Specification properties and intrinsic ergodicity for subshifts

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- Today: studying specification properties on $\mathbb{Z}$ subshifts

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- In fact, any mixing SFT has (strong) specification for some N

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- However, specification is incredibly restrictive!

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  - Example: all β-shifts with constant mistake function $g(n) = 1$

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- Answer: Neither one does! (independently answered by Kwietniak-Oprocha-Rams and Pavlov)

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- "Phase transition" somewhere btwn. $g = 1$ and $g = 4$
- Open question: does low enough $g(n)$ preclude multiple MMEs completely?

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  - (No restrictions on infinite runs)

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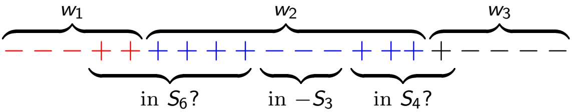

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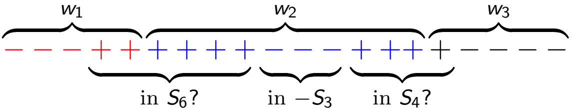

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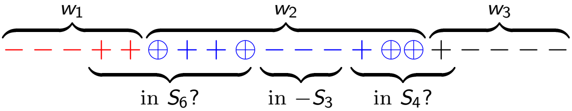

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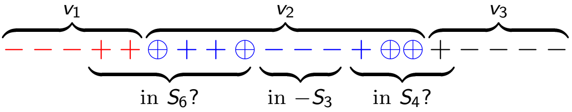

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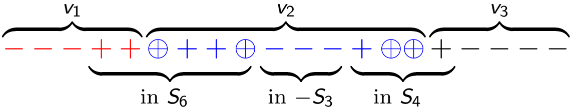

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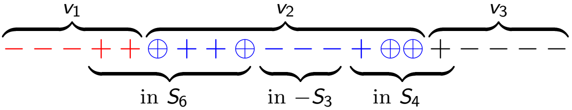

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    - Then, uniform Bernoulli measures on $\{1, \ldots, N\}$ and $\{-1, \ldots, -N\}$ are MMEs
- To show this, need to know: how small can we make sets S_n?

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Thanks for listening!