Automorphisms of symbolic dynamical systems

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December, 2014

Next topics

- Subactions, normal subgroups and associated representations
- Characteristic subsystems
- Finite index normal subgroups, finite orbits, and the associated representations
- Gyration and orbit-sign homomorphisms
- ~~Dimension representation~~

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- If (X, T) is a $\mathbb{G}$-SFT and $\mathbb{H}$ is finitely generated then $(X^+_{\mathbb{H}}, T)$ is a $\mathbb{G}/\mathbb{H}$-SFT.

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- A formula for the gyration function in terms of return vectors:

Theorem ([BK])

$$g(U) = \sum_{\hat{x} \in \hat{X}_U} R_U(\hat{x})$$

Proof of gyration formula:

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- Orbit-sign of a shift σ_n is zero.
- Gyration of the $\phi(x)_n = \phi(x)_n + 1 \mod L$ on $\{1, \dots, L\}^{\mathbb{Z}^d}$:

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Some results obtained using gyration and orbit-sign:

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- Given $m = 2^k q \in \mathbb{Z}$ where q is odd,

$$SG(U,T)(2^k q) := g(U,T)(2^k q) + \sum_{j=0}^{k-1} sign(U,T)(2^j q)$$

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- The observation extends to inert automorphisms of mixing $\mathbb{Z}$-SFTs [KRW1].

Stablizers and subsactions

The following lemma relates orbits of a sub-action to orbits of an action:

Lemma (Orbits of subactions Lemma)

Let $\mathbb{G}, \mathbb{G}_2$ *be topological groups,* T *a* $\mathbb{G}$ *action on* X, $\mathbb{H} \lhd \mathbb{G}_2$ *be a closed normal subgroup,* $\alpha \in \mathrm{Hom}(\mathbb{G}_2, \mathbb{G})$, *set*

$$\mathcal{J} = \{K \lhd \mathbb{G} \ : \ \alpha(\mathbb{H}) = K \cap A(\mathbb{G}_2)\},$$

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Proof.

For $x \in X$, $stab_{T(\alpha)}(x) = \mathbb{H}$ if and only if $stab_T(x) \cap \alpha(\mathbb{G}_2) = \alpha(\mathbb{H})$. Thus

$$X_{\mathbb{H},T(\alpha)} = \{x \in X \ : \ stab_{T(\alpha)}(x) = \mathbb{H}\} =$$

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□

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Where the sum is over all $K \lhd \mathbb{G}$ *such that* $\alpha(\mathbb{H}) < K$ *and* $\alpha(\mathbb{H}) = K \cap A(\mathbb{G}_2)$.

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Where the sum is over all $K \lhd \mathbb{G}$ *such that* $\alpha(\mathbb{H}) < K$ *and* $\alpha(\mathbb{H}) = K \cap A(\mathbb{G}_2)$. *Here* $\alpha^{-1} : (\mathbb{G}/K) \to (\mathbb{G}_2/\mathbb{H})$ *is the inverse of the homomorphism* $\alpha : (\mathbb{G}_2/\mathbb{H}) \to (\mathbb{G}/K)$ *induced by* α.

References

[BK] M. Boyle and W. Krieger, Periodic points and automorphisms of the shift, Trans. Amer. Math. Soc., 1987

[BLR] M. Boyle, D. Lind, and D. Rudolph, The automorphism group of a shift of finite type, Trans. Amer. Math. Soc. 1988

[Har] Y. Harman, Large semigroups of cellular automata, Ergodic Theory Dynam. Systems, 2012

[KR1] K. H. Kim and F. W. Roush, Williams's conjecture is false for reducible subshifts, J. Amer. Math. Soc, 1992

[KR2] K. H. Kim and F. W. Roush, The Williams conjecture is false for irreducible subshifts, Ann. of Math., 1999

[KRW1] K. H. Kim, F. W. Roush, and J. B. Wagoner, Automorphisms of the dimension group and gyration numbers, J. Amer. Math. Soc., 1992

[KRW2] K. H. Kim, F. W. Roush, and J. B. Wagoner, Characterization of inert actions on periodic points (I + II), Forum Math., 2000