Automorphisms of symbolic dynamical systems

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- By inspecting the action of $\langle \phi_1, \phi_2, \phi_3 \rangle$ on the point $\ldots 000 * 000 \ldots$ we see that it generates a group isomorphic to the free product $C_2 * C_2 * C_2$.

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- $h(X,\sigma)$ is invariant under isomorphism, and more generally decreases under factor maps.

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- Say that a subshift $X \subset A^{\mathbb{G}}$ is a topological Markov field (TMF) with respect to W or "has the Markov property with respect to W": If for every $F \subset \mathbb{G}$ and any $x, y \in X$ if $x|_{\partial_W F} = y|_{\partial_W F}$ then there exists (a unique) $z \in X$ such that $z|_F = x|_F$ and $z|_{F^c} = y|_{F^c}$.

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- Exercise: Any $\mathbb{Z}$-subshift which is a TMF is also an SFT. Note: We did not restrict F to be a finite set in the definition above.
- Fact: For any $d \geq 2$ there are $\mathbb{Z}^d$-subshifts which are TMF but not of finite type [CM],[Hoc].

Strongly Irreducible subshifts

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- For the full shift with $\mathbb{G} = \mathbb{Z}$ - due to Hedlund [H].

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- Based on construction of automorphisms using Markers.
- Under the same assumptions $Aut(X, \sigma)$ embeds the direct sum of every countable collection of finite groups.
- More assumptions are needed to embed free groups
- Different sorts of "Markers" are used in the proof of many important results in ergodic theory and in topological dynamics.

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Lemma (Existence of Markers, [Hoc])

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