Four-Cycle Free Graphs and Entropy Minimality

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Outline

- Entropy Minimality and Hom Shifts
- Mixing Conditions and Entropy Minimality
- The Space of 3-Colourings
- Four-cycle free graphs

Hom shifts

Given a finite undirected graph $\mathcal{H}$ without multiple edges, we can define a shift space $X_{\mathcal{H}} \subset \mathcal{H}^{\mathbb{Z}^d}$ such that symbols on adjacent vertices of $\mathbb{Z}^d$ form an edge in the graph $\mathcal{H}$.

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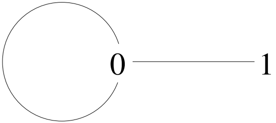
Examples:(Hard Square model)



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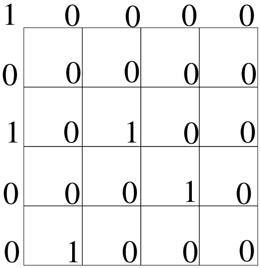

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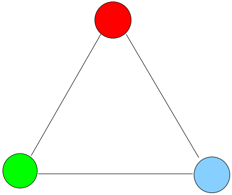

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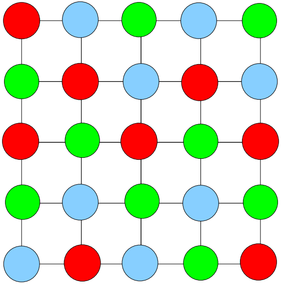

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$$h_{top}(X) := \lim_{n \longrightarrow \infty} \frac{\log |\mathcal{B}(X) \cap \mathfrak{A}^{\{1,2,\ldots,n\}^d}|}{n^d}.$$

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Theorem

A shift of finite type is entropy minimal if and only if every measure of maximal entropy is fully supported.

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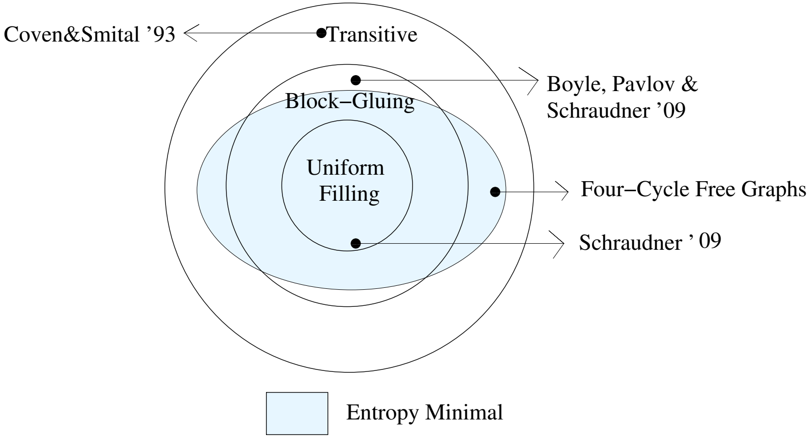
Theorem (Chandgotia '14)

If $\mathcal{H}$ is a four-cycle free graph then $X_{\mathcal{H}}$ is entropy minimal.

Mixing Conditions and Entropy Minimality



Height Functions and X_{C_3}

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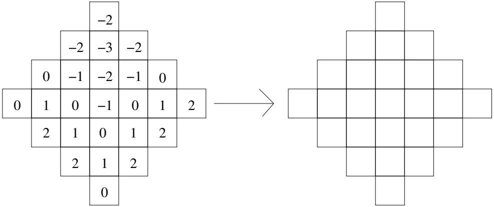

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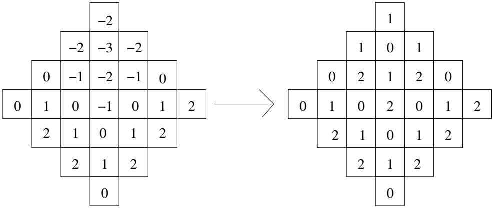

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The slope may be different in different directions.

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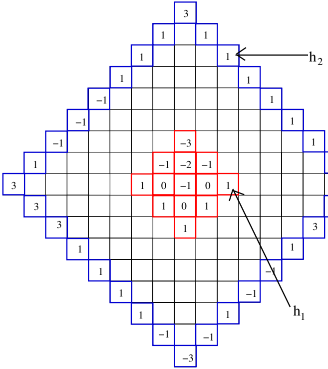

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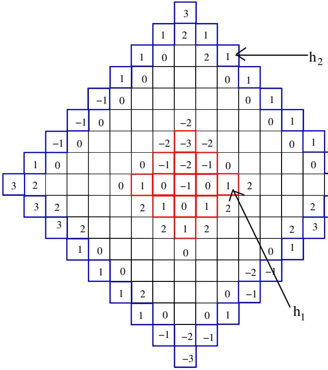

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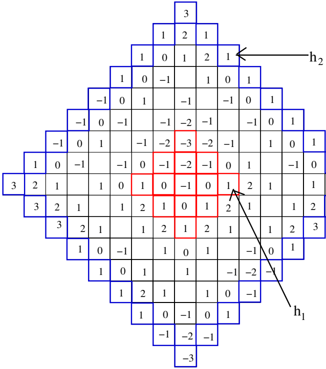

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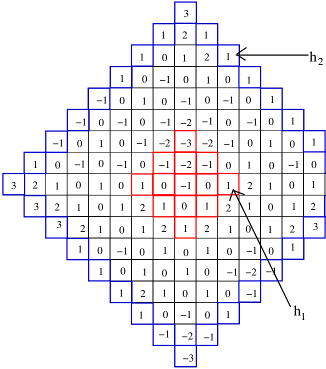

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What if C_3 is replaced by some other four-cycle free graph $\mathcal{H}$?

Universal Covers

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Let $\mathcal{H}$ be a graph without self-loops. A non-backtracking walk on $\mathcal{H}$ is a sequence of vertices $v_1, v_2, \ldots v_n \in \mathcal{H}$ such that $v_i \sim_{\mathcal{H}} v_{i+1}$ but $v_{i-1} \neq v_{i+1}$.

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Choose a vertex $u \in \mathcal{H}$. Denoted by $E_{\mathcal{H}}$, the universal cover of $\mathcal{H}$ is a tree where the vertex set is the set of all non-backtracking walks on $\mathcal{H}$ starting with u; two such walks are adjacent if one extends the other by a single step.

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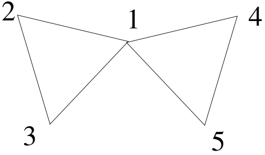
The universal cover of C_3 is $\mathbb{Z}$ (segments of the walks $1, 2, 3, 1, 2, 3, \ldots$ and $1, 3, 2, 1, 3, 2, \ldots$).

Universal Covers: An Example



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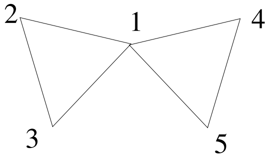

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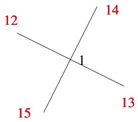

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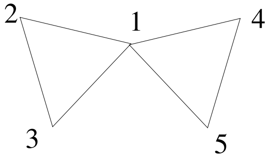

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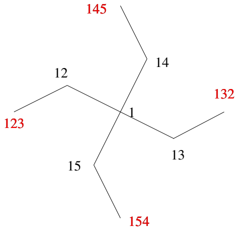

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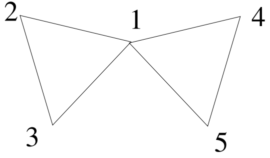

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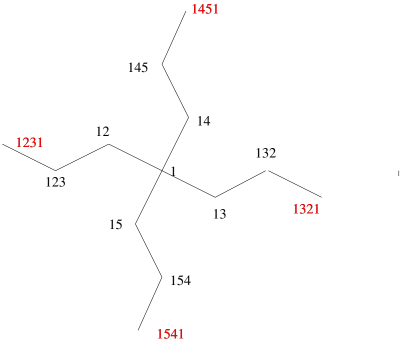

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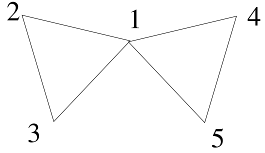

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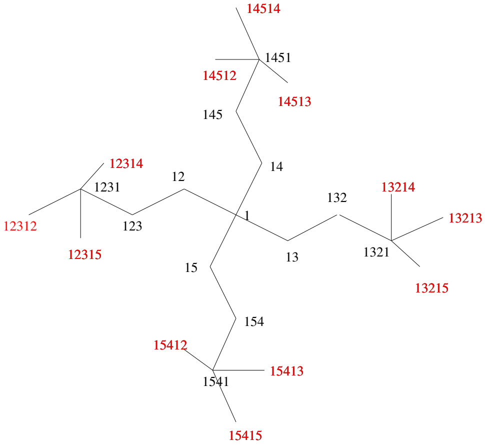

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The map is surjective. Given $x \in X_{\mathcal{H}}$, a lift of x is a configuration $\tilde{x} \in X_{E_{\mathcal{H}}}$ such that $\pi(\tilde{x}) = x$. There is a unique lift once we fix the lift at a single vertex.

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Given a configuration $x \in X_{\mathcal{H}}$, we can now construct a corresponding generalised height function

$$h_x : \mathbb{Z}^d \times \mathbb{Z}^d \longrightarrow \mathbb{Z}^+$$

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For any $\mathbb{Z}^d$-ergodic measure on $X_{\mathcal{H}}$, the subadditive ergodic theorem implies that the generalised slope exists (and is a constant) almost everywhere.

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The proof for the entropy minimality of X_{C_3} can be reworked to prove that $X_{\mathcal{H}}$ is entropy minimal for all four-cycle free graphs $\mathcal{H}$; the subadditivity has to be taken into account.

A Couple of Questions

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Conjecture: If $\mathcal{H}$ is any connected graph then $X_{\mathcal{H}}$ is entropy minimal. (d=2)

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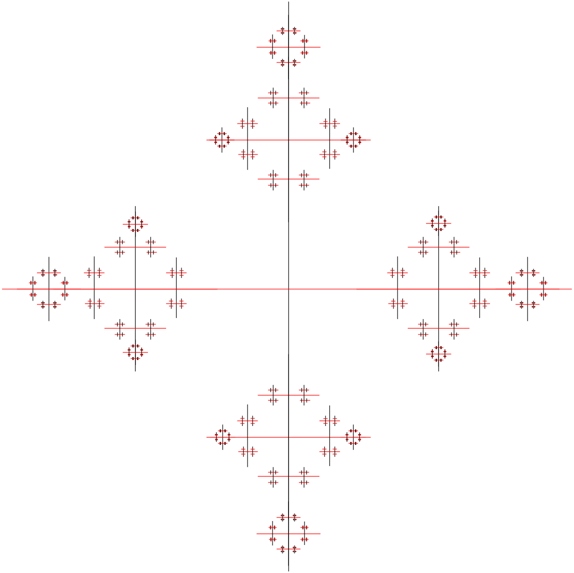
Question: What shift spaces are conjugate to $X_{\mathcal{H}}$ for some graph $\mathcal{H}$?



Thank You!