A strongly aperiodic Heisenberg shift of finite type

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Joint work with M. Schraudner and I. Ugarcovici.

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Questions

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- Block and Weinberger, Mozes (certain classes of Lie groups - semisimple, simple..)
- Weakly aperiodic SFT for Heisenberg and sol: Nowak and Weinberger, using cohomological techniques.

A group G is called an ascending HNN extension of $\mathbb{Z}^2$ if there exists a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad a_{ij} \in \mathbb{Z}$$

with $\det A \neq 0$ and

$$G \cong \left\langle \mathbf{x}, \mathbf{y}, \mathbf{z} \middle| [y,z] = 1, x^{-1}yx = y^{a_{11}} z^{a_{21}}, x^{-1}zx = y^{a_{12}} z^{a_{22}} \right\rangle$$

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$$G \cong \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\}.$$

Equivalently the discrete Heisenberg group is a semi-direct product $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ with matrix $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$:

$$\begin{aligned}(x, y, z) \cdot (x', y', z') &= (x + x', y + y', z + z' + xy') \\ &= (x + x', (y + y', z + z' + xy')) \\ &= \left(x + x', \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} y' \\ z' \end{pmatrix} + (y, z)\right)\end{aligned}$$

In particular: If $|\det A| = 1$ then the semi-direct product $\mathbb{Z}^2 \ltimes_A \mathbb{Z}$ (using notation $(x, (y, z)))$ is an ascending HNN extension of $\mathbb{Z}^2$ with matrix A and the identification

$$\mathbf{x} = (1, 0, 0) \qquad \mathbf{y} = (0, 1, 0) \qquad \mathbf{z} = (0, 0, 1).$$

So we have in particular:

- $\mathbb{Z}^3$, for $A = Id$.
- The discrete Heisenberg group, for $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.
- The group *sol*, for $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

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(Ş, Schraudner, Ugarcovici) The discrete Heisenberg group admits a strongly aperiodic SFT.

Both results use the same framework: construct G SFT with well chosen (in particular strongly aperiodic SFT) projective dynamics on the $<\mathbf{y}, \mathbf{z}>$ subgroup.

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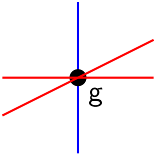
Projective subdynamics: Johson and Madden, Schraudner, Schraudner and Pavlov.

Recall that to define a G SFT we can define *nearest neighbor* rules by choosing a symmetric set S of generators and constructing a Wang tiling of the S Cayley graph, $\mathcal{G}_S$, of G.
For any matrix A with $\det A \neq 0$ we consider the generator set $S = \{\mathbf{x}^{\pm 1}, \mathbf{y}^{\pm 1}, \mathbf{z}^{\pm 1}\}$. The *unit cube* for $\mathcal{G}_S$ is the following:



Red edges correspond to the generators **y** and **z**. Blue edges correspond to the generator x, the stable element.

Proof of first result:

Let $\Omega_{\mathbb{Z}^2}$ be a $\mathbb{Z}^2$ SFT, and let $\mathcal{W}_{\mathbb{Z}^2}$ be a Wang tiling of the "standard" Cayley graph of $\mathbb{Z}^2$.

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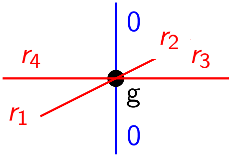

The r_i are the labels for a Wang tile from $\mathcal{W}_{\mathbb{Z}^2}$, 0 is a symbol that does not appear in the alphabet of $\Omega_{\mathbb{Z}^2}$.

Let Ω be the G-SFT obtained by the Wang tilings with $\mathcal{W}$. If $\Omega_{\mathbb{Z}^2}$ is non-empty, then so is Ω.

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If Stab(ω) has finite index for some $\omega \in \Omega$, then Stab(ω)$\cap < \mathbf{y}, \mathbf{z} > \neq \{e\}$, a contradiction.

Fix a non-invertible matrix A, and let G be the associated ascending HNN extension of $\mathbb{Z}^2$.

Question

Can we characterize $\mathbb{Z}^2$ SFT $\Omega_{\mathbb{Z}^2}$ for which there exists a non-trivial G extension?

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More on this later.

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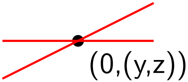
$$(1, (0, 0)) \cdot (0, (y, z)) =$$



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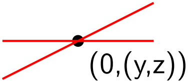

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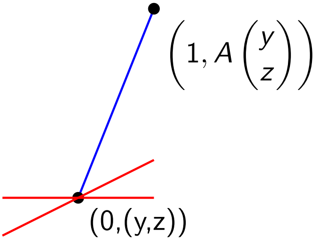
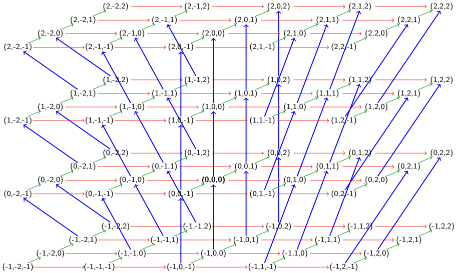


The S-Cayley graph of the Heisenberg group (left labeled):



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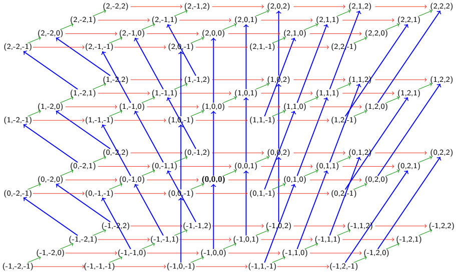

The slope of the blue arrows are a function of the y-coordinate of the vertex:

$$(1,(0,0))\,(x,(y,z)) = (x+1,(y,z+y))$$

More importantly for us:

$$(2,(0,0))\,(x,(y,z)) = (x+2,(y,z+2y))$$

Namely: if we look at every other $<\mathbf{y},\mathbf{z}>$ coset, the blue arrow is skewed by $2y$ in the $\mathbf{z}$ direction.
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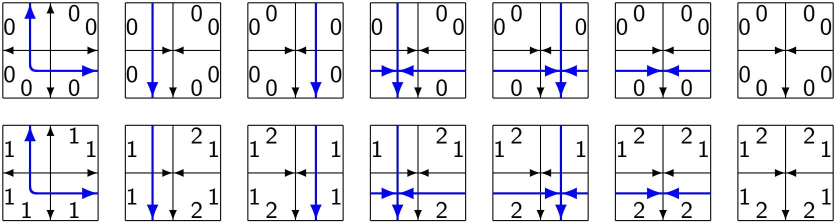
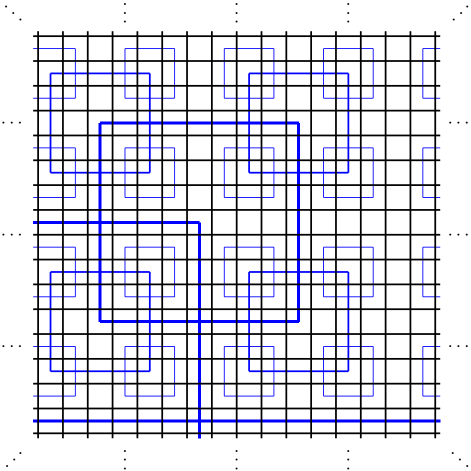
This idea is present in a $\mathbb{Z}^3$ construction of Kari and Culik: does not work verbatim in Heisenberg case: information transmitted by skewing can be conflicting.

Recall the Robinson tilings (N. Auburn mini-course):
Robinson's tiles:



Robinson's tilings have a *hierarchical* structure:



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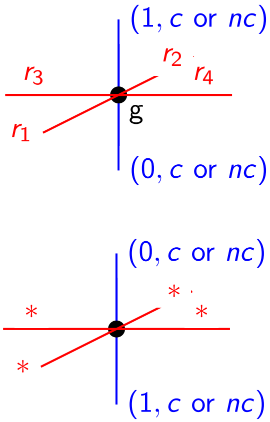

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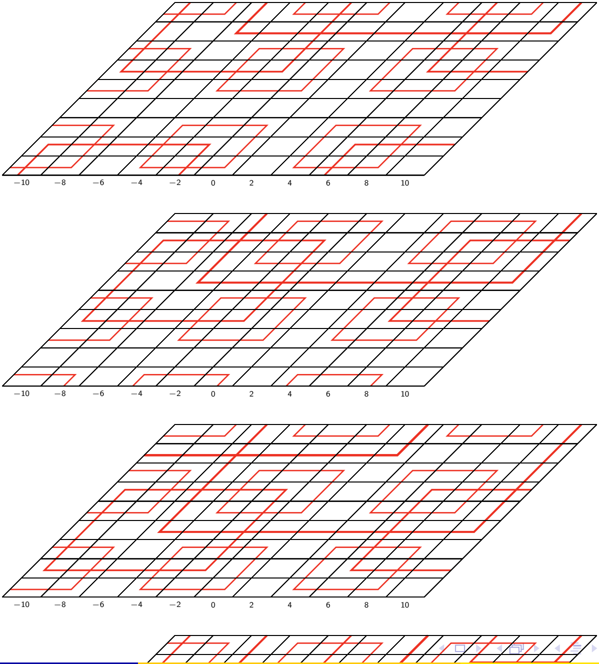
This is the Kari-Culik $\mathbb{Z}^3$ construction framework.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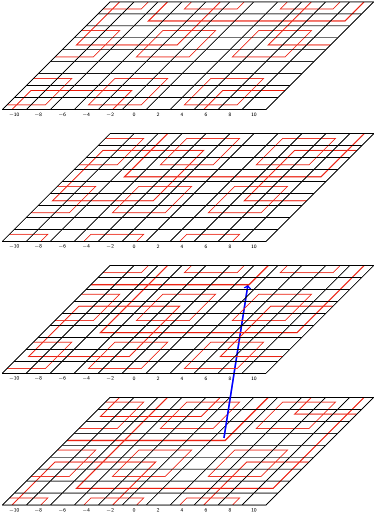
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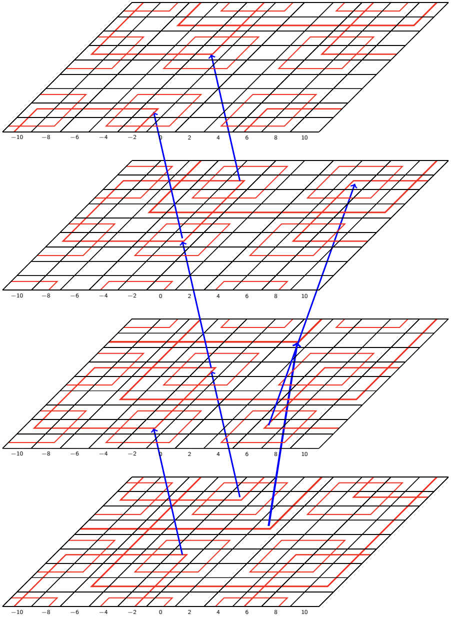
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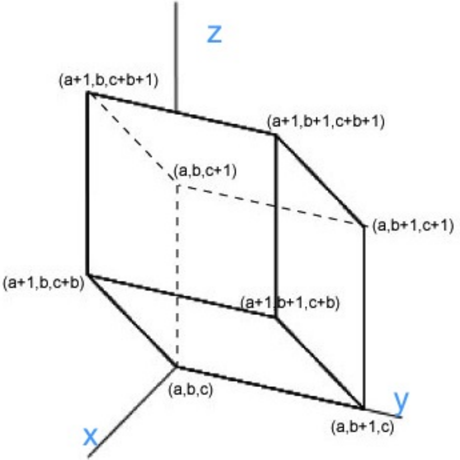
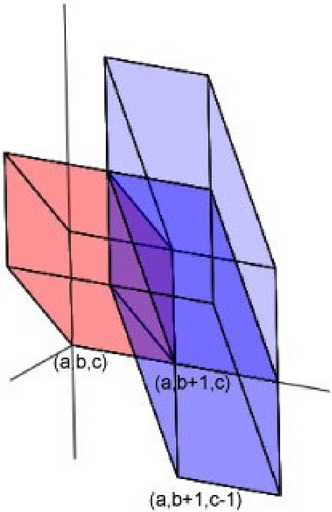
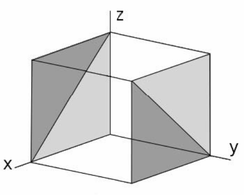
We can also define a Heisenberg Wang tile by using the fundamental domain of the quotient space H/G where H is the real-Heisenberg group (see for example Bertazzon) which coincides topologically with the unit cube $[0,1]^3$ with some identifications.
In particular, the action of Γ "skews" the cube as it is translated:



and the skewing causes each translate $C\gamma$ to intersect more than 6 other translates:



Thus Each H-Wang tile requires 8 labels:



Notice that the partitions are those induced by the action of $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ on the x, z-faces.