Turing degrees of limit sets of cellular automata

Alex Borello, Julien Cervelle and Pascal Vanier

LACL, Université Paris-Est Créteil

some monday in december, 2014

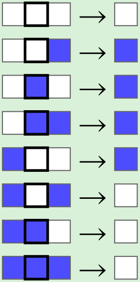
1. Some definitions and examples

2. Turing degrees of Limit Sets

Cellular automata (CAs)

A **finite** alphabet :

A **local function** f :



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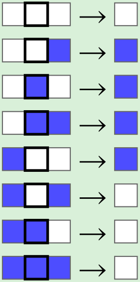

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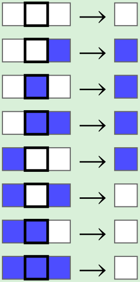

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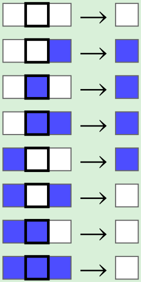
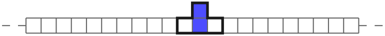


One time-step : apply uniformly the local function f, leading to a global function F



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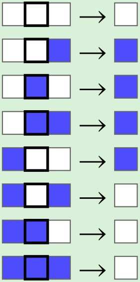

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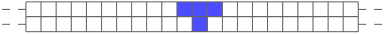

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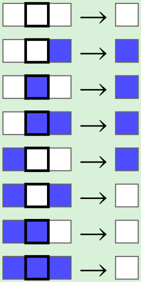
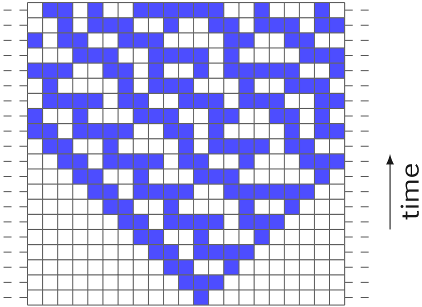


The corresponding space-time diagram:



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The limit set of a CA F is the set of configurations having an infinite sequence of preimages :

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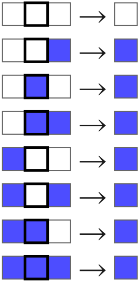

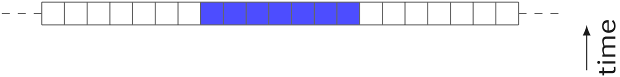

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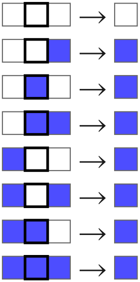

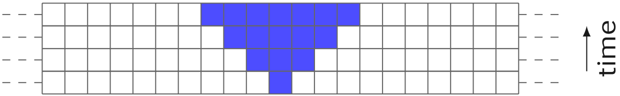

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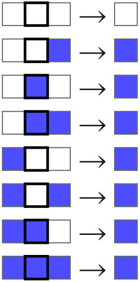

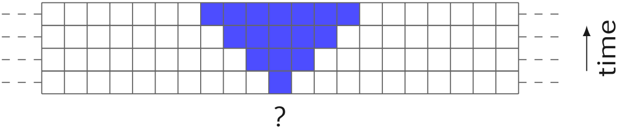

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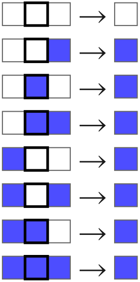

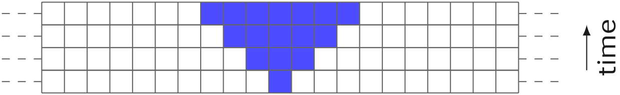

?

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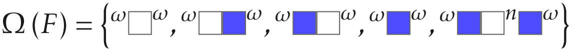


Limit sets are **subshifts**.

Limit sets

What do we now about them ?

Theorem [Kari 1992] It is **undecidable** to know if a limit set is a **singleton**.

Theorem [Kari 1994] **Any** non-trivial **property** on limit sets is **undecidable**.

Ok... let's look at them from a computability point of view then

Effectively closed sets

What kind of objects are limit sets ?

For each CA F there is a Turing Machine M_F which **halts only** on oracles **not in** $\Omega(F)$.

Definition An **effectively closed set** (Π_1^0 **class**) is a subset of $S \subseteq \Sigma^{\mathbb{Z}}$ for which there exists a Turing machine M_S such that

$$M \text{ halts with oracle } s \Leftrightarrow s \notin S$$

Theorem **Limit sets are effectively closed sets.**

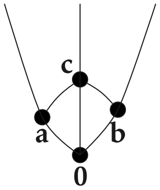
Turing degrees

How do we compare the computational power of subsets of $\Sigma^{\mathbb{Z}}$?

Definition (Turing degree)

- $x \leq_T y$ if there exists ϕ recursive which **computes x with oracle y**.
- $x \equiv_T y$ if $x \leq_T y$ and $x \geq_T y$.
- A Turing degree is an equivalence class for $\equiv_T$.

We note $\deg_T x$ the degree of x.



$\mathbf{0}$ is the degree of computable sequences

For $S \subseteq \Sigma^{\mathbb{Z}}$, we note $\deg_T S$ the set $\{\deg_T s \ : \ s \in S\}$

1. Some definitions and examples

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Limit sets and effectively closed sets

Can we realise any set of Turing degrees (of effectively closed sets) with a limit set ?

Limit sets always contain a uniform configuration ${}^{\omega}a^{\omega}$

Corollary For any CA F, $\mathbf{0} \in \deg_T \Omega(F)$

This is not the case for all effectively closed sets :

Theorem [Jockush-Soare 1972] There **exists effectively closed sets** containing **only non-computable points.**

But can we realise any set of Turing degrees containing $\mathbf{0}$?

Turing degrees of limit sets

Theorem For any effectively closed set $S \subseteq \Sigma^{\mathbb{Z}}$ there exists a CA F such that:

$$\deg_T \Omega(F) = \deg_T S \cup \{\mathbf{0}\}$$

The sets of Turing degrees of limit sets are exactly the sets of Turing degrees of effectively closed sets with a computable point.

Simulating Turing machines

Turing machines can be simulated by cellular automata.

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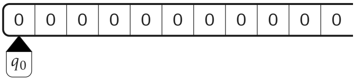

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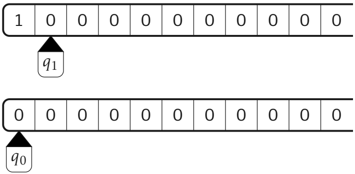

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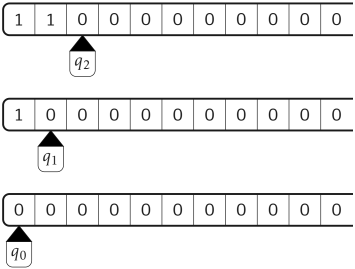

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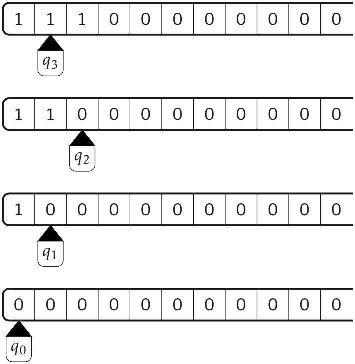

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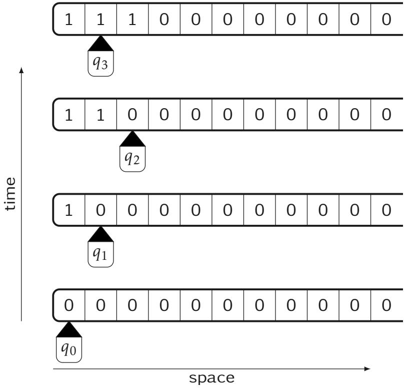

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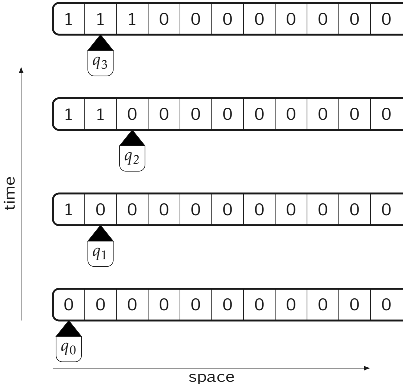

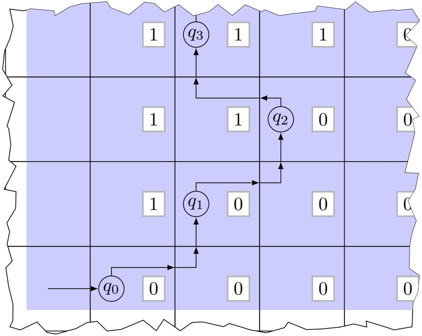

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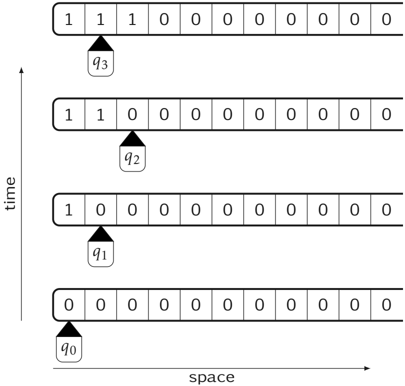

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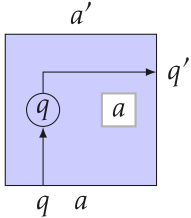

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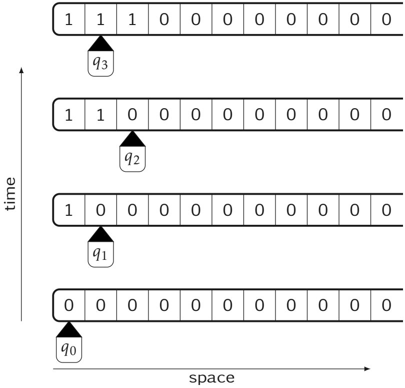

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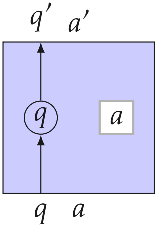

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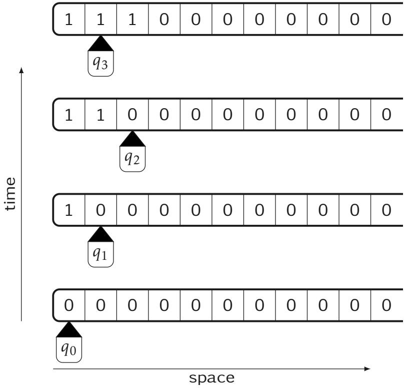

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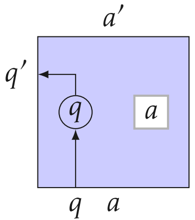

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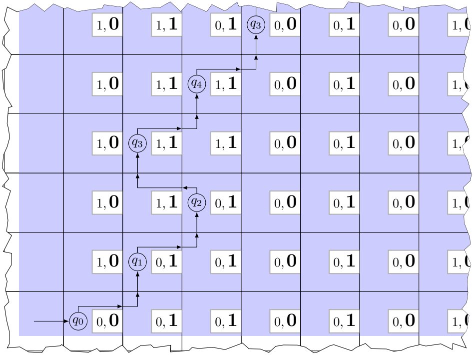

Infinite computations of M_S correspond to elements of S

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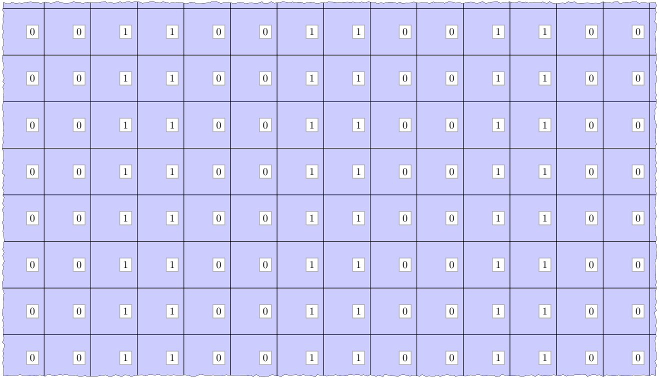


Not all configurations of the limit set correspond to a valid oracle.

Constructin' Constructin'

What we want:

For any machine M_S we want to construct a CA F_{M_S} such that

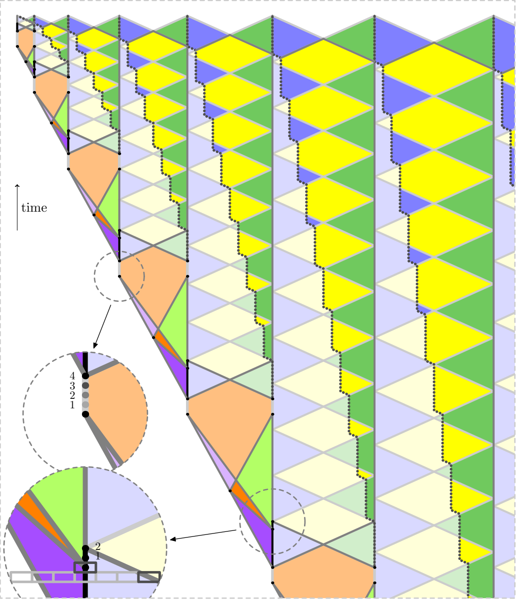
$$\deg_T S = \deg_T \Omega\left(F_{M_S}\right) \cup \{\mathbf{0}\}$$

What we need in order to get what we want:

- Limit sets containing only accepted oracles and computable configurations
- A construction ensuring we only have correct computations:
  - Always a head when there is a computation
  - The oracle needs to appear non-deterministically
  - The computation needs to have a starting point
  - Bad oracles need to be pruned from the limit set
  - No computation = computable

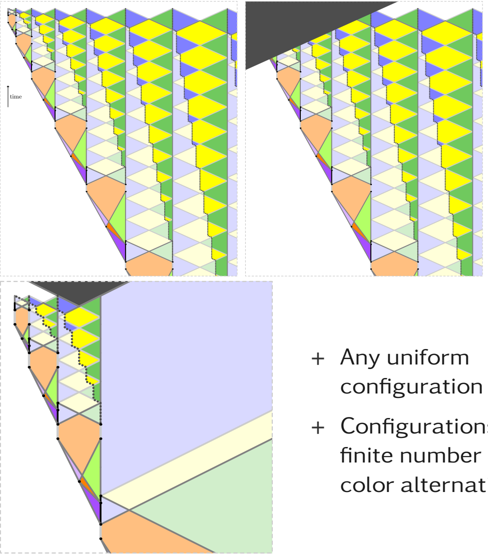
A self-vanishing sparse grid

- We need a grid to embed the computation
- We reverse the computation : only infinite computations have an infinite sequence of preimages
- Columns of squares of increasing size and number
- **Error** : make **killer state** appear.



Corresponding limit set

Any horizontal line of the following classes of configurations



+ Any uniform configuration
+ Configurations with a finite number of color alternations

Embedding computations in the grid

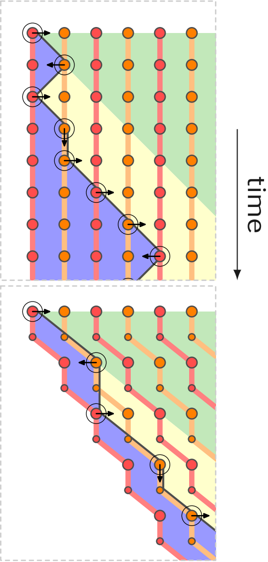
A "symbolic" space-time diagram of a Turing machine:

- Green : zone never reached by the head, no work tape.
- Blue : left of the head.
- Yellow : right of the head.

The same diagram but one out of two steps shifts the tape.

We **can embed** this **in the sparse grid** if the **Turing machine** is **reversible**.

The green zone can be detected from the grid without knowing the computation.



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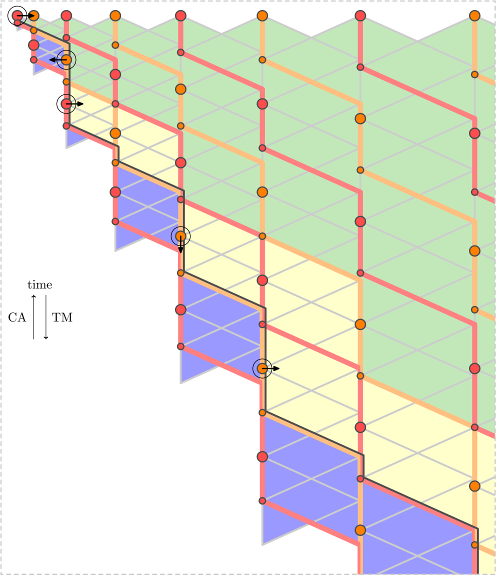

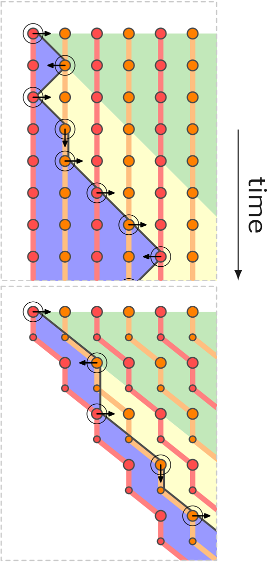

The computation itself

It is still not enough:

Remark What if the work tape inherited from the past is not well formed ?

Solution: Use a robust Turing machine.

Definition A robust Turing machine is a Turing machine that constantly rechecks all computations steps already done.

Theorem Any Turing machine can be "robustified" [Hooper 1966] in a reversible way [Kari-Ollinger 2008].

Consequence: If there is an error in the computation it should already have occured an infinity of times.

Conclusion

The only configurations containing something complicated in the oracle are parts of a grid containing a valid computation on this oracle.

Theorem For any effectively closed set $S \subseteq \Sigma^{\mathbb{Z}}$ there exists a CA F such that:

$$\deg_T \Omega(F) = \deg_T S \cup \{\mathbf{0}\}$$