Automorphisms of symbolic dynamical systems

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- Kreiger's property (A) and the syntactic semigroup.
- $Aut(X, \sigma)$ acting on the syntactic semigroup and applications.

Example 1: Dyck shifts

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- Formally: The k-Dyck shift $D_k \subset \{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k\}^{\mathbb{Z}}$. Consider the monoid (semigroup with 1) with 0 generated by $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k$ with the relations $\alpha_k \beta_k = 1$ and $\alpha_k \beta_j = 0$ for $k \neq j$.Forbidden words: Those that simplify to 0.

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- Generalizations: Restrict the "scenery" to come from an SFT, replace $\{+1,-1\}$ by different subset of $\mathbb{Z}$ or $\mathbb{Z}^d$ or some other group $\mathbb{G}$ ...

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- Both have a certain "synchronization property" defined by Kreiger, called Property A.

Local coordinates and presynchronized points

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- Following [Fi], a point $x \in X$ is δ-presynchronized if for all $n \in \mathbb{Z}$, and $y \in X$ such that $d(T^n x, y) < \delta$ $[T^n x, y]_\epsilon \neq \emptyset$ and $[y, T^n x]_\epsilon \neq \emptyset$.

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- $x \in X$ is presynchronized if it is δ-presynchronized for some $\delta > 0$.

An equivalence relation on presynchronized periodic points

- Now suppose T is expansive and $p, q \in X$ are presynchronized periodic points, write $p \to q$ if there exists a presynchronized point z which is left-asymptotic to p and right asymptotic to q.

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- Examples: SFTs, transitive sofic shifts, synchronized subshifts, Dyck shift, ..

Property (A) of subshifts

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- For $j_- < j_+ \in \mathbb{Z}$, define the (j_-, j_+)-context of a point $x \in X$:

$$C(x, j_-, j_+) := \{(y_{(-\infty, j_)}, z_{(j_+, +\infty)}) \ : \ y_{(-\infty, j_)} x_{[j_-, j_+]} z_{(j_+, +\infty)} \in X\}$$

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- Claim: Property (A) is a conjugacy invaraint.

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- Let X be subshift with property (A).
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$$(ab\vec{c}aab\bar{b}c) \cdot (\bar{a}ab\vec{b}caa) = (ab\vec{c}\bar{a}abbcaa)$$

The homomorphism from $Aut(X, \sigma)$ to $Aut(M(X, \sigma))$

- Theorem [Kr2]: Any isomorphism $\phi : (X, \sigma) \to (Y, \sigma)$ between property (A) subshifts induces an isomorphism of semigroups $\phi^* : M(X, \sigma) \to M(Y, \sigma)$.

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- Application: Necessary and sufficient conditions for embedding SFTs.
- Exercise: For any $\mathbb{G}$-flow (X, T), $Aut(X, T)$ acts on the ergodic measures of maximal entropy.
- Observation: For the Dyck shift, and for the RWRS shift, the two ergodic measures of maximal entropy μ_+ and μ_- are both invariant for the action of the automorphism group.

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- Application: Necessary and sufficient conditions for embedding SFTs.
- Exercise: For any $\mathbb{G}$-flow (X, T), $Aut(X, T)$ acts on the ergodic measures of maximal entropy.
- Observation: For the Dyck shift, and for the RWRS shift, the two ergodic measures of maximal entropy μ_+ and μ_- are both invariant for the action of the automorphism group.
- To a periodic point can invariantly associate the local entropies with respect to ergodic measures of maximal entropy.

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