Automorphisms of symbolic dynamical systems

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Recall: Topological Markov fields

For $W \Subset \mathbb{G}$:

$$\partial_W F := \left\{ g \in F \ : \ F^c \cap gW^{-1} \neq \emptyset \right\}$$

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Lemma (Existence of Markers, [Hoc])

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- For expansive flows, there is at least one invariant measure μ for which $h_{\mu}(X,\sigma) = h(X,\sigma)$. Such a measure is called a measure of maximal entropy (MME).

Proof of Existence of Markers ($\mathbb{Z}^d$-TMF with positive entropy)

Theorem (Shannon-McMillan-Breiman)

$$\forall \epsilon > 0 \ \lim_{k \to \infty} \mu \left(\left| \frac{1}{|B_k|} \log \mu([x]_{B_k}) - h_\mu(X, \sigma) \right| < \epsilon \right) = 1$$

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Lemma (Big overlap Lemma)

For any $v \in B_{k+m}$ there exists $w \in \mathbb{Z}^d$ such that

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- By SMB and the Big Overlap Lemma, the Marker property happens with high probability.

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Auxiliary result: Endomorphisms from (S,M)-Markers:

- Let $a^{(1)}, \dots, a^{(N)} \in \Sigma^M$ be compatible (S, M)-Markers for X.
- Define an endomorphism $\phi : X \to X$ by the following rule: "Replace every occurrence of $a^{(i)}$ in x by $a^{(1)}$".
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- It follows that the map $p \mapsto \phi_p$ is an embedding of S_N into $Aut(X, \sigma)$.

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Let (X,σ) be a $\mathbb{G}$-subshift which is a Topological Markov Field and suppose $h(X,\sigma) > 0$, then for any $n \in \mathbb{N}$ there exists finite sets $S \subset M \subset B \Subset \mathbb{G}$ and an N-tuple of compatible (S, M, B)-markers for X.

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The idea of the proof:
If you choose $x \in X$ randomly in a (suitable sense), then as $S, M, B \to \infty$ (in an appropriate sense) with high probability choosing $x|_{B \setminus int_W S}$ has N admissible completions which make up compatible (S, M, B)-markers.

Proof of the Existence of Markers Lemma, Part 1

Lemma ("Multiple completions Lemma")

Let μ be a σ-invariant, ergodic measure of positive entropy on X. Then for any $N \in \mathbb{N}$ and finite $W \subset X$

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Proof: Let

$$A_{g,M} := \left\{x \in X \ : \ x_{g^{-1}h} = x_h \text{ iff } h \notin M\right\},$$

If $\mu(A_{g,M}) > 0$ it follows from Poincaré recurrence that for μ-almost any $x \in A_{g,M}$, there is an infinite $V \subset \mathbb{G}$ so that $x \in \sigma_h A_{g,M}$ for all $h \in V$. Note that $\sigma_h^{-1} A_{g,M} = A_{h^{-1}gh,h^{-1}M}$ and that $A_{g,M_1} \cap A_{g,M_2} = \emptyset$ if $M_1 \neq M_2$. It follows that the elements $(h^{-1}gh)_{h \in V}$ are all distinct, so $|qstab(x)| = \infty$.

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- It follows that for sufficiently big $B \Subset \mathbb{G}$, these completions of $x|_{B \setminus int_W S}$ will be (S, M, B)-Markers.