Automorphisms of shifts with low complexity

Bryna Kra
Northwestern University

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Automorphisms of shift systems

(X, σ) is a shift system.

- $\mathcal{A}$ is a finite alphabet.
- $\sigma\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is the left shift $(\sigma x)(n) = x(n+1)$.
- $X \subset \mathcal{A}^{\mathbb{Z}}$ is closed and invariant under σ.

An **automorphism** of (X, σ) is a homeomorphism $\phi\colon X \to X$ such that

$$\sigma \circ \phi = \phi \circ \sigma.$$

$\mathrm{Aut}(X)$ denotes the group of all automorphisms of X.

Thus $\langle \sigma \rangle$ lies is in the center of $\mathrm{Aut}(X)$.

Question

What can be said about $\mathrm{Aut}(X)$?

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Automorphisms are sliding block codes

A map $\phi\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is a **sliding block code** if there exists $R \in \mathbb{N}$ such that for all $x \in X$, the value of $(\phi x)(0)$ depends only on the window

$$x(-R), x(-R+1), \ldots, x(0), \ldots, x(R-1), x(R).$$

Curtis, Hedlund & Lyndon Theorem, 1969

For a shift (X, σ), any element of $\mathrm{Aut}(X)$ is a sliding block code.

It follows that $\mathrm{Aut}(X)$ is countable. But....

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Lots of automorphisms

Hedlund: $\mathrm{Aut}(\mathcal{A}^{\mathbb{Z}})$ contains:

- any countably generated free group,
- any finite free product of cyclic groups,
- every finite group.

Ryan: The center of $\mathrm{Aut}(\mathcal{A}^{\mathbb{Z}})$ is exactly $\langle \sigma \rangle$.

Boyle, Lind & Rudolph: For a mixing SFT, $\mathrm{Aut}(X)$ contains:

- free group on two generators,
- direct sum of countably many copies of $\mathbb{Z}$,
- direct sum of any countable collection of finite groups.

In these cases,

- $\mathrm{Aut}(X)$ is not finitely generated, is not amenable.
- Positive topological entropy.

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Very low complexity

The **block complexity function** $P_X\colon X \to \mathbb{N}$ counts the number of words of length n in the language of X.

If $P_x(n)$ is the complexity of a fixed $x \in X$, then

$$\sup_{x \in X} P_x(n) \leq P_X(n).$$

Morse & Hedlund Theorem

The word $x \in \mathcal{A}^{\mathbb{Z}}$ is periodic if and only if there exists $n \in \mathbb{N}$ such that $P_x(n) \leq n$.

- If there exists $n \in \mathbb{N}$ with $P_X(n) \leq n$, then X is periodic (every element has finite order).
- If $P_X(n)/n \to 0$ as $n \to \infty$, then $|\mathrm{Aut}(X)| < \infty$.
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Linear complexity:

$$\limsup_{n\to\infty} \frac{P_x(n)}{n} < k.$$

Includes:

- Sturmian shifts: $P_X(n) = n + 1$ for all n.
- Quasi-sturmian shifts: $n + 1 \leq P_X(n) \leq n + c$ for all n and for some constant $c \geq 1$.
- Rauzy-Arnoux shifts: minimal shifts with $P_X(n) = 2n + 1$ for all n.
- Many morphic systems, such as Thue-Morse substitution.
- Interval Exchange Transformations

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Shift systems with linear growth (general case)

A group G is **virtually** H if G contains H as a subgroup of finite index.

Theorem (Cyr and Kra)

If (X, σ) is a shift system such that there exists $k \in \mathbb{N}$ with

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then every finitely generated subgroup of $\mathrm{Aut}(X)$ is virtually $\mathbb{Z}^d$ for some $d < k$.

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(X, σ) is **transitive** if there exists $x \in X$ such that

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If (X, σ) is a transitive shift system such that

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(X, σ) is **minimal** if for every $x \in X$,

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If (X, σ) is a minimal shift system such that there exists $k \in \mathbb{N}$ with

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then $\mathrm{Aut}(X)/\langle\sigma\rangle$ is finite and $|\mathrm{Aut}(X)/\langle\sigma\rangle| < k$.

- Growth assumption is only $\liminf$.
- Answers a question of Salo and Törmä (for linearly recurrent).
- Another proof by Donoso, Durand, Maass, & Petite.

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Ideas in the proof: full group

The **full group** $[\sigma]$ of (X, σ) is the subgroup of homeomorphisms from $X \to X$ that are orbit preserving:

$$[\sigma] = \{\psi \in \mathrm{Hom}(X) \colon \psi(x) \in \mathcal{O}(x) \text{ for all } x \in X\}.$$

- For $\psi \in [\sigma]$, there is a function $k_\psi \colon X \to \mathbb{Z}$ such that

$$\psi(x) = \sigma^{k_\psi(x)}(x) \qquad \text{for all } x \in X.$$

- $\mathrm{Aut}(X) \cap [\sigma]$ is the centralizer of σ in $[\sigma]$.
- $\mathrm{Aut}(X) \cap [\sigma]$ is abelian.

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Assume $\limsup_{n\to\infty} \frac{P_x(n)}{n} < k$ and X has dense aperiodic points.

Then show:

- There exist $x_1, \ldots, x_{k-1} \in X$ such that

$$X = \overline{O}(x_1) \cup \overline{O}(x_2) \cup \ldots \cup \overline{O}(x_{k-1}).$$

- $\mathrm{Aut}(X)$ is locally a group of polynomial with rate $< k - 1$.
  Uses theorem of Cassaigne: a shift satisfies $P_X(n) = O(n)$ if and only if $P_X(N+1) - P_X(n)$ is bounded.

Use this to conclude:

- $\mathrm{Aut}(X) \cap [\sigma] \cong \mathbb{Z}^d$ for some $d < k$.
- $\mathrm{Aut}(X)/\mathrm{Aut}(X) \cap [\sigma]$ is finite.
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- There exist $x_1, \ldots, x_{k-1} \in X$ such that
$$X = \overline{O}(x_1) \cup \overline{O}(x_2) \cup \ldots \cup \overline{O}(x_{k-1}).$$
- $\mathrm{Aut}(X)$ is locally a group of polynomial with rate $< k - 1$.
  Uses theorem of Cassaigne: a shift satisfies $P_X(n) = O(n)$ if and only if $P_X(N+1) - P_X(n)$ is bounded.

Use this to conclude:

- $\mathrm{Aut}(X) \cap [\sigma] \cong \mathbb{Z}^d$ for some $d < k$.
- $\mathrm{Aut}(X)/\mathrm{Aut}(X) \cap [\sigma]$ is finite.
- In particular, $Aut(X)$ is virtually $\mathbb{Z}^d$.

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Ideas in the proof: transitive to general

Transitive shifts

- Any automorphism is determined by a transitive point.
- Analyze dense points to conclude

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- Orbit preserving automorphisms act like powers of the shift on transitive points, and so are elements of $\langle \sigma \rangle$.

General shifts

- Any element of $\mathrm{Aut}(X)$ preserves closure of aperiodic points.
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Assume $\liminf_{n\to\infty} \frac{P_X(n)}{n} < k$ and X is aperiodic and minimal.

- There are at most $k-1$ distinct elements of one-sided equivalence classes of $x \in X$ that are non uniquely left extendable.
- Use block codes to count.

Inspired by an argument of Boshernitzan...

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Number of ergodic measures

Theorem (Boshernitzan, 1984)

A minimal shift system (X, σ) such that there exists $k \in \mathbb{N}$ with

$$\liminf_{n\to\infty} P_X(n) - kn = -\infty,$$

has at most $k - 1$ invariant, ergodic probability measures.

A k-interval exchange has at most $k - 1$ ergodic measures.

(Weaker than Veech, Katok bound of $\lfloor k/2 \rfloor$.)

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- Is minimality needed?
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Generic measures

X is a compact metric space

μ is a Borel measure on X

$T: X \to X$ measurable map preserving μ

A point $x \in X$ is a **generic point for the measure μ** if for every continuous, compactly supported function $f: X \to \mathbb{R}$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x) = \int f \, d\mu.$$

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A bound on the number of generic measures

Theorem (Cyr and Kra)

If (X, σ) is a shift system and there exists $k \in \mathbb{N}$ such that

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there are at most $k-1$ nonatomic generic measures for (X, σ).

- Analog of Boshernitzan's result for generic measures.
- No minimality assumption.
- The result is sharp: there exists a minimal shift system with linear growth rate of k and exactly k ergodic measures.
- Proof uses counting arguments for non-preperiodic points.

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An application to Interval Exchange Transformations

Theorem (Chaika and Masur, 2014)

There exists an interval exchange transformation with a generic, but not ergodic, measure.

They ask: how many generic measures can you have?

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