Automorphisms of symbolic dynamical systems

Tom Meyerovitch

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Topological dynamical system, $G$-flows

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- A $G$-topological dynamical system (TDS) or $G$-flow is a pair $(X, T)$ where $X$ is a compact (Hausdorff) topological space and $T \in \text{Hom}(G, \text{Homeo}(X))$ is homomorphism of the group $G$ into the group of homeomorphisms.
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(\(X, T\)) is topologically isomorphic or topologically conjugate to
(\(Y, S\)) (denoted \((X, T) \cong (Y, S)\)) if there exists a homeomorphism
\(\Phi : X \to Y\) so that \(\Phi \circ T_g = S_g \circ \Phi\) for all \(g \in G\).
Isomorphism, factor maps, subsystems, subactions

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- \((Y, S)\) is a \textbf{topological factor} of \((X, T)\) if there exists a continuous surjective \(\Phi : X \to Y\) so that \(\Phi \circ T_g = S_g \circ \Phi\) for all \(g \in G\).
• \((X, T)\) is **topologically isomorphic** or **topologically conjugate** to \((Y, S)\) (denoted \((X, T) \cong (Y, S)\)) if there exists a homeomorphism \(\Phi : X \to Y\) so that \(\Phi \circ T_g = S_g \circ \Phi\) for all \(g \in G\).

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• A **subsystem** of \((X, T)\) is a \(G\)-flow \((Y, T)\) where \(Y \subset X\) is closed (topologically) and \(T\)-invariant \((T_g(Y) = Y\) for all \(g \in G\)).
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If \(\alpha \in \text{Hom}(H, G)\) is a group homomorphism and (\(X, T\)) is a \(G\)-flow, we can get an \(H\)-flow (\(X, T^{(\alpha)}\)) by \(T^{(\alpha)}_h := T^{(\alpha(h))}.\) When \(\alpha\) is injective, we say (\(X, T^{(\alpha)}\)) is a subaction of (\(X, T\)).
Recall that a $\mathbb{G}$-action $(X, T)$ is \textit{expansive} if there exists $\epsilon > 0$ so that for any distinct $x, y \in X$ there exists $g \in \mathbb{G}$ so that $d(T_g x, T_g y) > \epsilon$. 

An example for an expansive $\mathbb{Z}$-action: A hyperbolic automorphism of the torus $\mathbb{R}^2/\mathbb{Z}^2$, $T(x, y) = (y, x+y)$ is an expansive $\mathbb{Z}$-action.

Obviously, if a $\mathbb{G}$-flow $(X, T)$ has an expansive subaction, then $(X, T)$ is itself expansive.

Mañé's theorem [Man]: If $\mathbb{Z}$ acts on compact metric $X$ via an expansive $T$, then $X$ has finite topological dimension. Furthermore, if $(X, T)$ is also minimal (has no proper subsystems) then $X$ has zero topological dimension.
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$$[a]_F := \{ y \in X : y|_F = a \}, a \in \Sigma^F, F \subset \subset G.$$
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Some examples for $G$-SFTs

- Take a finite symmetric $S \subset G \setminus \{1\}$, and consider the Cayley graph $G_S$ whose vertex set is $G$. 
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- Take a finite symmetric $S \subset G \setminus \{1\}$, and consider the **Cayley graph** $G_S$ whose vertex set is $G$.
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Some examples for $\mathbb{G}$-SFTs

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- For any $n \in \mathbb{N}$ the collection of $n$-colorings of $G_S$ is an SFT.
- The collection of ordered perfect matchings in $G_S$ can be viewed as an SFT ("dimer models").
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The automorphism group of a $G$-flow

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Exercise: For a countable group $G$, the automorphism group of an expansive $G$-action is always finite or countable.
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(Some of the) Big questions about $Aut(X, \sigma)$

For a given subshift or class of subshifts, what can we say about $Aut(X, \sigma)$ as a group? Almost-commutative? Almost-nilpotent? Amenable? Sofic? Finitely generated? What is the center? What are the subgroups? What are the quotients? What are the irreducible representations? What are the finite dimensional representations?...

What do properties of the group $G$ say about properties of $Aut(X, \sigma)$ and vice versa? What do dynamical properties of $(X, \sigma)$ say about group-theoretic properties of $Aut(X, \sigma)$? How does $Aut(X, \sigma)$ act on $(X, \sigma)$? What are the $Aut(X, \sigma)$-invariant closed subsets? How does $Aut(X, \sigma)$ act on $\sigma$-invariant closed sets? How does $Aut(X, \sigma)$ act on $\sigma$-invariant measures? What are the $Aut(X, \sigma)$-invariant measures? $Aut(\{1, 2, 3\} \mathbb{Z}, \sigma) \sim = Aut(\{1, 2\} \mathbb{Z}, \sigma)$?
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$\text{Aut}({1, 2, 3}^\mathbb{Z}, \sigma) \cong \text{Aut}({1, 2}^\mathbb{Z}, \sigma)$?
Rule of thumb

If the dynamical system \((X,\sigma)\) is "sufficiently chaotic", \(\text{Aut}(X,\sigma)\) is "a big group" in an algebraic sense.

If the dynamical system \((X,\sigma)\) is "sufficiently degenerate", \(\text{Aut}(X,\sigma)\) is "a small group".

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Exercise: $\text{Aut}(X, T) = C(T, G)$, inside the group $\text{Homeo}(X)$. 

The Center of $G$: $Z(G) := C(G)$. 

Exercise: If $(X, T)$ is a $G$-flow and $T$ is a faithful action, then $Z(G)$ embeds in $\text{Aut}(X, T)$. 

If $G$ acts on $X$ via $T$, $\text{stab}_T(x) := \{g \in G : T^g(x) = x\}$. 

Exercise: $\text{stab}_T(T^g(x)) = g[\text{stab}_T(x)]g^{-1}$. 

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Reminder: Basic notions about groups

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Semi-direct products and wreath-products

For a group $H$ we denote by $\text{Aut}(H)$ the automorphisms of the group $H$ (bijective self-homomorphisms of $H$)

The wreath product $H \rtimes G$ of $H$ and $G$: This is the semi-direct product of $H$ and $G$ with respect to the shift-action.

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If $G$ acts on $H$ by $\alpha \in \text{Hom}(G, \text{Aut}(H))$ then the semi-direct product $G \rtimes_\alpha H$ is the group whose elements are $G \times H$ with multiplication rule:

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Let $\mathcal{O}$ denote the (finite) set of $\mathbb{G}$-orbits in $X$, and $S(\mathcal{O})$ denote the group of permutations of $\mathcal{O}$.
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In the above situation, $\text{Aut}(X, \sigma)$ is isomorphic as a group to the permutational wreath product $S(O) \wr \psi Z(G)$, where $\psi : S(O) \rightarrow O$ is the obvious action.
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Residually finite groups and finite orbits

A group $G$ is residually finite if for each $1 \neq g \in G$, there is a normal subgroup $H$ of finite index with $g \notin H$.

Theorem (Ceccherini-Silberstein and Coornaert [CSC]): If $G$ is residually finite and $(X, \sigma)$ is a strongly irreducible $G$-SFT which contains a point $x$ with a finite orbit, then the points with finite orbits are dense in $X$.

Theorem: (see [BLR], [CSC]): If $G$ is residually finite and $(X, \sigma)$ is a $G$-SFT with DPP then $\text{Aut}(X, \sigma)$ is residually finite.

Corollary: In the above situation, $\text{Aut}(X, \sigma)$ does not contain a divisible subgroup: for any $\phi \in \text{Aut}(X, \sigma) \backslash \{\text{id}\}$ there exists $n \in \mathbb{N}$ such that the equation $\psi^n = \phi$ has no solution $\psi \in \text{Aut}(X, \sigma)$. 

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Residually finite groups and finite orbits

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- Theorem (Ceccherini-Silberstein and Coornaert [CSC]): If $\mathbb{G}$ is residually finite and $(X, \sigma)$ is a strongly irreducible $\mathbb{G}$-SFT which contains a point $x$ with a finite orbit, then the points with finite orbits are dense in $X$ (($X, \sigma$) has dense periodic points (DPP) ).

- Theorem: (see [BLR],[CSC]): If $\mathbb{G}$ is residually finite and $(X, \sigma)$ is a $\mathbb{G}$-SFT with DPP then $\text{Aut}(X, \sigma)$ is residually finite.
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**Theorem:** (see [BLR],[CSC]): If $G$ is residually finite and $(X, \sigma)$ is a $G$-SFT with DPP then $\text{Aut}(X, \sigma)$ is residually finite.

**Corollary:** In the above situation, $\text{Aut}(X, \sigma)$ does not contains a divisible subgroup: For any $\phi \in \text{Aut}(X, \sigma) \setminus \{id\}$ there exists $n \in \mathbb{N}$ such that the equation $\psi^n = \phi$ has no solution $\psi \in \text{Aut}(X, \sigma)$. 

Residually finite groups and finite orbits
On the automorphism group of $\mathbb{Z}$-SFTs

[BLR], extending [H] Let $(X, \sigma)$ be an uncountable $\mathbb{Z}$-SFT. Then $\text{Aut}(X, \sigma)$:

- Embeds any finite group.
- Embeds the direct sum of every countable collection of finite groups.
- Embeds a free product of any number of 2-element groups, hence a free group on a countable number of generators.
- Embeds $\text{Aut}(\{1, \ldots, n\}^\mathbb{Z}, \sigma)$ for all $n$. [KR]
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If $(X, \sigma)$ is irreducible then the center of $Aut(X, \sigma)$ generated by $\sigma$. [R]
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References


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Let’s prove that $Aut(X, \sigma)$ embeds the free product of cyclic groups:

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- For $j \in \{1, 2, 3\}$ define $\phi_j \in Aut(X, \sigma)$ to be the automorphism which “swaps $sj$ with $s0$” for $s \not\in \{0, j\}$.
- By inspecting the action of $\langle \phi_1, \phi_2, \phi_3 \rangle$ on the point $\ldots 000 \ast 000 \ldots$ we see that it generates a group isomorphic to the free product $C_2 \ast C_2 \ast C_2$. 
Coming up next...

- Topological Markov Fields.
- Amenability and topological entropy.
- The Marker method for $\mathbb{Z}^d$-SFTs and for $G$-SFTs.