3-Colouring P_t-free graphs without short odd cycles

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Abstract

For any odd $t \geq 9$, we present a polynomial-time algorithm that solves the 3-colouring problem, and finds a 3-colouring if one exists, in P_t-free graphs of odd girth at least $t-2$. In particular, our algorithm works for (P_9, C_3, C_5)-free graphs, thus making progress towards determining the complexity of 3-colouring in P_t-free graphs, which is open for $t \geq 8$.

1 Introduction

A *k-colouring* of a graph G is an assigment of k distinct colours to the vertices of G such that all vertices are coloured and adjacent vertices have distinct colours. The *k-colouring problem* for a graph G and a natural number k consists in deciding whether G is k-colourable or not. For $k = 1$, the problem is equivalent to deciding whether G has an edge, and for $k = 2$, it is equivalent to deciding whether G is bipartite, which can be done in linear time using a BFS algorithm. For $k \geq 3$, Karp [19] showed that the problem is NP-complete.

There are some graph classes for which the k-colouring problem is polynomially time solvable, such as perfect graphs [13]. On the other hand, the problem remains NP-complete, even when restricted to the class of all triangle-free graphs with bounded maximum degree [23].

The class of H-free graphs, defined as the class of all graphs not containing H as an induced subgraph, is a very natural restriction and has appeared in many complexity results. An extension of the result from [23] is that the k-colouring problem is NP-complete for H-free graphs, for any graph H containing a cycle [18, 21]. Moreover, if H is a forest with a vertex of degree at least 3, then k-colouring is NP-complete for H-free graphs and $k \geq 3$ [16, 22]. Combining these results, we obtain that for $k \geq 3$, the k-colouring problem is NP-complete in H-free graphs, for all graphs H that are *not* a disjoint union of paths. So naturally, there has been much activity in determining for which k and t the k-colouring problem is polynomially time solvable or NP-complete in P_t-free graphs, where P_t is the path on t vertices. (There are also several recent results for graphs not having a disjoint union of paths as an induced subgraph, see [2, 3, 4, 5, 9, 20].)

Huang [17] proved that 4-colouring is NP-complete for P_7-free graphs, and that 5-colouring is NP-complete for P_6-free graphs (which implies that k-colouring is NP-complete for P_6-free graphs for $k \geq 5$). On the other hand, Hoàng, Kamiński, Lozin, Sawada, and Shu [15] showed that k-colouring can be solved in polynomial time on P_5-free graphs for any fixed k, and recently,

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Chudnovsky, Spirkl, and Zhong [6, 7] showed that the same is true for P_6-free graphs and $k = 4$. This means we have a complete classification of the complexity of k-colouring P_t-free graphs for any fixed $k \geq 4$.

For $k = 3$, less is known. Randerath and Schiermeyer [24] and Bonomo, Chudnovsky, Maceli, Schaudt, Stein, and Zhong [1] gave polynomial time algorithms for 3-colouring P_6- and P_7-free graphs, respectively. However, no such algorithm is known for P_8-free graphs (although some partial results are known for this class of graphs [8]). Interestingly, it is also unknown whether or not there exists $t \in \mathbb{N}$ such that 3-colouring is NP-complete on P_t-free graphs.

Some of the positive results from above are easier to prove if in addition to the path P_t, also the triangle, or other cycles, are excluded, and in fact, sometimes this was the first step for proving the full result. Let $\mathcal{C}$ be any fixed family of graphs (here, it will always be a set of cycles). Call a graph $\mathcal{C}$-free if it does not contain any member of the family $\mathcal{C}$, and call G $(P_t, \mathcal{C})$-free if in addition, G is P_t-free. We refer to [11] and [14] for an overview on the complexity of colouring $(P_t, \mathcal{C})$-free graphs.

In particular, in the case $k = 3$, Golovach, Paulusma and Song [12] showed that for any $t \geq 1$, the 3-colouring problem is polynomial time solvable for (P_t, C_4)-free graphs. The same authors show that for every $s \geq 3$ there is a $t = t(s)$ such that every $(P_t, \mathcal{C}_{\leq s})$-free graph is 3-colourable, where $\mathcal{C}_{\leq s}$ is the family of all cycles up to length s. Recall that by the results of [18, 21], it is also known that for all $s \geq 3$, 3-colouring is NP-complete for $\mathcal{C}_{\leq s}$-free graphs.

We focus on graphs having no short *odd* cycles. The *odd girth* of a graph G is the length of its shortest odd cycle (it is infinite if G is bipartite). Adhering to the terminology from above, we call a graph $\mathcal{C}^{odd}_{\leq \ell}$-free, for $\ell \geq 3$ odd, if it has odd girth at least $\ell + 2$.

The following is our main theorem.

Theorem 1.1. *For any odd $t \geq 9$, there is a polynomial time algorithm deciding whether a $(P_t, \mathcal{C}^{odd}_{\leq t-4})$-free graph is 3-colourable (and giving a 3-colouring if one exists).*

In particular, this implies:

Theorem 1.2. *There is a polynomial time algorithm deciding whether a (P_9, C_3, C_5)-free graph is 3-colourable (and giving a 3-colouring if one exists).*

We remark that the complexity of 3-colouring (P_t, C_ℓ)-free graphs is open for $t \geq 8$ and both $\ell = 3$, $\ell = 5$. So, Theorem 1.2 can be seen as a first step to determining the complexity of this problem, or even to resolving the problem for P_9-free graphs.

The paper is organised as follows. We give a quick overview of our proof in Section 2, and then go through some preliminaries in Section 3. We present the proof of Theorems 1.1 and 1.2 in Section 4. The proof relies on Lemma 4.1, which is also presented in Section 4. In Section 5, we prove Lemma 4.1 for all pairs t, G such that at least one of the following applies: $t = 9$; G is C_8-free; or G has an induced C_t. In particular, the proof of Theorem 1.2 is completed after Section 5. The more complicated proof for C_t-free graphs with $t > 9$ having an induced C_8 is postponed to Section 6. Finally, Section 7 is devoted to the running-time analysis of our algorithm, the running time being $O(n + m)$.

2 Overview of the proof

Our proof relies on a structural analysis of a longest odd cycle C of G and it surroundings. This is done in Section 5, distinguishing the two possible lengths of C: $t - 2$ or t (these cases are treated in Section 5.2 and 5.3, respectively). In the case that C has length $t-2$, we assume the cycle C_8 is forbidden or that $t = 9$ (for the time being).

We distinguish different sets D, T, T', S of neighbours of C, according to the location of their neighbours on C, and prove all other vertices lie at most at distance 2 from C (unless they are dominated by a non-neighbour, a case which can be safely ignored). We determine which sets can be adjacent to each other, and which cannot. The main results of the structural analysis in Sections 5.2 and 5.3 are summarised in Lemmas 5.12 and 5.24. Most importantly, these lemmas identify a set X of bounded size dominating important parts of the graph.

In Section 5.4, we prove Lemma 4.1, by colouring $V(C) \cup X$ in all possible ways. After doing so, all other vertices will either be adjacent to a coloured vertex, or are *reducible*, or belong to *reducible bipartite subgraphs*, by which we mean that they can be coloured in a canonical way (see Section 3 for a definition). This allows us to ultimately reduce the problem to a 2-list-colouring instance, which is known to be polynomial time solvable. This finishes the proof, unless $t = 9$ and there is a C_8.

We treat that case in Section 6. Our previous approach almost works in the same way. The only problem is that if in addition, C has length $t-2$, the structural analysis leaves us with one difficult case. Namely, there might be a set Y^* of vertices at distance 2 from C that cannot be dominated, or made reducible, in a similar way as the other sets. Our solution is that we test all colourings of C, and for each such colouring, we also colour the vertices from Y^*, and some of their neighbours and second neighbours 'by hand' in a canonical way (that will depend on the colouring of C). When doing so, we have to make sure that any vertex *not* coloured 'by hand' will not be affected by our colouring - either because it is already coloured in a compatible way, or more generally, because its list of available colours did not decrease. The main challenge in Section 6 is how to choose the vertices that will be coloured together with Y^*, in way that other vertices will not be affected. This is achieved by a structural analysis. We can then proceed as before. That is, we test all colourings of X, colour the reducible vertices and subgraphs, and thus reduce the problem to a 2-list-colouring instance.

3 Preliminaries

3.1 Basic definitions

For $n \in \mathbb{N}$, let $[n] = \{0, 1, ..., n\}$. A graph G has vertex set $V(G)$ and edge set $E(G)$. We only need to consider connected graphs G, since one can solve the 3-colouring problem on the components. We call G (or one of its components K) *trivial* if it has only one vertex.

If G does not contain another graph H as an induced subgraph, we say G is *H-free*. If $\mathcal{H}$ is a family of subgraphs, and G does not contain any of the graphs in $\mathcal{H}$ as an induced subgraph, we say G is $\mathcal{H}$-free. Sometimes we write $(H, \mathcal{H})$-free to mean that the graph is $\{H\} \cup \mathcal{H}$-free.

Let $v \in V(G)$ and $A, B \subseteq V(G)$. Then $N_B(v)$ is the set of all neighbours of v in B, and $N_B(A)$ is the set of all neighbours of vertices of A in $B \setminus A$. If $B = V(G)$, we omit the subscript. Note that $N(\emptyset) = \emptyset$. Also, $G[A]$ denotes the subgraph of G induced by A, and $G - v$ is the graph obtained from G by deleting v and its adjacent edges. We call a vertex v *complete* to A if all v-A edges are present, and if this holds for all $v \in B$, we say B is complete to A.

A *stable set* is a subset of pairwise non-adjacent vertices. A graph G is *bipartite* if $V(G)$ can be partitioned into two stable sets. A vertex $w \in V(G)$ *dominates* another vertex $v \in V(G)$ if

$N(v) \subseteq N(w)$. The following lemma is well-known and easy to prove.

Lemma 3.1. *Let G, H be graphs such that G is H-free, and let $v \in V(G)$. Then $G - v$ is H-free, and if v is dominated by one of its non-neighbours, then*

$$G \text{ is 3-colourable if and only if } G - v \text{ is 3-colourable.}$$

3.2 Palettes, updates and reducibility

In the context of 3-colouring, we call a family $\mathcal{L}$ of lists $L := \{L(v) : v \in V(G)\}$, where $L(v) \subseteq \{1, 2, 3\}$ for every $v \in V(G)$, a *palette* of the graph G. We call L *feasible* if no vertex has an empty list. A subpalette L' of L is a palette such that $L'(v) \subseteq L(v)$ for each $v \in V(G)$. A graph G and a palette L of G will often be written as a pair (G, L). We say that (G, L) is colourable if there is a subpalette L' of L such that $|L'(v)| = 1$ for every vertex $v \in V(G)$, and $L'(u) \neq L'(w)$ for every edge $uw \in V(G)$. We sometimes call a vertex with a list of size 1 a *coloured* vertex.

Updating the palette L means obtaining a subpalette L' of L by subsequently, for any vertex v having a list of size 1 in the current palette, deleting its colour from the list of each neighbour w of v. (For instance, if in a path all but one vertex v have the list $\{1, 2\}$, and v has list $\{1\}$, then updating this palette gives a colouring of the path.) If updating a palette L of a graph G results in the same palette L, we will say that L is *updated*.

Given (G, L), call a vertex $v \in V(G)$ *reducible (for colour α)* if $\alpha \in L(v)$ and $\alpha \notin L(w)$ for each $w \in N(v)$. A trivial subgraph $K = \{v\}$ of G is called reducible if v is reducible. The subpalette L' obtained from L by setting $L'(v) := \alpha$ and keeping all other lists will be called the *reduction palette* for the reducible component K.

A bipartite subgraph $K$of G, with partition classes U_1, U_2, is called *reducible (for colours α_1, α_2)* if there are distinct colours α_1, α_2 such that for $j = 1, 2$,

$$\alpha_j \in \bigcap_{u \in U_j} L(u) \setminus \bigcup_{w \in N(U_j) \setminus U_{3-j}} L(w).$$

The subpalette L' obtained from L by setting $L'(u) = \{\alpha_j\}$ for $j = 1, 2$ and all $u \in U_j$, and keeping all other lists will be called the *reduction palette* for the reducible component K.

We leave the proof of the following lemma as an exercise to the reader.

Lemma 3.2. *Let G be a graph and let L be palette of G. Suppose K is a reducible subgraph of G, and L' is the corresponding reduction palette. Then (G, L) is 3-colourable if and only if (G, L') is 3-colourable.*

4 The proof of Theorem 1.1

The heart of our proof is the following lemma, which we will prove in Sections 5 and 6. In order to state the lemma easily, let us define, for a graph G and a palette L, the set $V_3(G, L) := \{v \in V(G) : |L(v)| = 3\}$.

Lemma 4.1. *Let G be a $(P_t, \mathcal{C}^{odd}_{\leq t-4})$-free connected graph. Then there is a set $\mathcal{L}$ of updated feasible palettes such that*

- *$|\mathcal{L}| \leq 2^{O(d)}$ and $\mathcal{L}$ can be found in polynomial time;*

- *G is 3-colourable if and only if (G, L) is 3-colourable for some $L \in \mathcal{L}$; and*
- *for each $L \in \mathcal{L}$, every component of $G[V_3(G, L)]$ is reducible.*

We will also need a result on the list-colouring problem with lists of size at most 2. Given a graph G and a finite list $L(v) \subseteq \mathbb{N}$ for each vertex $v \in V(G)$ (with no restriction on the total number of colours) the *list-colouring problem* asks for a a colouring of all vertices with colours from their lists. If $|L(v)| \leq 2$ for each vertex $v \in V(G)$, the problem can be solved in $O(|V(G)|+|E(G)|)$ time, by reducing the 2-list-colouring instance to a 2-SAT instance [10, 25].

With this result and Lemma 4.1 at hand, we are ready to present the proof of our main result.

Proof of Theorem 1.1. Given G, we apply Lemma 4.1 to obtain a set $\mathcal{L}$ of palettes. For each $L \in \mathcal{L}$, and each reducible component of (G, L), we consider the reduction palette L' for L and solve the 2-list-colouring instance (G, L'). This either gives us a valid 3-colouring for one of the palettes L, and thus for our instance G, or proves that no such colouring exists. □

The running time of the algorithm and the time of finding the set $\mathcal{L}$ from Lemma 4.1 will be analysed in Section 7.

Sections 5 and 6 are devoted to the proof of Lemma 4.1. As discussed above we first concentrate on the easier case that either $t = 9$, or G is C_8-free, or G has an induced C_t, which we deal with in Section 5, and leave the more complicated general case for Section 6. For simplicity of notation, let $\mathcal{G}^*$ denote the class of all $(P_t, \mathcal{C}^{odd}_{\leq t-4})$-free graphs such that at least one of the following holds:

- $t = 9$; or
- G is C_8-free; or
- G has an induced C_t.

5 The proof of Lemma 4.1 for $G \in \mathcal{G}^*$

In this section, we will prove Lemma 4.1 for all $G \in \mathcal{G}^*$. In particular, as stated earlier, this section is all that is needed for the proof of Theorem 1.2.

The section is organised as follows. First, we investigate the structure of the graph G, proving some basic observations in Section 5.1, and then distinguishing two cases according to whether or not G has an induced cycle of length t. The structural analysis for the case that G is C_t-free is treated in Section 5.2, while the case that G does have a cycle of length t is treated in Section 5.3. All structural properties we find in these two sections will be conveniently summarised in two lemmas, Lemma 5.12 (at the end of Section 5.2) and Lemma 5.24 (end of Section 5.3). Then, in Section 5.4, where we will prove Lemma 4.1 for $G \in \mathcal{G}^*$, we only need to refer to these two lemmas.

5.1 The base cycle and its neighbours

We start with a general definition describing the neighbours and second neighbours of a cycle C of G.

Definition 5.1. *Let $\ell \in \mathbb{N}$ and let $C = c_0, c_1, ..., c_\ell, c_0$ be a cycle in G. For $i \in [\ell]$, define*

- $D_i := \big\{v \in N(C) : vc_j \in E(G)$ *if and only if* $j = i\big\}$;

- $T_i := \{v \in N(C) : vc_j \in E(G)$ *if and only if* $j \in \{i, i+2\}\}$;
- $T'_i := \{v \in N(C) : vc_j \in E(G)$ *if and only if* $j \in \{i, i+4\}\}$; *and*
- $S_i := \{v \in N(C) : vc_j \in E(G)$ *if and only if* $j \in \{i, i+2, i+4\}\}$.

Set

$$D := \bigcup_{i \in [\ell]} D_i, \quad T := \bigcup_{i \in [\ell]} T_i, \quad T' := \bigcup_{i \in [\ell]} T'_i \quad \text{and} \quad S := \bigcup_{i \in [\ell]} S_i,$$

and let Y *denote the set of vertices located at distance two from* C.

Clearly, since we can assume G is not bipartite (as otherwise the 3-colouring problem is well known to be polynomial), G has an odd cycle C. Because of the forbidden subgraphs, the length of the longest odd cycle is either $t-2$ or t. We will see now that in each of these two cases, $V(G)$ can be decomposed into some of the sets that Definition 5.1 gives for C.

Claim 5.2. *Suppose that no vertex in* G *is dominated by any of its non-neighbours.*[1] *Let* C *be a longest odd cycle in* G.

(a) If $|V(C)| = t-2$*, then* $V(G) = V(C) \cup N(C) \cup Y$*, with* $N(C) = D \cup T$.

(b) If $|V(C)| = t$*, then* $V(G) = V(C) \cup N(C) \cup Y$*, with* $N(C) = T \cup T' \cup S$.

Furthermore, for each $i \in [|V(C)| - 1]$*, each of the sets* D_i*,* T_i*,* T'_i *and* S_i *is stable.*

Proof. Since G has no odd induced cycles of length up to $t-4$, and no induced P_t, it is not hard to see that in case (a), $N(C) = D \cup T$, and in case (b), $N(C) = T \cup T' \cup S$. Moreover, each of the sets D_i, T_i, T'_i and S_i, for every $i \in [t-3]$, or $i \in [t-1]$, is stable, as G is triangle-free.

It only remains to show that no vertex of G lies at distance 3 from $V(C)$. Assume otherwise, and let $v_1v_2v_3$ be an induced path such that v_i lies at distance i from $V(C)$. Suppose $C = c_0, c_1, ..., c_\ell, c_0$, and that v_1 is adjacent to c_0, but not adjacent to any c_i with $i \leq t-5$. Since v_1 does not dominate v_3, there is a neighbour w of v_3 that is not a neighbour of v_1, and since G is triangle-free, w is not adjacent to v_2. Furthermore, since v_3 is at distance 3 from C, we know that w does not have any neighbours on C. Thus $w, v_3, v_2, v_1, c_0, c_1, c_2, ..., c_{t-5}$ is an induced P_t, which is impossible. □

We can deduce some useful information on the components of $G[Y]$.

Claim 5.3. *In the situation of Claim 5.2, every nontrivial component* K *of* $G[Y]$ *is bipartite and vertices of the same bipartition class of* K *have identical neighbourhoods in* $N(C)$.

Proof. Since G is (P_t, C_3)-free, we know that for every induced path $y_1y_2y_3$ of length 3 in $G[Y]$, vertices y_1 and y_3 have the same neighbours in $N(C)$. In order to see that $G[Y]$ is bipartite, apply the observation from the previous sentence to any odd cycle in $G[Y]$, thus generating a C_3 and therefore, a contradiction. □

[1] Because of Lemma 3.1, we need not worry about dominated vertices, and Claim 5.2 is the only place where we need to exclude them. In an algorithmic implementation of our method, one will ignore such vertices if detected by Claim 5.2, and colour them at the very end (in case a colouring is found).

5.2 The structure of G if it is C_t-free

In this subsection we asume that G is C_t-free, but does have an induced cycle $C = c_0, c_1, ..., c_{t-3}, c_0$ of length $t-2$. Let D, T and Y be the sets given by Definition 5.1 and Claim 5.2 for cycle C. Indices are taken modulo $t-2$.

For understanding the structure of $G[N(C) \cup Y]$, we first analyse the edges between Y and $N(C)$.

Claim 5.4. *No vertex from Y can have neighbours both in Y and in D.*

Proof. This follows directly from the fact that G is (P_t, C_3)-free. □

Claim 5.5. *For each $y \in Y$ there is an $i \in [t-3]$ such that at least one of the following holds:*

(a) $N(y) \setminus Y \subseteq D_j \cup T_i \cup T_{i+2}$ for some $j \in \{i-2, i, i+2, i+4\}$; or

(b) $t = 9$ and $N(y) \subseteq D_i \cup T_i \cup T_{i+2} \cup D_{i+4}$.

Proof. Because of the forbidden cycles (and observing that if $t = 9$ then $\ell - 4 = \ell + 3$), we know that if $y \in N(D_\ell)$, then all neighbours of y in D lie in $D_\ell \cup D_{\ell+4}$ or in $D_{\ell-4} \cup D_\ell$, and all neighbours of y in T lie in $T_{\ell-4} \cup T_{\ell-2} \cup T_\ell \cup T_{\ell+2}$. Moreover, if y has a neighbour in T_h, then $N(y) \cap T$ is contained either in $T_h \cup T_{h+2}$ or in $T_{h-2} \cup T_h$. This gives (a), unless y has neighbours in both D_i and in D_{i+4}. However, in that case, the neighbours of y in T must belong to $T_i \cup T_{i+2}$, and hence, there is an induced C_8 going through y, which is only allowed if $t = 9$. Putting these observations together, and using Claim 5.4 for (b), the statement follows. □

Let Y_i' be the set of all $y \in Y$ which have neighbours in both D_i and D_{i+4}. (By Claim 5.5, this set is empty if $t > 9$.) Observe that if there are independent edges xy, $x'y'$ such that $y, y' \in Y_i'$, $x \in D_i$, $x' \in D_{i+4}$, then $y, x, c_i, c_{i+1}, ..., c_{i+4}, x', y'$ is an induced path of length t (i.e. of length 9), which is impossible. Therefore, for any two vertices $y, y' \in Y_i'$, we have that either $N(y) \cap D_i \subseteq N(y') \cap D_i$ or $N(y') \cap D_{i+4} \subseteq N(y) \cap D_{i+4}$ (or both). Choosing two vertices y_i, y_i' in each set Y_i' such that $N(y) \cap D_i$ and $N(y') \cap D_{i+4}$ are inclusion-minimal, then choosing a neighbour q_i of y_i in D_i, and a neighbour q_i' of y_i' in D_{i+4}, and letting $Q := \bigcup_{i \in [t-3]} \{q_i, q_i'\}$, we obtain the following claim.

Claim 5.6. *There is a set Q of size at most 14 such that every vertex in $\bigcup_{i \in [t-3]} Y_i'$ has a neighbour in Q.*

Let us now analyse possible edges inside Y.

Claim 5.7. *For any $i \in [t-3]$, if $yy' \in E(G[Y])$ and $y \in N(T_i)$, then*

$$N(y') \setminus Y \subseteq \bigcup_{j \in \{i-3, i-1, i+1, i+3\}} (T_j \cup T_{j+2}).$$

Proof. This follows from the fact that G has no induced odd cycles of length up to $t-4$. □

Claims 5.4 and 5.7 immediately imply the following.

Claim 5.8. *If $y, y' \in Y$ are adjacent, then*

(i) there is a vertex c on C such that $N(y) \subseteq N(c)$; and

(ii) there are two consecutive vertices c, c' on C such that $z \in N(N(y))$ and $z' \in N(N(y'))$.

We now define a set W of certain vertices of Y.

Definition 5.9. *Let $W \subseteq Y \cap N(T)$ be the set of all vertices $y \in Y$ for which there is an $i \in [t-3]$ such that $y \in N(T_i)$ and one of the following holds:*

(i) $y \in N(D_{i-2} \cup D_{i+4})$; or

(ii) y has a neighbour in $Y \cap N(T_{i-3} \cup T_{i+3})$.

Observe that by Claim 5.4, the vertex set of any component of $G[Y]$ is either contained in W or disjoint from W.

Claim 5.10. *Let $y, y' \in Y \setminus W$ with $yy' \in E(G)$. Then for any $z \in N(y) \setminus Y$ and $z' \in N(y') \setminus Y$, we have that $zz' \in E(G)$.*

Proof. Suppose otherwise. By Claim 5.7 there is an index $i \in [t-3]$ such that $z \in T_i$ and $z' \in T_{i+1}$ (after possibly changing the roles of z and z'). Then $z, y, y', z', c_{i+3}, c_{i+4}, ..., c_{i-1}, c_i, z$ is an induced cycle of length t, a contradiction. □

It turns out that W is dominated by a set X of bounded size.

Claim 5.11. *There exist a set $R \subseteq N(C)$ such that $W \subseteq N(R)$ and $|R| \leq 2t-4$.*

Proof. Let $i \in [t-3]$. We claim that for any two vertices $y, z \in W \cap N(T_i)$ we have

$$N_{T_i}(y) \subseteq N_{T_i}(z), \text{ or } N_{T_i}(z) \subseteq N_{T_i}(y), \text{ or } N(y) \cap D = N(z) \cap D \neq \emptyset. \tag{1}$$

Then we can take, for each $i \in [t-3]$ with $T_i \neq \emptyset$, a vertex $y \in W$, with the property that among all such choices, $N_{T_i}(y)$ is inclusion-minimal, and choose an arbitrary vertex $a_i \in N_{T_i}(y)$. If $N(y) \cap D \neq \emptyset$, we also choose a arbitrary vertex $b_i \in N(y) \cap D$. Then by (1), the set R consisting of all a_i and all existing b_i is as desired.

It remains to prove (1). For contradiction suppose (1) fails for vertices $y, z \in W \cap N(T_i)$. Then there are $t_1 \in N_{T_i}(y) \setminus N_{T_i}(z)$ and $t_2 \in N_{T_i}(z) \setminus N_{T_i}(y)$.

We distinguish two cases. First assume z is as in Definition 5.9 *(i)*, that is, $z \in N(D_{i-2} \cup D_{i+4})$. Then there is $d \in N_{D_{i-2} \cup D_{i+4}}(z)$, say $d \in D_{i+4}$ (the other case is symmetric). Since we assume (1) does not hold, we may assume that $d \notin N(y)$ (after possibly swapping the roles of y and z). Consider the induced path $y, t_1, c_{i+2}, t_2, z, d, c_{i+4}, c_{i+5}, c_{i+6}, ..., c_{i-1}$, which has length t, a contradiction.

Now assume z is as in Definition 5.9 *(ii)*. Then z has a a neighbour $z' \in Y \cap N(T_{i-3} \cup T_{i+3})$, say $z' \in Y \cap N(T_{i+3})$. Let $t_3 \in N_{T_{i+3}}(z')$. Consider the path $y, t_1, c_{i+2}, t_2, z, z', t_3, c_{i+5}, c_{i+6}, c_{i+7}, ..., c_{i-1}$. As this path is induced ($yt_3 \notin E(G)$ because of Claim 5.5) and has length t, we again obtain a contradiction to G being P_t-free. This proves (1), and thus concludes the proof of the lemma. □

We now resume the main results of this subsection in the following lemma.

Lemma 5.12. *There is a set $X \subseteq N(C)$ with $|X| \leq 2t+10$ such that for every component K of $G[Y]$ one of the following holds.*

(I) $V(K) \subseteq N(X)$;

(II) K is trivial and there is a vertex c on C such that $N(K) \subseteq N(c)$; or

(III) K is bipartite with partition classes U_1, U_2, and

- *(A) there are vertices $b_1, b_2 \in V(C)$ such that $N(U_j) \subseteq N(b_j)$ for $j = 1, 2$;*
- *(B) there are $x_j \in N(N(U_j)) \cap V(C)$ for $j = 1, 2$ such that $x_1x_2 \in E(G)$; and*
- *(C) $N(U_1) \setminus Y$ is complete to $N(U_2) \setminus Y$.*

Proof. Apply Claim 5.11 to obtain a set R of size at most $2t-4$ that dominates W. Apply Claim 5.6 to obtain a set Q of size at most 14 that dominates $\bigcup_{i\in[t-3]} Y'_i$. Set $X := R \cup Q$.

Now, consider any vertex $y \in Y \setminus W$. By Claim 5.5, there is an $i \in [t-3]$ such that all neighbours of y in $N(C)$ belong to $D_i \cup T_{i-2} \cup T_i$. If y is isolated in $G[Y]$, we take $c(y) := c_i$ which is as required for (II). If y is not isolated in $G[Y]$, then Claim 5.3 implies that K is bipartite and the vertices of each partition class have identical neighbourhoods in $N(C)$. So by Claim 5.8 and Claim 5.10, we have (III). □

5.3 The structure of G if it contains a C_t

In this subsection, we assume that G has a cycle $C = c_0, c_1, ..., c_{t-1}$ of length t, and we let T, T' and S be the sets given by Definition 5.1 and Claim 5.2 for cycle C, while Y denotes the set of vertices located at distance two from C. In this subsection, all indices are taken modulo t.

We start by understanding some basic adjacencies.

Claim 5.13. *If $uv \in E(G[N(C)])$ and $i \in [t-1]$, then the following holds.*

(a) If $u \in T_i$ then $v \in T_{i+3} \cup \bigcup_{j\in\{i+1,i-1,i-3\}} \left(T_j \cup T'_j \cup S_j\right)$; and

(b) if $u \in T'_i$ then $v \in T_{i+3} \cup \bigcup_{j\in\{i+1,i-1\}} \left(T_j \cup T'_j \cup S_j\right)$.

Proof. Any case not covered would lead to an induced odd cycle of length $\leq t-4$ on vertices from $V(C) \cup \{u, v\}$. But such a cycle is forbidden in G. □

Claim 5.13 easily implies the following claim.

Claim 5.14. *If $uv \in E(G[N(C)])$, then there is an edge between $N(v) \cap V(C)$ and $N(u) \cap V(C)$.*

We now turn to the edges between $N(C)$ and Y. Our first claim follows directly from the fact that G is P_t-free.

Claim 5.15. *There are no edges between Y and T.*

There may, however, exist edges between Y and $T' \cup S$. Let us see what we can say about these.

Claim 5.16. *For each $y \in Y$ there is an $i \in [t-1]$ such that*

(a) $N(y) \subseteq Y \cup \bigcup_{j\in\{i,i+2\}}(T'_j \cup S_j)$; and

(b) c_{i+2} is complete to $N(y) \setminus (Y \cup T'_i)$ and c_{i+4} is complete to $N(y) \setminus (Y \cup T'_{i+2})$.

Proof. Note that if $y \in Y \cap N(T'_i \cup S_i)$, for some $i \in [t-1]$, then, by Claim 5.15 and since there are no induced odd cycle of length at most $t-4$ on vertices from $V(C) \cup N(y) \cup \{y\}$, we obtain that $N(y) \subseteq Y \cup \bigcup_{j\in\{i,i+2,i-2\}}(T'_j \cup S_j)$. This gives (a), which implies (b). □

Claim 5.17. *Let $y \in Y$ and let i be as in Claim 5.16. If $T'_i \neq \emptyset \neq T'_{i+2}$ and $y \in N(T'_i) \cup N(T'_{i+2})$, then y is complete to $T'_i \cup T'_{i+2}$.*

Proof. Assume $y \in N(T'_i)$ (the other case is symmetric). Let $t_1 \in N(y) \cap T'_i$ and let $t_2 \in T'_{i+2}$. Consider the path $y, t_1, c_{i+4}, c_{i+3}, c_{i+2}, t_2, c_{i+6}, c_{i+7}, ..., c_{i-2}, c_{i-1}$. This path has length t, so it cannot be induced. Note that the only one possible chord is yt_2. As the vertex t_2 was chosen arbitrarily from T'_{i+2}, this means y is complete to T'_{i+2}.

Changing the roles of the sets T'_i and T'_{i+2} in the above argument we see that y is also complete to T'_i. □

Claim 5.17 enables us to prove the following claim.

Claim 5.18. *There is a set $M \subseteq N(C)$ with $|M| \leq 2t$ such that*

(i) if $i \in [t-1]$ is such that $T'_{i-2} \cup T'_{i+2} \neq \emptyset$, then $N(T'_i) \cap Y \subseteq N(M)$; and

(ii) $M \cap T'_i \neq \emptyset$ for all i with $T'_i \neq \emptyset$.

Proof. Choose for M one vertex from each of the non-empty sets T'_i, for $i \in [t-1]$. Then, $|M| \leq 2t$, and (ii) clearly holds. In order to see (i), note that by Claim 5.17 any neighbour of T'_i in Y is complete to T'_i, and thus sends an edge to x_i. □

We now check edges in Y and their neighbours in $N(C)$.

Claim 5.19. *For any $i \in [t-1]$, if $yy' \in E(G[Y])$ and $y \in N(T'_i \cup S_i)$, then*

$$N(y') \setminus Y \subseteq \bigcup_{j \in \{i-3, i-1, i+1, i+3\}} (T'_j \cup S_j).$$

Proof. Any case not covered would lead to an induced odd cycle on at most $t-4$ vertices from $V(C) \cup N(y) \cup N(y') \cup \{y, y'\}$. □

Now, we will identify a useful subset $W \subseteq Y$.

Definition 5.20. *We define $W \subseteq Y \cap N(T' \cup S)$ as the set of endvertices of all edges yz such that there is $i \in [t-1]$ with $y \in Y \cap N(T'_i \cup S_i)$ and $z \in Y \cap \bigcup_{j \in \{i-3, i+3\}} N(T'_j \cup S_j)$.*

Note that by Claim 5.3, any component of $G[Y]$ has either all or none of its vertices in W.

Claim 5.21. *For any $i \in [t-1]$, $yy' \in E(G[Y])$ and $y \in N(T'_i \cup S_i) \setminus W$ the following hold.*

(a) If $N(y) \cap N(T'_{i+2}) \neq \emptyset$, then $N(y') \setminus Y \subseteq T'_{i+1} \cup S_{i+1}$.

(b) Each $w \in N(N(y') \setminus Y) \setminus Y$ is adjacent to some neighbour of $N(y)$ in $V(C)$.

(c) There are consecutive vertices c, c', c'' on C such that $N(y) \setminus Y \subseteq N(c) \cup N(c'')$ and $N(y') \setminus Y \subseteq N(c')$.

(d) If $y, y' \notin \bigcup_{i \in [t-1]} (N(T'_i) \cap N(T'_{i+2}))$, then there are consecutive vertices c, c' on C such that $N(y) \setminus Y \subseteq N(c)$ and $N(y') \setminus Y \subseteq N(c')$.

Proof. Item (a) is straightforward from Claim 5.19. For $w \in N(N(y') \setminus Y) \cap V(C)$, item (b) follows from Claim 5.19 and the definition of W, and for $w \in N(N(y') \setminus Y) \cap N(C)$, item (b) follows from Claim 5.14. For (c) and (d), we use Claim 5.16. □

We now show that W is dominated by a set of bounded size, B.

Claim 5.22. *There is a set $B \subseteq N(C)$ such that $W \subseteq N(B)$ and $|B| \leq t$.*

Proof. Let $yz, y'z' \in E(G[W])$ belong to different components from Y, and assume $i \in [t-1]$ is such that $y, y' \in N(T'_i \cup S_i)$ and $z, z' \in N(T'_{i+3} \cup S_{i+3})$. We claim that at least one of the following holds.

$$N_{T'_i \cup S_i}(y) \subseteq N_{T'_i \cup S_i}(y'), \text{ or } N_{T'_i \cup S_i}(y') \subseteq N_{T'_i \cup S_i}(y). \tag{2}$$

Indeed, if (2) is false, then there are vertices $s_1 \in N_{T'_i \cup S_i}(y) \setminus N_{T'_i \cup S_i}(y')$, $s_2 \in N_{T'_i \cup S_i}(y') \setminus N_{T'_i \cup S_i}(y)$, and $s_3 \in N_{T'_{i+3} \cup S_{i+3}}(z')$. Consider the path $y, s_1, c_{i+4}, s_2, y', z', s_3, c_{i+7}, c_{i+8}, c_{i+9}, ..., c_{i-1}$. This path has length t, and is induced, a contradiction. This proves (2).

Now, for each $i \in [t-1]$ with $T'_i \cup S_i \neq \emptyset$, take an edge $yz \in E(G[W])$, with $y \in Y \cap N(T'_i \cup S_i)$ and $z \in Y \cap N(T'_{i+3} \cup S_{i+3})$, and with the property that among all such choices, $N_{T'_i \cup S_i}(y)$ is inclusion-minimal. Choose an arbitrary vertex $x_i \in N_{T'_i \cup S_i}(y)$. Then by (2), $B := \{x_0, x_1, ..., x_{t-1}\}$ is as desired. □

We now show an interesting fact about the neighbourhoods of components of $G[Y]$.

Claim 5.23. *Let K be a bipartite component of $G[Y \setminus W]$, with partition classes U_1, U_2, and let $w \in N(U_1) \cap N(C)$. If there is no edge from w to $N(U_2) \cap N(C)$, then $N(w) \cap Y \subseteq V(K)$.*

Proof. Otherwise, w has neighbours $y \in U_1$, $y'' \in Y \setminus V(K)$ and a non-neighbour $t \in N(U_2) \cap N(C)$, and there is a vertex $y' \in N(y) \cap U_2$ such that $y't \in E(G)$. By Claim 5.19, if $y \in N(T'_i \cup S_i)$, then $N(y') \setminus Y \subseteq T'_{i-1} \cup S_{i-1} \cup T'_{i+1} \cup S_{i+1}$. Assume $t \in T'_{i+1} \cup S_{i+1}$ (the other case is symmetric). Then $y'', r, y, y', t, c_{i+5}, c_{i+6}, \ldots, c_{i-2}, c_{i-1}$ is an induced path of length t, a contradiction. □

We are now ready for the final result of this subsection, which resumes all important properties we will need later on.

Lemma 5.24. *There is a set $X \subseteq N(C)$ with $|X| \leq 3t$ such that for every component K of $G[Y]$ at least one of the following holds:*

(I) $V(K) \subseteq N(X)$;

(II) *K is trivial and there is a vertex c on C such that $N(K) \subseteq N(c)$; or*

(III) *K is bipartite with partition classes U_1, U_2, and for each $j = 1, 2$, and each $w \in N(U_j) \cap N(C)$ having no neighbours in $N(U_{3-j}) \cap N(C)$, we have*

  (A) $w \notin X$; *and*

  (B) *each $z \in N(w) \setminus U_j$ is adjacent to some neighbour of $N(U_{3-j})$ on C.*

  Moreover, there are consecutive $c, c', c'' \in V(C)$ on C such that one of the following holds:

  (C) $N(U_1) \subseteq U_2 \cup N(c)$ *and* $N(U_2) \subseteq U_1 \cup N(c')$; *or*

(D) $N(U_1) \subseteq U_2 \cup N(c) \cup N(c'')$ *and* $N(U_2) \subseteq U_1 \cup N(c')$*, and furthermore,*

$$N(U_1) \cap X \cap N(c) \neq \emptyset \neq N(U_1) \cap X \cap N(c'').$$

Proof. We let $X' := M \cup B$, where M is the set from Claim 5.18 and B is the set from Claim 5.22. Now, for each $x \in X'$ we check whether there is a bipartite component K of $G[Y]$, with partition classes U_1, U_2, such that $N(x) \cap Y \subseteq U_j$, for some $j \in \{1, 2\}$. For any such x and K, we check whether U_{3-j} has a neighbour in X'. If this is not the case, we choose an arbitrary vertex from $N(U_{3-j}) \setminus Y$. Add all these vertices to X', which gives us the set X. Note that for all $x \in X$ and any bipartite component K of $G[Y]$,

$$\text{if } N(x) \cap Y \subseteq V(K) \text{ then } V(K) \subseteq N(X). \tag{3}$$

Indeed, to see this note that if one of the newly added vertices fits the role of x in the condition of (3), then the corresponding component K has been used for defining X (coming from the other side). Furthermore, the size of X is bounded as desired, and by Claim 5.18 (i) and Claim 5.22,

$$N(X) \supseteq W \cup \big(Y \cap \bigcup_{i \in [t-1]} N(T'_i) \cap N(T'_{i+2})\big). \tag{4}$$

Now, consider a trivial component $K = \{y\}$ of $G[Y]$. By (4), if there is an $i \in [t-1]$ such that $y \in N(T'_i) \cap N(T'_{i+2})$, then $y \in N(X)$ and hence (I) holds for K. Otherwise, by Claim 5.16 (a) and (b), we know that (II) holds for K.

Let us now turn to the non-trivial components of $G[Y]$. By Claim 5.3, any such component K is bipartite with partition classes U_1, U_2, and the vertices from each U_i have identical neighbourhoods in $N(C)$. In particular, K either is contained in $G[W]$ or does not meet W at all. For the former type of components K, (I) holds because of (4), so let us assume that $V(K) \cap W = \emptyset$. Then by Claim 5.23, by Claim 5.21 (b), and by (3), (A) and (B) hold.

If there are no $j \in \{1, 2\}$ and $i \in [t-1]$ such that $U_j \subseteq N(T'_i) \cap N(T'_{i+2})$, then by Claim 5.21 (d), we have (C). So assume there are j, i such that $U_j \subseteq N(T'_i) \cap N(T'_{i+2})$. Then by Claim 5.21 (c) there are three vertices c, c', c'' as desired, and by Claim 5.17 and since M meets each non-empty T'_i (this is guaranteed by Claim 5.18 (ii)), it follows that (D) holds for K. □

5.4 The proof of Lemma 4.1 for $G \in \mathcal{G}^*$

This section is devoted to the proof of Lemma 4.1 for all $G \in \mathcal{G}^*$. We will use Lemmas 5.12 and 5.24.

Case 1: G is C_t-free.

In this case, G has an induced cycle C of length $t-2$. We apply Lemma 5.12 to obtain a set X. We let $\mathcal{L}$ be the set of all feasible palettes obtained by first precolouring $V(C) \cup X$ (in all possible ways) and then updating. Now, given a palette $L \in \mathcal{L}$, let $V_3 = V_3(G, L)$ be as in the lemma, that is, $V_3 = \{v \in V(G) : |L(v)| = 3\}$. Then since $V(C)$ is coloured, $V_3 \subseteq Y$.

Let K' be a component of $G[V_3]$. Then K' must be a subgraph of a component K of $G[Y]$, as in Lemma 5.12 (II) or (III). If K is as in (II), then $K' = K = \{y\}$ for some $y \in Y$, and there is a vertex $c \in V(C)$ dominating y. Thus, the colour of c is missing on the list of each neighbour of y, implying that y is a reducible vertex, which is as desired.

So assume K is as in (III) of Lemma 5.12, with bipartition classes U_1, U_2. If K' is non-trivial, then, since L is updated, we know that each vertex in $N(K') \cap N(C)$ has a list of size 2. More precisely, by (III)(A), for $j = 1, 2$, there is a colour α_j missing on the lists of each vertex $v \in N(U_j) \cap N(C)$, and all neighbours of v on C must have colour α_j. By property (III)(B), it is clear that $\alpha_1 \neq \alpha_2$. Therefore K' is reducible, which is as desired.

It remains to treat the case that K is as in Lemma 5.12 (III) and $K' = \{y\}$ is trivial. Assume $y \in U_1$ (the other case is symmetric). Then, since L is updated, and $|L(y)| = 3$, the vertices in U_2 have lists of size 2, and there must be a vertex $z \in N(U_2) \cap N(C)$ having list size 1. Say z is coloured α_0. By (III)(C), vertex z is complete to $N(y) \cap N(C)$, meaning that α_0 is missing on the lists of all neighbours of y. So y is reducible, as desired.

Case 2: G has an induced cycle C of length C_t.
We apply Lemma 5.24. Let $\mathcal{L}$ be the set of all feasible palettes obtained by precolouring $V(C) \cup X$ in all possible ways and then updating. Given a palette $L \in \mathcal{L}$, we consider $V_3 = \{v \in V(G) : |L(v)| = 3\}$. Since $V(C)$ is coloured, $V_3 \subseteq Y$. Consider any component K' of $G[V_3]$. Then K' must be a subgraph of a component K of $G[Y]$ as in (II) or (III) of Lemma 5.24. If K is as in (II), then there is a vertex c on C such that $N(K) \subseteq N(c)$. So, since L is an updated palette, the colour of c is missing in the list of every neighbour of K. Hence K is reducible, which is as desired.

So we can assume K is as in (III), with partition classes U_1, U_2. Observe that vertices from Y can only have lost colours from their list by updating. So, by Claim 5.3, if K' is non trivial, then $K = K'$. Moreover, if $K' = \{y\}$ is trivial, then $y \in U_j$ for some $j \in \{1, 2\}$, and all vertices in U_{3-j} have list size 2 (they cannot have list size 1, since $|L(y)| = 3$).

If K satifies (III)(C), we consider the vertices c, c'. Being adjacent, they must have been assigned distinct colours α_1, α_2. Since the palette L is updated, α_1 is missing in the lists of all neighbours of U_1 in $N(C)$, and α_2 is missing in the lists of all neighbours of U_2 in $N(C)$. Therefore, if K' is non-trivial, then, since the list of each vertex in $K' = K$ contains all three colours, K' is reducible, which is as desired.

So, we can assume $K' = \{y\}$ is trivial. Assume $y \in U_1$ (the other case is symmetric). Note that

$$\text{every neighbour of } N(U_1) \text{ on } C \text{ is coloured } \alpha_1. \tag{5}$$

Let α_0 be the colour missing in the lists of the vertices from U_2. If $\alpha_0 = \alpha_1$, then y is reducible, which is as desired. We claim that this is the case. So assume otherwise, that is, assume $\alpha_0 \neq \alpha_1$. There must be a vertex w in $N(U_2) \cap N(C)$ coloured α_0. Note that w is not adjacent to any vertex $v \in N(y) \cap N(C)$, since v then would miss two colours on its list, which is impossible (as $|L(y)| = 3$). Now, (III)(A) implies that $w \notin X$, and therefore, w has a neighbour z that is coloured α_1 (as there is no other possible reason for w to be coloured α_0). But then, by (III)(B), and by (5), z is adjacent to a vertex coloured α_1. Since L is feasible and updated, we arrive at a contradiction.

If K satisfies (III)(D), we consider the vertices c, c', c''. Belonging to $V(C)$, they must have been coloured in L, say they were assigned colours $\alpha_1, \alpha_2, \alpha_3$. Because of the adjacencies of c, c', c'', we have $\alpha_1 \neq \alpha_2 \neq \alpha_3$. If $\alpha_1 = \alpha_3$, then we can argue exactly as in the previous case that K' is a bipartite reducible subgraph of G, or that $K' = G[y]$ is trivial and y reducible, which is as desired. So, we assume $\alpha_1 \neq \alpha_3$.

Let $x_1 \in N(U_1) \cap X \cap N(c)$, and let $x_2 \in N(U_1) \cap X \cap N(c'')$. If x_1 and x_2 have been assigned different colours, then the vertices in U_1 have list size 1, and therefore the vertices in U_2 have list size at most 2, a contradiction. So we can assume x_1 and x_2 have been assigned the same colour. Since colour α_1 is missing in the list of x_1, colour α_3 is missing in the list of x_2, and L is feasible, that colour has to be α_2. This means α_2 is missing in the list of all vertices in U_1. Thus $K' = \{y\}$ has to be trivial, with $y \in U_2$. Furthermore, y is reducible (for colour α_2), which is as desired.

6 Adjustments of the proof of Lemma 4.1 for $G \notin \mathcal{G}^*$

In this section, we modify the proof of the previous section to a proof of Lemma 4.1 for all G.

6.1 The strategy

No modifications have to be made if G has an induced C_t, so we only need to focus on the case treated in Subsection 5.2. Because of the argument there, we can now assume that $t > 9$. We let $C = c_0, c_1, ..., c_{t-3}, c_0$ be an induced cycle in G, and let D_i, T_i and Y be the sets from Claim 5.1 for C. As noted in Subsection 5.2, the only use of the fact that the graph G was supposed to be C_8-free if $t > 9$ was in the proof of Claim 5.5. It is easy to see that in order to apply to arbitrary G, Claim 5.5 has to be rewritten as follows.

Claim 6.1. *For each $y \in Y$ there is an $i \in [t-3]$ such that either*

(a) $N(y) \setminus Y \subseteq D_j \cup T_i \cup T_{i+2}$ for some $j \in \{i-2, i, i+2, i+4\}$; or

(b) $N(y) \subseteq D_i \cup T_i \cup T_{i+2} \cup D_{i+4}$ and $N(y) \cap D_i \neq \emptyset \neq N(y) \cap D_{i+4}$.

This leads to the following modification in the statement of Lemma 5.12. Property (II) has to be replaced with the following property, where we let Y^* denote the set of all vertices of Y that are as in Claim 6.1(b) for some $i \in [t-3]$.

(II') $K = \{y\}$ is trivial and one of the following holds:

- (a) there is a vertex $c \in V(C)$ such that $N(K) \subseteq N(c)$; or
- (b) $y \in Y^*$.

In order to deal with the vertices in Y^*, we will colour them, their neighbours, and some of their second neighbours, by assigning colours synchronously to whole sets, in a similar way as we colour bipartite reducible components. For this, consider (G, L), and an updated feasible subpalette L' of L. If there is $S \subseteq V(G)$ such that $|L'(v)| = 1$ for each $v \in S$ and $L(v) = L'(v)$ for each $v \in V(G) \setminus S$, then we say L' is a *nice* reduction of L. It easy to see that the following lemma holds.

Lemma 6.2. *Given two palettes L and L' of a graph G such that L' is a nice reduction of L, we have that (G, L) is 3-colourable if and only if (G, L') is 3-colourable.*

6.2 Preliminaries for generating the set of palettes $\mathcal{L}$

We first need a little more structural analysis. We start with possible edges inside $N(C)$.

Claim 6.3. *Let $uv \in E(G[N(C)])$ and let $i \in [t-3]$ such that $u \in D_i$. Then*

$$v \in D_{i+3} \cup \bigcup_{j \in \{i-3, i-1, i+1\}} \{D_j \cup T_j\}.$$

Proof. Vertex v being in any other set would lead to an induced odd cycle of length $\leq t-4$ on vertices from $V(C) \cup \{u, v\}$. But such a cycle is forbidden in G. $\square$

We now explore the first and second neighbourhoods of vertices from Y^*. For each $i \in [t-3]$, let $Y_i^* \subseteq Y^*$ denote the set of all vertices from Y that are adjacent to both D_i and D_{i+4}.

Claim 6.4. *Let $i \in [t-3]$, and let $y \in Y_i^*$. Then, setting $D_i^* := N(y) \cap D_i$ and $D_{i+4}^* := N(y) \cap D_{i+4}$,*

(a) $D_{i-2} = D_{i+2} = D_{i+6} = \emptyset$;

(b) $N(D_i^*) \subseteq V(C) \cup D_{i+3} \cup T_{i-1} \cup T_{i+1} \cup Y$; and

(c) $N(D_{i+4}^*) \subseteq V(C) \cup D_{i+1} \cup T_{i+1} \cup T_{i+3} \cup Y$.

Moreover, letting D_j^+, for $j \in \{i, i+1, i+3, i+4\}$, denote the set of all vertices from D_j belonging to components of $G_i^+ := G[D_i \cup D_{i+1} \cup D_{i+3} \cup D_{i+4}]$ that contain a vertex from $D_i^* \cup D_{i+4}^*$, we have

(d) $N(D_{i+1}^+) \setminus (D_i^+ \cup D_{i+4}^+) \subseteq V(C) \cup T_{i-2} \cup T_i \cup T_{i+2} \cup Y$;

(e) $N(D_{i+3}^+) \setminus (D_i^+ \cup D_{i+4}^+) \subseteq V(C) \cup T_i \cup T_{i+2} \cup T_{i+4} \cup Y$; and

(f) $N(D_i^+ \cup D_{i+4}^+) \setminus (D_{i+1}^+ \cup D_{i+3}^+) \subseteq V(C) \cup T_{i-1} \cup T_{i+1} \cup T_{i+3} \cup (Y \cap N(D_i^* \cup D_{i+4}^*))$.

Proof. By Claim 6.1, we know that y only has neighbours in $D_i^* \cup T_i \cup T_{i+2} \cup D_{i+4}^*$. This fact will be implicitly used below for seeing that some of the paths we present are induced.

In order to see (a), note that any vertex $d_{i-2} \in D_{i-2}$ would lead to the induced path $y, d_i^*, c_i, c_{i+1}, c_{i+2}, ..., c_{i-2}, d_{i-2}$, where $d_i^* \in D_i^*$. This path has length t, which is impossible. Using symmetric arguments, we see that D_{i+2} and D_{i+6} are empty, too. This proves (a).

Furthermore, note that

$$\text{there are no edges between } D_i^* \text{ and } D_{i-1} \cup D_{i+1}. \tag{6}$$

Indeed, any such edge, say $d_i^* d_{i+1}$, with $d_i^* \in D_i^*$ and $d_{i+1} \in D_{i+1}$, gives rise to the path $c_{i-1}, c_{i-2}, ..., c_{i+2}, c_{i+1}, d_{i+1}, d_i^*, y$. This path is induced and has length t, which is forbidden. In the same way, we can show that

$$\text{there are no edges between } D_{i+4}^* \text{ and } D_{i+3} \cup D_{i+5}. \tag{7}$$

Next, in order to see (b), consider an edge from $d_i^* \in D_i^*$ to $x \in N(C) \setminus (D_{i+3} \cup T_{i-1} \cup T_{i+1})$. By Claim 6.3 and because of (6), we have $x \in D_{i-3} \cup T_{i-3}$ and $xd_{i+4} \notin E(G)$. So the cycle $x, d_i^*, y, d_{i+4}, c_{i+4}, c_{i+5}, ..., c_{i-3}, x$ is induced and of length $t-4$. This proves (b), and a symmetric argument shows (c). Also, observe that (d) and (e) follow directly from Claim 6.3 and from (a).

So it only remains to prove (f). For this, consider a possible edge from a vertex $d_i^+ \in D_i^+ \cup D_{i+4}^+$ to a vertex $x \notin V(C) \cup D_{i+1}^+ \cup D_{i+3}^+ \cup T_{i-1} \cup T_{i+1} \cup T_{i+3} \cup (Y \cap N(D_i^* \cup D_{i+4}^*))$. Take a shortest path $P = d_i^+, w_1, w_2, ..., w_m$ from d_i^+ to $D_i^* \cup D_{i+4}^*$ in G_i^+. We may assume that the only neighbour of x in $V(P) \cap (D_i^+ \cup D_{i+4}^+)$ is d_i^+ (after possibly replacing the vertex playing the role of d_i^+). So, by Claim 6.3, and by Claim 6.1, and since we assume that $t > 9$,

$$x \text{ has no neighbours in } V(P) \setminus \{d_i^+\}, \tag{8}$$

Also, observe that because of (b) and (c), we know that $x \notin N(D_i^* \cup D_{i+4}^*)$, and thus, $d_i^+ \notin D_i^* \cup D_{i+4}^*$. So, $w_1 \in D_{i+1}^+ \cup D_{i+3}^+$ and $m \geq 2$.

Because of symmetry, we may assume that $d_i^+ \in D_i^+$. By Claim 6.3,

$$x \in D_{i-3} \cup T_{i-3} \cup D_{i-1} \cup Y \setminus N(D_i^* \cup D_{i+4}^*).$$

First we treat the case $x \in D_{i-3} \cup T_{i-3}$. If $w_1 \in D_{i+3}^+$, then the induced cycle $x, d_i^+, w_1, c_{i+3}, c_{i+4}, ..., c_{i-3}, x$ has length $t-4$, which is forbidden. So $w_1 \in D_{i+1}^+$. If $w_2 \in D_{i+4}$, we obtain the forbidden induced cycle $x, d_i^+, w_1, w_2, c_{i+4}, ..., c_{i-3}, x$ of length $t-4$. So we can assume $w_2 \notin D_{i+4}$, and hence, $w_2 \in D_i$. In particular, by (6), $w_2 \in D_i^+ \setminus D_i^*$, and so, by the choice of P, w_1 is not adjacent to $D_i^* \cup D_{i+4}^*$. Consider the path $c_{i+2}, c_{i+1}, w_1, d_i^+, x, c_{i-3}, c_{i-4}, ..., c_{i+4}, d_{i+4}^*, y, d_i^*$, where $d_j^* \in N(y) \cap D_j^*$ for $j = i, i+4$. This path has length t, and it is induced, as we already saw that the edges $d_{i+4}^* w_1$, $d_i^* w_1$ are not present, and by (b) and (c), the edges $d_i^* x$, $d_{i+4}^* x$ are not present. We arrived at a contradiction. This proves that $x \notin D_{i-3} \cup T_{i-3}$.

Now, let us consider the possibility that $x \in D_{i-1}$. First assume $w_m \in D_i^*$. Then by (6), vertex w_{m-1} lies in D_{i+3}^+ and no vertex in $\{w_1, ..., w_m\}$ is adjacent to D_{i+4}^*. Then, by the choice of P, and by (8), we know that $x, d_i^+, w_1, w_2, ..., w_m, y, d_{i+4}^*, c_{i+4}, c_{i+5}, ..., c_{i-1}, x$ (where $d_{i+4}^* \in D_{i+4}^*$) is an induced cycle of length at least t, a contradiction. So we can assume $w_m \in D_{i+4}^*$, which, by (7), implies that w_{m-1} lies in D_{i+1}^+. Because of the induced cycle $x, d_i^+, w_1, w_2, ..., w_m, c_{i+4}, c_{i+5}, ..., c_{i-1}, x$, which has length $t-4+m$, we are done unless $m < 4$. So assume that $m < 4$, and note that then $m = 2$, by Claim 6.3. Consider the cycle $x, d_i^+, w_1, c_{i+1}, c_{i+2}, ..., c_{i-1}, x$, which is induced and has length t, a contradiction. We conclude that $x \notin D_{i-1}$.

It remains to eliminate the case that $x \in Y \setminus N(D_i^* \cup D_{i+4}^*)$. In this case, consider the path $x, d_i^+, w_1, w_2, ..., w_m, y, d_{i+4}^*, c_{i+4}, c_{i+5}, ..., c_{i-1}$ (where $d_{i+4}^* \in D_{i+4}^*$) if $w_m \in D_i^*$, and the path $x, d_i^+, w_1, w_2, ..., w_m, c_{i+4}, c_{i+5}, ..., c_{i-1}$ if $w_m \in D_{i+4}^*$ and $m \geq 4$. By (8), and as $x \notin N(D_i^* \cup D_{i+4}^*)$, both these paths are induced and have length at least t, which is forbidden. So we can assume that $w_m \in D_{i+4}^*$ and $m < 4$, implying that $m = 2$ and $w_1 \in D_{i+1}^+$. Consider the induced path $x, d_i^+, w_1, c_{i+1}, c_{i+2}, ..., c_{i-1}$ of length t, a contradiction. Hence $x \notin Y \setminus N(D_i^* \cup D_{i+4}^*)$, as desired. □

Claim 6.5. *Let $y \in Y_i^*$, and let D_i^*, D_{i+4}^*, D_{i+3}^+ and D_{i+1}^+ be as in Claim 6.4. Let $y' \in Y$.*

(a) If $N(y') \cap D_{i+1}^+ \neq \emptyset$, then either $N(y') \setminus D_{i+1}^+ \subseteq T_{i-1} \cup T_{i+1}$ or $N(y') \setminus D_{i+1}^+ \subseteq T_{i+1} \cup T_{i+3}$.

(b) If $N(y') \cap D_{i+3}^+ \neq \emptyset$, then either $N(y') \setminus D_{i+3}^+ \subseteq T_{i-1} \cup T_{i+1}$ or $N(y') \setminus D_{i+3}^+ \subseteq T_{i+1} \cup T_{i+3}$.

Proof. We only prove (b), since (a) is symmetric. Let $d_{i+3}^+ \in N(y') \cap D_{i+3}^+$, and let $P = d_{i+3}^+, w_1, w_2, ..., w_m$ be a shortest path from d_{i+3}^+ to $D_i^* \cup D_{i+4}^*$ in G_i^+ (where G_i^+ is as in Claim 6.4). Because of Claim 6.1, and since we assume $t > 9$, we may assume that y' has no neighbours in $V(P) \setminus \{d_{i+3}^+\}$ (after possibly changing the vertex in D_{i+3}^+ playing the role of d_{i+3}^+). Also, note that the only neighbour of y in $V(P)$ is w_m, and that

$$\text{no vertex in } \{w_1, ..., w_{m-2}\} \text{ is adjacent to } D_i^* \cup D_{i+4}^*. \tag{9}$$

We claim that for all $\ell < m$,

$$w_\ell \notin D_i^+ \cup D_{i+4}^+. \tag{10}$$

Indeed, if this is not true, then let $\ell < m$ be the smallest index such that $w_\ell \in D_i^+ \cup D_{i+4}^+$. Let $d_j^* \in D_j^*$, for $j = i, i+4$. If $w_\ell \in D_i^+$, then consider the path $y', d_{i+3}^+, w_1, w_2, ..., w_\ell, c_i, c_{i-1}, ..., c_{i+5}, c_{i+4}, d_{i+4}^*, y$, which, by (9), is induced and has length at least t. If $w_\ell \in D_{i+4}^+$, then consider the path $y', d_{i+3}^+, w_1, w_2, ..., w_\ell, c_{i+4}, c_{i+5}, ..., c_i, d_i^*, y$, which has length t and by (9), is induced. As such paths are forbidden, we proved (10).

Note that (10) together with Claim 6.3 implies that $m = 1$ and

$$w_1 \in D_i^*.$$

We will now show that

$$y' \text{ has no neighbours in } D_{i-1} \cup (D_{i+3} \setminus D_{i+3}^+). \tag{11}$$

For contradiction, suppose y' has a neighbour $x \in D_{i-1} \cup (D_{i+3} \setminus D_{i+3}^+)$. Then $x, y', d_{i+3}^+, w_1, y, d_{i+4}^*, c_{i+4}, c_{i+5}, ..., c_{i-1}$ is a path, where $d_{i+4}^* \in D_{i+4}^*$. This path has length t, and because of Claim 6.3 and Claim 6.4 (b), the only possible edge is xc_{i-1}, leading to an induced cycle of the same length, a contradiction. This proves (11).

Next, we show that

$$y' \text{ has no neighbours in } T_{i+5} \cup D_{i+7}. \tag{12}$$

In order to see (12), assume y' has a neighbour $x \in T_{i+5} \cup D_{i+7}$. Consider the cycle $x, y', d_{i+3}^+, w_1, c_i, c_{i-1}, ..., c_{i+7}, x$. By Claim 6.4(b), this cycle is induced, and furthermore, it has length $t-4$. This is a contradiction, which proves (12).

By (11) and (12), and because of Claims 5.4 and 6.1, it only remains to show that y' cannot have neighbours in both T_{i-1} and T_{i+3}. Suppose otherwise, and let $t_j \in N(y') \cap T_j$, for $j = i-1, i+3$. Then $c_{i-1}, t_{i-1}, y', t_{i+3}, c_{i+3}, c_{i+4}, ..., c_{i-1}$ is an induced cycle of length $t-4$, a contradiction. □

Claim 6.6. *Let $y \in Y_i^*$, $y' \in Y \setminus Y_i^*$ such that y' has a neighbour in $D^* := N(y) \cap (D_i \cup D_{i+4})$. Then*

(a) $N(y') \subseteq D_i \cup T_i \cup T_{i+2} \cup D_{i+4}$; and

(b) if $d \in D \cap N(y') \setminus N(y)$, then $N(d) \subseteq N(D^*)$.

Proof. Because of symmetry, we can assume that $y' \in N(D_i)$. Let $d_i \in N(y) \cap N(y') \cap D_i$ and let $d_{i+4} \in N(y) \cap D_{i+4}$. Note that by Claim 6.4 (b), y has no neighbours in $D_{i-4} \cup T_{i-4} \cup T_{i-2}$. Also, note that by Claim 5.4, y' has no neighbours in Y, and as $y' \notin Y_i^*$, we know that $y'd_{i+4} \notin E(G)$.

We first show (a). If y' has a neighbour $t_{i-4} \in D_{i-4} \cup T_{i-4}$, then $t_{i-4}, y', d_i, y, d_{i+4}, c_{i+4}, c_{i+5}, ..., c_{i-4}, t_{i-4}$ is an induced cycle of length $t-4$, a contradiction. If y' has a neighbour $t_{i-2} \in T_{i-2}$, then $c_{i+6}, c_{i+7}, ..., c_{i-2}, t_{i-2}, y', d_i, y, d_{i+4}, c_{i+4}, c_{i+3}, c_{i+2}, c_{i+1}$ is an induced path of length t, a contradiction. So by Claim 6.1, item (a) follows.

Let us now show (b). For this, let $d \in D \cap N(y') \setminus N(y)$ and note that as $y' \in N(D_i)$ and $y' \notin Y_i^*$ we have $d \in D_i$. Assume xd is an edge, with $x \in V(G) \setminus N(D^*)$. Consider the path $x, d, y', d_i, y, d_{i+4}, c_{i+4}, c_{i+5}, ..., c_{i-1}$, which has length t, and therefore cannot be induced. The only possible chords are of the form xc_j, for $j = i+5, i+6, ..., i-1$. Morover, if the only chord is xc_{i-1}, we obtain an induced cycle of length t, which is forbidden. Therefore, and because of Claim 6.3, we know that $x \in D_{i-3} \cup T_{i-3}$. Then, consider the path $c_{i+1}, c_{i+2}, c_{i+3}, c_{i+4}, d_{i+4}, y, d_i, y', d, x, c_{i-3}, c_{i-4}, ..., c_{i+6}$. This path is induced and has length t, a contradiction. □

6.3 The proof of Lemma 4.1 for all G

Given a $(P_t, \mathcal{C}_{\leq t-4}^{odd})$-free graph G, we can assume G is C_t-free, and so, G has an induced cycle C of length $t-2$. We find, as before, a set X as in Lemma 5.12, but we will not colour all of it yet. Instead, for every colouring of $V(C)$, we generate a palette. We take $\mathcal{L}'$ as the set of all such palettes that are feasible. For each $L' \in \mathcal{L}'$, we proceed as follows.

For each $i \in [t-3]$, we consider the set $Y_i^* \subseteq Y^*$. Set

$$D_j^{*(i)} := \bigcup_{y \in Y_i^*} (N(y) \cap D_j) \text{ for } j = i, i+4.$$

For $j \in \{i, i+1, i+3, i+4\}$, let $D_j^{+(i)}$ denote the set of all vertices from D_j belonging to components of

$$G_i^{+(i)} := G[D_i \cup D_{i+1} \cup D_{i+3} \cup D_{i+4}]$$

that contain some vertex from $D_i^{*(i)} \cup D_{i+4}^{*(i)}$. Finally, we let $Z^{*(i)}$ be the set of all $y \in Y \setminus Y_i^*$ that are adjacent to $V(G_i^{+(i)})$. By Claim 6.4 (f), Claim 6.5 and Claim 6.6, we know that $Z^{*(i)}$ partitions into three sets $Z_1^{*(i)}$, $Z_2^{*(i)}$ and $Z_3^{*(i)}$, such that for $j = 1, 2, 3$,

$$N(Z_j^{*(i)}) \setminus D \subseteq T_{i+j-2} \cup T_{i+j}. \tag{13}$$

Moreover,

$$N(Z_j^{*(i)}) \cap D \subseteq D_{i+1}^{+(i)} \cup D_{i+3}^{+(i)} \text{ for } j = 1, 3, \tag{14}$$

and $N(Z_2^{*(i)}) \cap D \subseteq D_i \cup D_{i+4}$. Further, for $j = i, i+4$, setting

$$D_j^{++(i)} := N(Z_2^{*(i)}) \cap (D_j \setminus D_j^{+(i)})$$

we have, by Claim 6.6 (b), and by Claim 6.4 (b) and (c),

$$N(D_j^{++(i)}) \subseteq Y_i^* \cup Z_2^{*(i)} \cup \{c_j\} \cup T, \tag{15}$$

with

$$N(D_i^{++(i)}) \cap T \subseteq T_{i-1} \cup T_{i+1} \text{ and } N(D_{i+4}^{++(i)}) \cap T \subseteq T_{i+1} \cup T_{i+3}. \tag{16}$$

Set

$$F(Y_i^*) := Y_i^* \cup Z^{*(i)} \bigcup_{j=i,i+1,i+3,i+4} D_j^{+(i)} \cup \bigcup_{j=i,i+4} D_j^{++(i)}.$$

By Claim 6.1, Claim 6.4, Claim 6.5 and Claim 6.6 and because of (13), (14) and (15), we know that $N(F_i^*) \subseteq V(C) \cup T$, and moreover, $F(Y_i^*) \cap F(Y_j^*) = \emptyset$ for distinct $i, j \in [t-3]$. So we can treat Y_i^* and Y_j^* independently.

For $i \in [t-3]$, let α_i be the colour assigned to c_i in L'. Colour, for each $i \in [t-3]$ with $Y_i^* \neq \emptyset$, all vertices from the set $F(Y_i^*)$ as follows:

- Assign colour α_{i+1} to all vertices in $D_i^{+(i)} \cup D_i^{++(i)} \cup Z_1^{*(i)}$;
- assign colour α_{i+2} to all vertices in $D_{i+1}^{+(i)} \cup D_{i+3}^{+(i)} \cup Y_i^* \cup Z_2^{*(i)}$; and
- assign colour α_{i+3} to all vertices in $D_{i+4}^{+(i)} \cup D_{i+4}^{++(i)} \cup Z_3^{*(i)}$.

After updating, we call the obtained palette L''. Note that for $j = 1, 2, 3$ we have that $\alpha_{i+j} \notin L'(t)$ for $t \in T_{i+j-2} \cup T_{i+j}$. Hence, by (13), (14), (15) and (16), and by Claims 6.4, 6.5 and 6.6, this colouring is valid, and moreover, the neighbours of $F(Y_i^*)$ have not lost any colours in their lists, for $i \in [t-3]$. So by Lemma 6.2, (G, L'') is 3-colourable if and only if (G, L') is 3-colourable, which means that we can work with L'' instead of L'. For each L'' obtained in this way, we colour $X \setminus V(C)$ in all possible ways, update the palette, and, if feasible, we add it to the set $\mathcal{L}$.

Now, given a palette $L \in \mathcal{L}$, let $V_3 = V_3(G, L) \subseteq Y$ be as in the lemma and let K' be a component of $G[V_3]$. Then K' must be a subgraph of a component K of $G[Y]$, as in Lemma 5.12. Note that K is not as in (II') with $K = \{y\}$ and $y \in Y^*$, since then $|L(y)| = 1$, by our previous arguments. Hence we can proceed exactly as in the case when $G \in \mathcal{G}^*$. This finishes the proof.

7 The overall complexity of the algorithm

In this section we analyse the complexity of the algorithm. Let m, n be the number of edges and vertices of G. We assume $t \in \mathbb{N}$, $t \geq 9$ is odd.

We start by analysing the easier case that $G \in \mathcal{G}^*$, and leave the analysis for the case $G \notin \mathcal{G}^*$ to the end.

Finding the cycle C and the set X. It can be checked in $O(m)$ time whether G is bipartite. If it is not, we will find an induced cycle C that either has length $t-2$ or t. If C has length $t-2$, we will work with C as if the graph G was C_t-free – should we find, at any point in our process, an induced cycle C' of length t, we abort the process, and start afresh with C'.

We check the vertices in $N(C)$ and partition them into sets D_i, T_i, T_i', S_i. We check the remaining vertices, and if they have a neighbour in $N(C)$, we put them into Y. By Claim 5.2 we know that any vertex not qualifying for Y is dominated by one of its non-neighbours. So we can put such vertices aside in order to colour them at the very end. This takes $O(m)$ time.

Now, if $|V(C)| = t-2$, we find sets Q from Claim 5.6 and R from Claim 5.11, which together form X (in this case). This can be done in $O(n+m)$ time, as we only need to find inclusion minimal neighbourhoods in disjoint sets D_i or T_i (and then pick a vertex for Q or R).

If $|V(C)| = t$, we need to find sets M from Claim 5.18 and B from Claim 5.22, which also takes $O(n+m)$ time. Together, M and B form the set X', from which we obtain the set X of Lemma 5.24 by adding some extra vertices. More precisely, we add one new vertex each time we find a bipartite component K of $G[Y]$, and a vertex $x \in X'$, such that the only neighbours of x in Y belong to K, and, at the same time, one of the bipartition classes of K does not send any edge to X'. We can explore in $O(n+m)$ time all components K to check if they are as required, and add the extra vertex to X' if needed.

The total time for finding C, analysing the structure, and finding X is $O(n+m)$.

Checking all precolourings of X. We need to consider all distinct feasible colourings of $V(C) \cup X$. This set has at most $4t$ vertices, which form a connected set, so we will need to check $3 \cdot 2^{4t-1}$ many colourings. Updating can be done in $O(m)$ time.

We then go through the components of $G[V_3]$ (where V_3 are the vertices having list size 3). Any trivial reducible component can be coloured by checking which colour is missing at any of its neighbours' list. Bipartite reducible components can be dealt with similarly. This takes $O(n)$ time.

Finally, we need to solve a list-colouring instance with lists of size at most 2, which can be done in $O(n+m)$ time. If a colouring is found, we add back to G the vertices dominated by non-neighbours, suitably coloured. In conclusion, if $G \in \mathcal{G}^*$, then the overall complexity of the algorithm is $2^{O(t)} \cdot O(n+m)$.

Variation with extra precolouring in case $G \notin \mathcal{G}^*$. We will know we are in this case if Y^* turns out to be non-empty. We need to determine all sets Y_i^* and the corresponding $F(Y_i^*)$. Note that these sets can be found in $O(m)$ time, before we colour the cycle C. We then go through all feasible colourings of C. There are $3 \cdot 2^{t-3}$ colourings we need to check. For each of these we colour all $F(Y_i^*)$, and then update, colour the reducible subgraphs and run the 2-list colouring instance as before. The overall complexity of the algorithm stays at $2^{O(t)} \cdot O(n+m)$.

References

[1] F. Bonomo, M. Chudnovsky, P. Maceli, O. Schaudt, M. Stein, and M. Zhong. Three-coloring and list three-coloring of graphs without induced paths on seven vertices. *Combinatorica*, 38(4):779–801, 2018.

[2] N. Brettell, J. Horsfield, D. Paulusma. Colouring $(sP_1 + P_5)$-Free Graphs: a Mim-Width Perspective. Preprint, arXiv:2004.05022 (2020).

[3] H. Broersma, F.V. Fomin, P.A. Golovach, D. Paulusma. Three complexity results on coloring P_k-free graphs. *Eur. J. Comb.* 34(3), 609–619, 2013.

[4] H. Broersma, P.A. Golovach, D. Paulusma, J. Song. Updating the complexity status of coloring graphs without a fixed induced linear forest. *Theor. Comput. Sci.* 414(1), 9–19, 2012.

[5] M. Chudnovsky, S. Huang, S. Spirkl, and M. Zhong. List-three-coloring graphs with no induced P_6+rP_3. *Algorithmica*, published online 07/2020. https://doi.org/10.1007/s00453-020-00754-y

[6] M. Chudnovsky, S. Spirkl, and M. Zhong. Four-coloring P_6-free graphs: Extending an excellent precoloring. arXiv:1802.02282 (2018).

[7] M. Chudnovsky, S. Spirkl, and M. Zhong. Four-coloring P_6-free graphs: Finding an excellent precoloring. Preprint arXiv:1802.02283 (2018).

[8] M. Chudnovsky, J. Stacho. 3-Colorable subclasses of P_8-free graphs. *SIAM J. Discrete Math.*, 32(2), 1111–1138, 2018.

[9] J.-F. Couturier, P.A. Golovach, D. Kratsch, D. Paulusma. List coloring in the absence of a linear forest. *Algorithmica* 71(1), 21–35, 2015.

[10] P. Erdős, A. Rubin, and H. Taylor. Choosability in graphs. *Congressus Numerantium*, 26:125–157, 1979.

[11] P. A. Golovach, M. Johnson, D. Paulusma, and J. Song. A survey on the computational complexity of colouring graphs with forbidden subgraphs. *Journal of Graph Theory*, 84(4):331–363, 2017.

[12] P.A. Golovach, D. Paulusma, and J. Song. Coloring graphs without short cycles and long induced paths. *Discrete Applied Mathematics*, 167:107–120, 2014.

[13] M. Grötschel, L. Lovász, and A. Schrijver. The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica*, 1:169–197, 1981.

[14] P. Hell and S. Huang. Complexity of coloring graphs without paths and cycles. *Discrete Applied Mathematics*, 216(1):211–232, 2017.

[15] C.T. Hoàng, M. Kamiński, V.V. Lozin, J Sawada, and X. Shu. Deciding k-colorability of P_5-free graphs in polynomial time. *Algorithmica*, 57:74–81, 2010.

[16] I. Holyer. The NP-completeness of edge-coloring. *SIAM J. Comput.* 10(4), 718–720, 1981.

[17] S. Huang. Improved complexity results on k-coloring P_t-free graphs. *European Journal of Combinatorics*, 51:336–346, 2016.

[18] M. Kamiński and V.V. Lozin. Coloring edges and vertices of graphs without short or long cycles. *Contributions to Discrete Mathematics*, 2:61–66, 2007.

[19] R. Karp. Reducibility among combinatorial problems. In R. Miller and J. Thatcher, editors, *Complexity of Computer Computations*, pages 85–103. Plenum Press, New York, 1972.

[20] T. Klimošová, J. Malík, T. Masarík, J. Novotná, D. Paulusma, and V. Slívová. Colouring $(P_r + P_s)$-free graphs. *Algorithmica*, 82, 1833–1858, 2020.

[21] D. Král, J. Kratochvíl, Zs. Tuza, and G.J. Woeginger. Complexity of coloring graphs without forbidden induced subgraphs. In M.C. Golumbic, M. Stern, A. Levy, and G. Morgenstern, editors, *Proceedings of the International Workshop on Graph-Theoretic Concepts in Computer Science 2001*, volume 2204 of *Lecture Notes in Computer Science*, pages 254–262, 2001.

[22] D. Leven and Z. Galil. NP-completeness of finding the chromatic index of regular graphs. *Journal of Algorithms*, 4:35–44, 1983.

[23] F. Maffray and M. Preissmann. On the NP-completeness of the k-colorability problem for triangle-free graphs. *Discrete Mathematics*, 162:313–317, 1996.

[24] B. Randerath and I. Schiermeyer. 3-Colorability $\in$ P for P_6-free graphs, *Discrete Applied Mathematics*, 136(2-3): 299-313, 2004.

[25] V. Vizing. Coloring the vertices of a graph in prescribed colors. *Metody Diskretnogo Analiza*, 29:3–10, 1976.