4 and thus contains a $(1 - (3\delta)^{1/6})$ -spanning subgraph Y'' of Y with $(1 - 2(3\delta)^{1/6})$ -complete degree, by Lemma 3.2. By removing at most $(3\delta)^{1/6}n$ vertices from Y'' we can assure that Y'' is balanced. If Y'' can be $(1 - 10(3\delta)^{1/6})$ -spanned by two disjoint monochromatic connected matchings, we are done, since in that case, we found five matchings which together $(1 - 11(3\delta)^{1/6})$ -span H . Otherwise, as the edges of Y'' are green and blue the colouring of Y'' is $4(3\delta)^{1/6}$ -split in Y'' , by Lemma 3.9. We denote

its blue and green components by B'_1, B'_2 , respectively G'_1, G'_2 , with $\overline{B'_1} = \overline{G'_1}$, $\overline{B'_2} = \overline{G'_2}$, $\underline{B'_1} = \underline{G'_2}$, and $\underline{B'_2} = \underline{G'_1}$.

Since Y'' is $(1 - (3\delta)^{1/6})$ -spanning in Y' it is also $(1 - (3\delta)^{1/6})$ -spanning in Y . Therefore the subgraph

$$B'_1 \cup B'_2 \cup M_1 \cup M_2 \cup M_3 \text{ is } (1 - (3\delta)^{1/6})\text{-non-trivial in } H. \quad (28)$$

By Lemma 3.9, Y'' can be $(1 - 4(3\delta)^{1/6})$ -spanned by two blue matchings $M_4 \subseteq B'_1, M_5 \subseteq B'_2$ and an additional green matching. If any of the matchings M_i has less than γn edges, we can ignore it and still have a sufficiently large cover of H . Thus we get that

$$B'_1, B'_2, G'_1, G'_2, M_1, M_2, \text{ and } M_3 \text{ are } \gamma\text{-non-trivial in } H. \quad (29)$$

Moreover, let B_1 and B_2 be the blue components in H that contain B'_1 and B'_2 , respectively. We define G_1 and G_2 analogously. If $B_1 = B_2$, we are done as $M_4 \cup M_5$ is a connected matching. This and symmetry implies

$$B_1 \neq B_2 \text{ and } G_1 \neq G_2. \quad (30)$$

Claim 3.15. *For each $i = 1, 2, 3$ we have that*

- (a)
 - if $|\overline{M_i} \setminus \overline{G_1 \cup G_2}| > 6\epsilon n$, then $\underline{B'_1} \subseteq \underline{R_i}$ or $\underline{B'_2} \subseteq \underline{R_i}$;
 - if $|\overline{M_i} \setminus \overline{B_1 \cup B_2}| > 6\epsilon n$, then $\underline{G'_1} \subseteq \underline{R_i}$ or $\underline{G'_2} \subseteq \underline{R_i}$;
- (b)
 - if $|\overline{M_i} \setminus \overline{G_1 \cup G_2}| > 6\epsilon n$, then $\overline{B'_1} \subseteq \overline{R_i}$ or $\overline{B'_2} \subseteq \overline{R_i}$;
 - if $|\overline{M_i} \setminus \overline{B_1 \cup B_2}| > 6\epsilon n$, then $\overline{G'_1} \subseteq \overline{R_i}$ or $\overline{G'_2} \subseteq \overline{R_i}$;
- (c)
 - if $|\overline{M_i} \setminus \overline{G_1 \cup G_2 \cup B_1 \cup B_2}| > 2\epsilon n$, then $\underline{B'_1 \cup B'_2} = \underline{G'_1 \cup G'_2} \subseteq \underline{R_i}$;
 - if $|\overline{M_i} \setminus \overline{G_1 \cup G_2 \cup B_1 \cup B_2}| > 2\epsilon n$, then $\overline{B'_1 \cup B'_2} = \overline{G'_1 \cup G'_2} \subseteq \overline{R_i}$.

Proof. For the first part of (a), assume $|\overline{M_1} \setminus \overline{G_1 \cup G_2}| > 6\epsilon n$. Note that there is no green edge between $\overline{M_1} \setminus \overline{G_1 \cup G_2}$ and $\underline{G'_1}$. First assume that $\overline{M_1 \cap B_1} \setminus \overline{G_1 \cup G_2}$ has a size of at least $2\epsilon n$. Then, by (30), any edge between $\overline{M_1 \cap B_1} \setminus \overline{G_1 \cup G_2}$ and $\underline{B'_2} = \underline{G'_1}$ is red. So, by Lemma 3.3 and (29) the result follows. So we can assume that this is not true. Similarly, the result holds if $|\overline{M_1 \cap B_2} \setminus \overline{G_1 \cup G_2}| \geq 2\epsilon n$. Therefore, we can assume that $\overline{M_1} \setminus \overline{B_1 \cup B_2 \cup G_1 \cup G_2}$ has a size of at least $2\epsilon n$. In this case, since all edges between $\overline{M_1} \setminus \overline{B_1 \cup B_2 \cup G_1 \cup G_2}$ and $\underline{B'_1}$ are red, the result follows again by Lemma 3.3 and (29). Item (b) and the second part of (a) follow similarly.

For the first part of (c), note that any edge between $\overline{M_i} \setminus \overline{G_1 \cup G_2 \cup B_1 \cup B_2}$ and $\underline{B'_1 \cup B'_2} = \underline{G'_1 \cup G'_2}$ has to be red and use Lemma 3.3 with (29). The second part of (c) is analogous. \square

By Claim 3.11 there are green and blue γ -non-trivial components $G_3 \neq G_1, G_2$ and $B_3 \neq B_1, B_2$ in H .

Claim 3.16. *It holds that $|V(G_3 \cap B_3 \cap (M_1 \cup M_2 \cup M_3))| > 36\epsilon n$.*

Proof. Assume otherwise. That is, assume

$$|V(G_3 \cap B_3 \cap (M_1 \cup M_2 \cup M_3))| \leq 36\epsilon n.$$

The components B_3 and G_3 do not meet with $B'_1 \cup B'_2 = G'_1 \cup G'_2$ and by (28), there are not more than $2(3\delta)^{1/6}n$ vertices outside of $B'_1 \cup B'_2 \cup M_1 \cup M_2 \cup M_3$. As $\gamma > 2(3\delta)^{1/6} + \delta$ by (22), we conclude that $B_3 \cap (M_1 \cup M_2 \cup M_3)$ and $G_3 \cap (M_1 \cup M_2 \cup M_3)$ are each δ -non-trivial. Hence there are indices i, i', j, j' such that there is a blue 37ϵ -non-trivial subgraph $B'_3 \subseteq B_3$ and a green 37ϵ -non-trivial subgraph $G'_3 \subseteq G_3$ such that $\overline{B'_3} \subseteq \overline{M_i}$ and $\overline{B'_3} \subseteq \overline{M_{i'}}$, and $\overline{G'_3} \subseteq \overline{M_j}$ and $\overline{G'_3} \subseteq \overline{M_{j'}}$. Actually, we can choose these indices such that $i \neq i'$ and $j \neq j'$. Since if $i = i'$, say, Claim 3.13 yields that $(B_3 \cap H) \setminus M_i$ is not $\gamma/4$ -empty and therefore, by (22) and (28), there is some index $k \neq i$ such that $B_3 \cap M_k$ is not 37ϵ -empty, which allows us to swap i' for k .

For an index $k \neq i$, the edges between $\overline{B'_3 \cap M_i}$ and $G'_3 \cap M_k$ are blue and green. As by our initial assumption $|V(G_3 \cap B_3 \cap (M_1 \cup M_2 \cup M_3))| \leq 36\epsilon n$, this implies that $|G_3 \cap M_k| \leq 36\epsilon n$. In the same way we obtain that $|G_3 \cap M_k| \leq 36\epsilon n$ for $k \neq i'$ or $|\overline{B'_3 \cap M_i}| \leq 36\epsilon n$, but the latter cannot happen by the choice of B'_3 . Hence we have $i = j'$ and $i' = j$; in other words,

$$|M_i \cap G_3| \geq 37\epsilon n, \quad |\overline{M_j \cap G_3}| \geq 37\epsilon n, \quad |\overline{M_i \cap B_3}| \geq 37\epsilon n \text{ and } |M_j \cap B_3| \geq 37\epsilon n.$$

So by Claim 3.15 (a) and (b), either we have $B'_1 \subseteq R_i$ and $B'_2 \subseteq R_j$, or we have $G'_1 \subseteq R_i$ and $G'_2 \subseteq R_j$. Indeed, the fact that $|M_i \cap G_3| \geq 37\epsilon n$ together with Claim 3.15 (b) implies that $\overline{B'_1} = \overline{G'_1} \subseteq \overline{R_i}$ or $\overline{B'_2} = \overline{G'_2} \subseteq \overline{R_i}$. Without loss of generality, we assume the latter. Next, since $|M_i \cap B_3| \geq 37\epsilon n$, and by Claim 3.15 (a), we get that $\overline{G'_1} = \overline{B'_2} \subseteq \overline{R_i}$ or $\overline{G'_2} = \overline{B'_1} \subseteq \overline{R_i}$. Without loss of generality, we assume the former. We repeat the same with index j , but as we already have $B'_2 \subseteq R_i$, the output of Claim 3.15 has to be $\overline{B'_1} = \overline{G'_2} \subseteq \overline{R_j}$ for $|\overline{M_j \cap G_3}| \geq 37\epsilon n$ and $\overline{B'_1} = \overline{G'_1} \subseteq \overline{R_j}$ for $|M_j \cap B_3| \geq 37\epsilon n$. For the remainder of the proof, let us assume that $B'_1 \subseteq R_i$ and $B'_2 \subseteq R_j$. Then $G'_1 \cap R_k = \emptyset = G'_2 \cap R_k$, where k is the third index, which together with Claim 3.15 (a) and (b) gives that $R_k \cap (G_3 \cup B_3)$ is 6ϵ -empty. The edges between $\overline{B'_2} = \overline{G'_1} \subseteq \overline{G_1 \cap R_j}$ and $\overline{B'_3 \cap R_i}$ have to be green, which implies that $\overline{B'_3} \subseteq \overline{G_1}$. As any edge between $\overline{B'_3}$ and $\overline{R_k - B_3}$ has to be green we deduce that $|\overline{R_k \cap G_1}| \geq 2\epsilon n$ since R_k is γ -non-trivial and $|\overline{R_k \cap B_3}| \leq 6\epsilon n$. This also implies that $|\overline{R_k - G_1}| \leq 6\epsilon n$.

By repeating the same argument with $\overline{B'_1} = \overline{G'_1} \subseteq \overline{G_1}$ and $\overline{B'_3}$, it follows that $|\overline{R_k \cap G_1}| \geq 2\epsilon n$ and $|\overline{R_k - G_1}| \leq 6\epsilon n$. So $R_k \cap G_1$ is 2ϵ -non-trivial and $R_k - G_1$ is 6ϵ -empty, a contradiction to Claim 3.13. \square

Claim 3.16 allows us assume that without loss of generality

$$|\overline{M_3 \cap G_3 \cap B_3}| > 6\epsilon n. \quad (31)$$

This implies $|\overline{M_3} \setminus \overline{G_1 \cup G_2 \cup B_1 \cup B_2}| > 2\epsilon$ and thus by Claim 3.15(c) with $i = 3$ we obtain

$$\overline{B'_1 \cup B'_2} = \overline{G'_1 \cup G'_2} \subseteq \overline{R_3}. \quad (32)$$

This implies that $(R_1 \cup R_2) \cap (G'_1 \cup G'_2) = \emptyset$. Since the edges between $\overline{M_3 \cap G_3 \cap B_3}$ and $\overline{R_1 \cup R_2}$ are coloured green and blue, we have by (31) and Lemma 3.3 that

$$\overline{M_1 \cup M_2} \subseteq \overline{R_1 \cup R_2} \subseteq \overline{G_3 \cup B_3}. \quad (33)$$

So, by (29) and Claim 3.15(b) with $i = 1$, we can assume that without loss of generality

$$\overline{B'_1} = \overline{G'_1} \subseteq \overline{R_1}, \quad (34)$$

and hence, by (29) and Claim 3.15(b) with $i = 2$ it follows that

$$\overline{B'_2} = \overline{G'_2} \subseteq \overline{R_2}. \quad (35)$$

The last two assertions imply that $\overline{R_3} \cap \overline{G'_1 \cup G'_2} = \emptyset$. Suppose that there is an $x \in \overline{R_1 \cup R_2} \setminus \overline{G_1 \cup G_2 \cup B_1 \cup B_2}$. By (32), the edges between x and $\overline{G'_1 \cup G'_2} = \overline{B'_1 \cup B'_2}$ are not red, and neither green or blue by choice of x . As G'_1 and G'_2 are both γ -non-trivial in H by (29) and H has $(1 - \epsilon)$ -complete degree, we obtain a contradiction. Hence

$$\overline{M_1 \cup M_2} \setminus \overline{G_1 \cup G_2 \cup B_1 \cup B_2} = \emptyset. \quad (36)$$

In the same fashion, suppose there is an $x \in (M_3 \setminus \overline{G_1 \cup G_2}) \cup (M_3 \setminus \overline{B_1 \cup B_2})$. By (34) and (35), and by the choice of x , the edges between x and $\overline{B'_1} = \overline{G'_1}$ respectively $\overline{B'_2} = \overline{G'_2}$ are neither green nor blue. Again, using (29) and the $(1 - \epsilon)$ -completeness of H , we obtain

$$\overline{M_3} \setminus \overline{G_1 \cup G_2} = \overline{M_3} \setminus \overline{B_1 \cup B_2} = \emptyset. \quad (37)$$

Finally, suppose there is an $x \in \overline{B_3 \cup G_3} \cap \overline{M_1 \cup M_2}$. By (29) and the $(1 - \epsilon)$ -completeness of H , x sees vertices in $\overline{M_3}$. This, however, contradicts (37) and thus

$$\overline{B_3 \cup G_3} \cap \overline{M_1 \cup M_2} = \emptyset. \quad (38)$$

Now let us turn to back the graph H , for reasons that will become clear below. Assume that H has a red edge vw outside of $M_1 \cup M_2 \cup M_3$. By maximality of the matchings M_i , vw is not part of R_1 , R_2 or R_3 . By (29), (34) and (35) we have $\overline{vw} \in G_1 \cap B_2$ or $\overline{vw} \in G_2 \cap B_1$. However, both cases contradict (31). This yields

$$V(H) = V(B'_1) \cup V(B'_2) \cup V(M_1) \cup V(M_2) \cup V(M_3). \quad (39)$$

Next, we restore the symmetry between the colours.

Claim 3.17. *Each colour has exactly three components.*

Proof. By (39) there are no red edges in $Y = H - V(M_1 \cup M_2 \cup M_3)$ and hence $Y = Y' = Y''$. By (32), (34) and (35) R_1 , R_2 and R_3 are the only red components in H .

Suppose there is a (possibly trivial) green component G_4 distinct from G_1 , G_2 and G_3 . Assume first that $\overline{G_4} \neq \emptyset$. Note that any edge between $\overline{G_4}$ and $\overline{G'_1 \cup G'_2}$ is red or blue. By (30), no vertex of $\overline{G_4}$ can send blue edges to both $\overline{G'_1}$ and $\overline{G'_2}$. Moreover, by (34) and (35), no vertex of $\overline{G_4}$ can send red edges to both $\overline{G'_1}$ and $\overline{G'_2}$. Since H has $(1 - \epsilon)$ -complete degree and $\overline{G'_1} = \overline{B'_1}$ and $\overline{G'_2} = \overline{B'_2}$

are γ -non-trivial, we derive $\underline{G}_4 \subseteq \underline{R}_1 \cup \underline{R}_2 \cap \underline{B}_1 \cup \underline{B}_2$. But this contradicts (31), because H is $(1 - \epsilon)$ -complete.

Now let us assume that $\underline{G}_4 = \emptyset$, and so, $\overline{G}_4 \neq \emptyset$. In other words, G_4 consists of a single vertex with no incident green edges. Suppose that $\overline{G}_4 \cap \overline{M}_3 = \emptyset$. So by (29) and (32), the edges between \overline{G}_4 and $\underline{G}'_1 \cup \underline{G}'_2$ are blue, which contradicts that \underline{B}'_1 and \underline{B}'_2 lie in distinct blue components, as asserted by (30). Therefore $\overline{G}_4 \subseteq \overline{M}_3$. So as $\underline{G}_4 = \emptyset$, all edges between \overline{G}_4 and $\underline{M}_1 \cup \underline{M}_2$ are blue. By (37), (38) and (39), $B_3 \subseteq \overline{[M_1 \cup M_2, M_3]}$. Since H is $(1 - \epsilon)$ -complete and B_3 is γ -non-trivial, we obtain that $\overline{G}_4 \subseteq \overline{B}_3$. We also have that $G_3 \subseteq \overline{[M_1 \cup M_2, M_3]}$ by (37), (38) and (39). Since G_3 is γ -non-trivial it follows that, $\overline{G_3 \cap M_1 \cup M_2}$ has a size of at least γn . Since the edges between \overline{G}_4 and \underline{G}_3 are blue, we obtain that $\underline{M}_1 \cup \underline{M}_2 \cap \underline{G}_3 \cap \underline{B}_3 \neq \emptyset$. But this represents a contradiction to (34) or (35), since there is no colour left for the edges between $\underline{G}_3 \cap \underline{B}_3$ and $\underline{B}'_1 \cup \underline{B}'_2$. Since a fourth blue component would behave the same way as G_4 , this finishes the proof of the claim. \square

By (32) and (39) it follows that $\underline{R}_i = \underline{M}_i$ for $i = 1, 2$. In the same way (34), (35) and (39) imply that

$$\overline{R}_3 = \overline{M}_3. \quad (40)$$

For $1 \leq i, j, k \leq 3$ we denote $\overline{i|j|k} := \overline{R_i \cap G_j \cap B_k}$ and $\underline{i|j|k} := \underline{R_i \cap G_j \cap B_k}$. From (29), (31), (34) and (35) we obtain that

$$|\overline{1|1|1}|, |\overline{2|2|2}|, |\overline{3|3|3}| > 6\epsilon n. \quad (41)$$

Note that by definition and $(1 - \epsilon)$ -completeness it follows that for all i, i', j, j', k, k' with $i \neq i', j \neq j'$ and $k \neq k'$ we have (modulo switching biparts)

$$\text{if } |\overline{i|j|k}| \geq \epsilon n, \text{ then } |\underline{i'|j'|k'}| = 0. \quad (42)$$

Let us show that $\underline{i|j|k} = \emptyset$, unless i, j, k are pairwise different. Indeed, otherwise, if say $\underline{1|1|k} \neq \emptyset$ for $k = 1, 2$ or 3 , we obtain a contradiction to (42) as $|\overline{2|2|2}|, |\overline{3|3|3}| \geq 6\epsilon n$ by (41). Then the edges of the graph $\overline{[1|1|k, 2|2|2 \cup 3|3|3]}$ are all blue as H has $(1 - \epsilon)$ -complete degree, implying that $2 = k = 3$, a contradiction. Hence \underline{H} can be decomposed into sets $\underline{i|j|k}$, where $1 \leq i, j, k \leq 3$ are pairwise different. So we have:

$$\underline{1|3|2} \cup \underline{1|2|3} \cup \underline{2|3|1} \cup \underline{2|1|3} \cup \underline{3|2|1} \cup \underline{3|1|2} = \underline{H}. \quad (43)$$

Claim 3.18. *We have $\overline{H} = \overline{1|1|1} \cup \overline{2|2|2} \cup \overline{3|3|3} \cup \overline{3|1|2} \cup \overline{3|2|1}$.*

Proof. First, we show there is no $\overline{i|j|k} \neq \emptyset$ such that exactly two of i, j, k are equal. If $\overline{3|1|1} \neq \emptyset$, say, then $|\underline{1|2|3}|, |\underline{1|3|2}| \leq \epsilon n$ by (42). Together with (43), this implies that \underline{R}_1 is not γ -non-trivial, a contradiction. Second, note that (32) implies that $\overline{3|1|2}$ and $\overline{3|2|1}$ have each a size of at least γn . Again, by (42), it follows that $\underline{i|j|k} = \emptyset$, if $i \neq 3$ and $3 \in \{j, k\}$. This proves the claim. \square

Claim 3.19. *We have $\overline{H} = \overline{1|1|1} \cup \overline{2|2|2} \cup \overline{3|3|3}$.*

Proof. By the previous claim it remains to show that $\overline{3|1|2} = \overline{3|2|1} = \emptyset$. To this end, suppose that $\overline{3|1|2} \neq \emptyset$ and thus $|1|2|3|, |2|3|1| \leq \epsilon n$ by (42). If $\overline{3|2|1} \neq \emptyset$ as well, then by (42) also $|1|3|2| \leq \epsilon n$ which, by Claim 3.18 and (43) gives the contradiction that $R_1 \subseteq \overline{1|1|1|}, \overline{1|2|3|} \cup \overline{1|3|2|}$ is not γ -non-trivial. So we have

$$\overline{H} = \overline{1|1|1|} \cup \overline{2|2|2|} \cup \overline{3|3|3|} \cup \overline{3|1|2|},$$

with $\overline{3|1|2} \neq \emptyset$. This partition is shown in Figure 3.

Ignoring from now on the matchings M_1 and M_2 , we aim at covering H with M_3 and four other matchings. To this end take a green matching M_1^{green} of maximum size in $G_1 - M_3$ and next a blue matching M_2^{blue} of maximum size in $B_2 - M_3 - M_1^{\text{green}}$. Denote

- $\overline{i|j|k}' := \overline{i|j|k} \setminus \overline{M_3 \cup M_1^{\text{green}} \cup M_2^{\text{blue}}}$ and
- $\underline{i|j|k}' := \underline{i|j|k} \setminus \underline{M_3 \cup M_1^{\text{green}} \cup M_2^{\text{blue}}}$.

We can assume that $M_3 \cup M_1^{\text{green}} \cup M_2^{\text{blue}}$ is not $(1 - \epsilon)$ -spanning. Thus, as H has $(1 - \epsilon)$ -complete degree, the maximality of the matchings M_3, M_1^{green} and M_2^{blue} implies that $\overline{3|1|2}', \overline{3|1|2}' = \emptyset$.

Moreover it follows that

- $|\overline{1|1|1}'| \leq \epsilon n$ or $|\underline{2|1|3}'| \leq \epsilon n$ by maximality of $M_1^{\text{green}} \subseteq G_1$,
- $|\overline{2|2|2}'| \leq \epsilon n$ or $|\underline{1|3|2}'| \leq \epsilon n$ by maximality of $M_2^{\text{blue}} \subseteq B_2$,
- $\overline{3|3|3}' = \emptyset$ as $\overline{R_3} = \overline{M_3}$ by (40).

If $|\overline{1|1|1}'|, |\underline{2|2|2}'| \leq \epsilon n$, then we have found three disjoint connected matchings that $(1 - 2\epsilon)$ -span H , contradicting our assumption. If $|\underline{2|1|3}'|, |\underline{1|3|2}'| \leq \epsilon n$, we take a green matching in G_2 and a blue maximum matching in B_1 , among the yet unmatched vertices. After this step, there are at most ϵn vertices of $\overline{3|2|1}'$ left uncovered and therefore all but at most $3\epsilon n$ vertices of \overline{H} are covered. Thus, as H is balanced, we have found five disjoint monochromatic connected matchings which together $(1 - 3\epsilon)$ -span H . So, either $|\overline{2|2|2}'|, |\underline{2|1|3}'| \leq \epsilon n$, or $|\overline{1|1|1}'|, |\underline{1|3|2}'| \leq \epsilon n$. In either case we can find two disjoint monochromatic connected matchings that cover all but at most $2\epsilon n$ vertices of the two other sets from the previous sentence and all but at most $2\epsilon n$ vertices of $\overline{3|2|1}'$. So we have five disjoint monochromatic connected matchings $(1 - 4\epsilon)$ -spanning H , a contradiction. \square

For ease of notation we set

$$X := \overline{1|1|1|}, Y := \overline{2|2|2|}, Z := \overline{3|3|3|} \text{ and}$$

$$A := \underline{1|3|2|}, B := \underline{1|2|3|}, C := \underline{2|3|1|}, D := \underline{2|1|3|}, E := \underline{3|2|1|}, F := \underline{3|1|2|}.$$

By Claim 3.19 and (43) we have $|\overline{H}| = X + Y + Z$ and $|\underline{H}| = A + B + C + D + E + F$. Note that the edges between any upper and lower part are monochromatic (see Figure 4). Also note that we reached complete symmetry between the colours

and the indices of the components, so we will from now on again treat them as interchangeable.

Observe that for (at least) one index $i \in \{1, 2, 3\}$ it holds that $|\overline{R_i}| \leq |R_i|$. We shall call such an index i a *weak* index for the colour red. If furthermore $|\overline{R_i}| < |R_i \cap B_j| = |R_i \cap G_k|$ and $|\overline{R_i}| < |R_i \cap B_k| = |R_i \cap G_j|$, where j, k are the other two indices from $\{1, 2, 3\}$, then we call i *very weak* for colour red. Analogously define (*very*) *weak* indices for colours blue and red.

Claim 3.20. *If index i is weak for colour c , then*

- (a) *the indices in $\{1, 2, 3\} - \{i\}$ are not weak for colour c , and*
- (b) *index i is very weak for colour c .*

Proof. Let us show this for $i = 2$ and colour red (the other cases are analogous). By assumption, $Y \leq C + D$. Since $X < A + B$ and $Z < E + F$ cannot both hold, we can assume without loss of generality that $Z \geq E + F$. Now if $X \leq A + B$, then we pick maximal red matchings in $\overline{1|1|1}, \overline{1|3|2} \cup \overline{1|2|3}$, $\overline{2|2|2}, \overline{2|3|1} \cup \overline{2|1|3}$ and $\overline{3|2|1} \cup \overline{3|1|2}, \overline{3|3|3}$, thus covering all but at most $3\epsilon n$ vertices of $\overline{1|1|1} \cup \overline{2|2|2} \cup \overline{3|2|1} \cup \overline{3|1|2}$. To finish we cover all but $4\epsilon n$ of the remaining vertices in $\overline{3|3|3} \cup (H \setminus R_3)$ with a blue and a green matching, a contradiction. Hence $X > A + B$. Using this fact, $Z > E + F$ follows by symmetry. This proves (a).

In order to show (b), let us first prove that $Y < C$. We pick a maximal red matching in each of R_1 and R_3 , thus covering all but at most $2\epsilon n$ vertices of $\overline{R_1} \cup \overline{R_3}$. Now if $Y \geq C$, then all but at most ϵn vertices of $\overline{2|3|1}$ are contained in a maximal red matching that also contains all but at most ϵn vertices of $\overline{2|2|2}$. We cover all but $4\epsilon n$ of the remaining vertices in $\overline{R_1} \cup \overline{R_3}$ with a blue and a green matching, a contradiction. The fact that $Y < D$ follows analogously. \square

Suppose two of the three indices 1, 2, 3 are weak for different colours, say 1 is weak for red and 2 is weak for green. Then Claim 3.20(b) gives that $X < A$ and $Y < E$. Thus we can match all but at most ϵn vertices of $\overline{1|1|1}$ into $\overline{1|3|2}$ and all but at most ϵn vertices of $\overline{2|2|2}$ into $\overline{3|2|1}$ with two matchings, one red and one green, and cover all but $6\epsilon n$ of the remaining vertices with three disjoint matchings, one from each of R_3, G_3, B_3 , a contradiction.

Hence, since each colour has a weak index, there is an index i that is weak for all three colours, $i = 2$ say. We match all but at most ϵn vertices of $\overline{2|2|2}$ into $\overline{3|1|2}$ with a blue matching M . Further choose a subset $F \subseteq \overline{3|1|2} \setminus V(M)$ of size $|\overline{2|2|2}| - |V(M)/2| \leq \epsilon n$, and let us from now work with the remaining set $\overline{3|1|2}' = \overline{3|1|2} \setminus (V(M) \cup F)$ of cardinality $F' = F - Y$. Set $n' = n - Y$. (So instead of five we will have to find four monochromatic connected matchings covering all but few vertices of $H - M$.) Without loss of generality assume $Z \geq X$. Claim 3.20(a) gives that

$$X > A + B, C + E, D + F' \text{ and } Z > A + C, B + D, E + F'. \quad (44)$$

Hence $X > n'/3$. So, one of the three sums $A+C, B+D, E+F'$ has to be strictly smaller than X , say $A+C < X$. Consequently, $Z = n' - X < B + D + E + F'$.

If $Z \geq D + E + F'$, then we cover all but at most ϵn vertices of $\overline{R_3} - M$ with a red matching, and cover all but at most ϵn vertices of the remains of

$\overline{3|3|3}$ with a blue matching that also covers all but at most ϵn vertices of $\overline{2|1|3}$. Now all that is left on the top is $\overline{1|1|1}$, which we can match with a red and a blue matching into the remains of $\overline{1|3|2} \cup \overline{1|2|3} \cup \overline{2|3|1}$ (except for ϵn vertices). Thus we found four connected matchings that cover all but at most γn vertices of $H - V(M)$, and are done.

So we may assume $Z < D + E + F'$ and thus $X > A + B + C$. If $X \leq A + B + C + E$, then we can proceed similarly as in the previous paragraph to find four matchings covering all vertices of H . Hence $X > A + B + C + E$, implying that $Z < D + F'$. But by (44) we have $D + F' < X$ a contradiction to our assumption that $X \leq Z$. This proves Lemma 3.1.

4 From connected matchings to cycles

In this section we prove Theorem 1.1(a). We basically follow the approach of Luczak [20], which has become a standard method in this field. Therefore we present only an outline of the proof, omitting most of the tedious details that have been discussed in earlier works in more general contexts. We refer the interested reader to [3, 5, 9, 12, 13, 21].

For a graph G the bipartite subgraph $H = [A, B] \subseteq G$ is (ϵ, G) -regular if

$$X \subseteq A, Y \subseteq B, |X| > \epsilon|A|, |Y| > \epsilon|B| \text{ imply } |d_G(X, Y) - d_G(A, B)| < \epsilon.$$

A vertex-partition $\{V_0, V_1, \dots, V_l\}$ of $l + 1$ clusters of a graph G is called (ϵ, G) -regular, if

- (a) $|V_1| = |V_2| = \dots = |V_l|$;
- (b) $|V_0| < \epsilon n$;
- (c) apart from at most $\epsilon \binom{l}{2}$ exceptional pairs, the graphs $[V_i, V_j]$ are (ϵ, G) -regular.

Lemma 4.1 (Regularity Lemma with prepartition and colours). *For every $\epsilon > 0$ and positive integers $m, r, s \in \mathbb{N}$ there are $m \leq M \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ the following holds. For any set of mutually edge-disjoint graphs G_1, G_2, \dots, G_r with $V(G_1) = V(G_2) = \dots = V(G_r) = V$, with $|V| = n$, and any partition $W_1 \cup \dots \cup W_s = V$, there is a partition $V_0 \cup V_1 \cup \dots \cup V_l$ of V into $l + 1$ clusters such that*

- (a) $m \leq l \leq M$;
- (b) for each $1 \leq i \leq l$ there is $1 \leq j \leq s$ such that $V_i \subseteq W_j$;
- (c) $V_0 \cup V_1 \cup \dots \cup V_l$ is (ϵ, G_i) -regular for each $1 \leq i \leq r$.

Let us now prove Theorem 1.1(a). Let $n \gg 0$ and $0 < \epsilon \ll 1$. Let the edges of $K_{n,n}$ with biparts W_1 and W_2 be coloured in red, green and blue. We denote by G_1, G_2 and G_3 the graphs induced by the edges of each of the colours.

For $m \gg 0$ and $\epsilon \ll d \ll 0$, Lemma 4.1 provides a vertex-partition V_0, V_1, \dots, V_l of $K_{n,n}$ satisfying Lemma 4.1(a)–(c) for some $M \geq m$. As usual, we define the (ϵ, d) -reduced graph Γ by identifying a new vertex v_i with each cluster V_i for $1 \leq i \leq l$. For $1 \leq i, j \leq l$ and $1 \leq q \leq 3$ we place an edge of colour q between

two vertices v_i, v_j if the subgraph $[V_i, V_j]$ of the respective clusters has edge-density at least d in G_q and is (ϵ, G_q) -regular. To get a simple graph, we keep an arbitrary edge from each multi-edge.

Since the clusters have the same size, we can, if necessary, remove some of them to obtain a balanced bipartite $(1 - 2\epsilon)$ -complete subgraph of Γ , which we will continue to call Γ . Therefore Lemma 3.1 can be used to cover all but at most $\rho|V(\Gamma)|$ vertices of Γ by five vertex-disjoint monochromatic connected matchings M_1, \dots, M_5 . We finish the proof by turning these five matchings into monochromatic cycles of $K_{n,n}$ using the following lemma from [3, 5, 9, 12, 13, 21].

Lemma 4.2. *Let $0 < \epsilon \ll \rho \ll d \leq 1$ and let Γ be the (ϵ, d) -reduced graph of G_1, G_2, \dots, G_r , obtained from Lemma 4.1. Assume that there is a set of disjoint monochromatic connected matchings \mathcal{M} in Γ . Let $U \subseteq V(G)$ be the set of vertices, which are in clusters associated to the vertices of $V(\mathcal{M})$. Then there are $|\mathcal{M}|$ monochromatic cycles in G partitioning all but $(1 - \rho)|U|$ vertices of U .*

5 Covering all vertices

5.1 Preliminaries

We call a balanced bipartite subgraph H of a $2n$ -vertex graph $(1 - \epsilon)$ -Hamiltonian, if any balanced bipartite subgraph of H with at least $2(1 - \epsilon)n$ vertices is Hamiltonian. The next lemma is a combination of results from [14, 22].

Lemma 5.1. *For any $1 > \gamma > 0$, there is an $n_0 \in \mathbb{N}$ such that any balanced bipartite graph on $2n \geq 2n_0$ vertices and of edge density at least γ has a $(1 - \gamma/4)$ -Hamiltonian subgraph of size at least $\gamma^{3024/\gamma}n/3$.*

We make no attempt to optimise the bounds in Lemma 5.1. For the proof, we need some definitions and tools. For a graph G , and disjoint $A, B \subseteq V(G)$ let $e(A, B)$ denote the number of edges in $[A, B]$. For $0 < \epsilon, \sigma < 1$, $[A, B]$ is called (ϵ, σ) -dense if $e(X, Y) \geq \sigma|A||B|$ for every $X \subseteq A, Y \subseteq B$ with $|X| \geq \epsilon|A|$ and $|Y| \geq \epsilon|B|$.

Theorem 5.2 (Peng et. al [22]). *Given a bipartite balanced graph of size $2n$ and edge density $0 < \gamma < 1$. Then for all $0 < \epsilon < 1$ there is an $(\epsilon, \gamma/2)$ -dense balanced subgraph on at least $\gamma^{12/\epsilon}n/2$ vertices.*

For $0 < \epsilon, \delta < 1$, we say that the balanced subgraph $H = [A, B]$ is (ϵ, δ) -uniform in G , if it has minimum degree at least $\delta|A|$, and any ϵ -non-trivial subgraph of H has at least one edge. The next result, due to Haxell, shows that sufficiently strong uniformity implies hamiltonicity.

Theorem 5.3 (Haxell [14]). *Let $\epsilon > 0$ be given, and suppose that $H = [A, B]$ is a bipartite graph with $|A| = |B| \geq \frac{1}{\epsilon}$ such that H is (ϵ, δ) -uniform for $\delta > 7\epsilon$. Then H is Hamiltonian.*

Proof of Lemma 5.1. Set $\epsilon := \gamma/253$ and $n_0 := 2\gamma^{-12/\epsilon}\epsilon^{-1}$. Let H be a balanced bipartite graph of density γ and size $2n \geq 2n_0$. Apply Theorem 5.2 to obtain a balanced $(\epsilon, \gamma/2)$ -dense subgraph $[A, B] \subseteq H$ of size at least $\gamma^{12/\epsilon}n/2$. Deleting at most $\epsilon|A|$ vertices on either side, we arrive at a $(2\epsilon, \gamma/3)$ -uniform subgraph $[X, Y] \subseteq [A, B]$ of size at least $\gamma^{12/\epsilon}n/3$.

In order to see that $[X, Y]$ is $(1-\gamma/4)$ -Hamiltonian, delete an arbitrary fraction of at most $\gamma/4 < 1/4$ vertices from each of X, Y . Clearly, the obtained subgraph $[X', Y']$ is $(3\epsilon, \frac{\gamma}{12})$ -uniform, and has size at least $\gamma^{12/\epsilon}n_0/4 \geq 1/(3\epsilon)$. Thus Theorem 5.3 applies and we are done. \square

Finally, we make use of the following lemma due to Gyárfás et al. It allows us to absorb small vertex sets with few monochromatic cycles.

Lemma 5.4 (Gyárfás et al. [10]). *There is a constant $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ and $m \leq \frac{n}{(8r)^{8(r+1)}}$, and for any r -colouring of $K_{n,m}$, there are $2r$ disjoint monochromatic cycles covering all m vertices on the smaller side.*

5.2 Proof of Theorem 1.1(b)

Let A and B be the two partition classes of the 3-edge-coloured $K_{n,n}$. We assume that $n \geq n_0$, where we specify n_0 later. Pick subsets $A_1 \subseteq A$ and $B_1 \subseteq B$ of size $\lceil n/2 \rceil$ each. Say red is the majority colour of $[A_1, B_1]$. Lemma 5.1 applied with $\gamma = 1/3$ yields a red $(1 - 1/12)$ -Hamiltonian subgraph $[A_2, B_2]$ of $[A_1, B_1]$ with

$$|A_2| = |B_2| \geq 3^{9999}|A_1| \geq 3^{-10^4}n.$$

Set $H := G - (A_2 \cup B_2)$, and note that each bipart of H has order at least $\lfloor n/2 \rfloor$. Let $\delta := 24^{-32} \cdot 3^{-10^4}$. Assuming n_0 is large enough, Theorem 1.1(a) yields five monochromatic vertex-disjoint cycles covering all but at most $2\delta n$ vertices of H . Let $X_A \subseteq A$ (resp. $X_B \subseteq B$) be the set of uncovered vertices in A (resp. B). Since we may assume none of the monochromatic cycles is an isolated vertex, we have $|X_A| = |X_B| \leq \delta n$.

By the choice of δ , and since we assume n_0 to be sufficiently large, we can apply Lemma 5.4 to the bipartite graphs $[A_2, X_B]$ and $[B_2, X_A]$. We obtain a union \mathcal{C} of twelve vertex-disjoint monochromatic cycles that together cover $X_A \cup X_B$. As $|X_A| = |X_B| \leq \delta n \leq 3^{-10^4}/12$, we know that $[A_2, B_2] - V(\cup \mathcal{C})$ contains a red Hamiltonian cycle. Thus, in total, we covered G with at most $5 + 12 + 1 = 18$ vertex-disjoint monochromatic cycles.

5.3 A remark on 3-coloured complete graphs

The number of 17 cycles needed to partition a 3-coloured complete graph, obtained by Gyárfás et al. [12], is not expected to be optimal. By a slight modification of their method, one can replace the number 17 with (the still not optimal number) 10.

Erdős et al. [6] have shown that any large enough 3-coloured K_n has a monochromatic *triangle cycle* of linear size. That is, a union of two cycles $(u_1, u_2, \dots, u_k, u_1)$ and $(u_1, v_1, u_2, v_2, \dots, u_k, v_k, u_1)$. Clearly, after the deletion of an arbitrary subset of the *outer vertices*, $\{v_1, \dots, v_k\}$, the triangle cycle still has a Hamiltonian cycle.

Given a sufficiently large 3-coloured K_n , we proceed as follows. First we reserve the vertex set of a linear sized monochromatic triangle cycle T for later use. We cover the remaining graph, except for some small set X , with three vertex-disjoint monochromatic cycles, using the result of Gyárfás et al. [12]. We then use Lemma 5.4 to cover all of X with six vertex-disjoint monochromatic cycles, which use some of the outer vertices of T (and X). This can be done

since T is of linear size while $|X|$ is a vanishing fraction of n . We finish by covering the remains of T with a monochromatic Hamiltonian cycle.

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