On intrinsic ergodicity of shift spaces with (almost) weak specification

Dominik Kwietniak
joint work with
Piotr Oprocha (AGH), Michał Rams (IM PAN)

JAGIELLONIAN
UNIVERSITY
IN KRAKOW

Santiago de Chile, December 18, 2014


Killing Two Birds......
BOINK!
BOINK!
with One Stone.
Reading in Paradise http://tayteh.blogspot.com

Example(s)

There exists a family of shift spaces which contains:

---

[1] Proved independently by Ronnie Pavlov.

Example(s)

There exists a family of shift spaces which contains:

- A shift space with almost specification and multiple MMEs[1].

[1] Proved independently by Ronnie Pavlov.

Example(s)

There exists a family of shift spaces which contains:

- A shift space with almost specification and multiple MMEs[1].
- A shift space with weak specification and multiple MMEs[1].

[1] Proved independently by Ronnie Pavlov.

Example(s)

There exists a family of shift spaces which contains:

- A shift space with almost specification and multiple MMEs[1].
- A shift space with weak specification and multiple MMEs[1].
- A shift space with almost specification but without weak specification.

[1] Proved independently by Ronnie Pavlov.

Example(s)

There exists a family of shift spaces which contains:

- A shift space with almost specification and multiple MMEs[1].
- A shift space with weak specification and multiple MMEs[1].
- A shift space with almost specification but without weak specification.
- An intrinsically ergodic shift space X and its factor Y with multiple MMEs such that there are Climenhaga-Thompson decompositions for X and Y, respectively

$$\mathcal{B}(X) = \mathcal{C}^{\mathsf{p}}_{\mathsf{X}} \cdot \mathcal{G}_X \cdot \mathcal{C}^{\mathsf{s}}_{\mathsf{X}}$$
$$\mathcal{B}(Y) = \mathcal{C}^{\mathsf{p}}_{\mathsf{Y}} \cdot \mathcal{G}_Y \cdot \mathcal{C}^{\mathsf{s}}_{\mathsf{Y}}$$

with $h(\mathcal{G}_X) > h(\mathcal{C}^{\mathsf{p}}_{\mathsf{X}} \cup \mathcal{C}^{\mathsf{s}}_{\mathsf{X}})$ and $h(\mathcal{G}_Y) < h(\mathcal{C}^{\mathsf{p}}_{\mathsf{Y}} \cup \mathcal{C}^{\mathsf{s}}_{\mathsf{Y}})$.

[1] Proved independently by Ronnie Pavlov.

Example(s)

There exists a family of shift spaces which contains:

- A shift space with almost specification and multiple MMEs[1].
- A shift space with weak specification and multiple MMEs[1].
- A shift space with almost specification but without weak specification.
- An intrinsically ergodic shift space X and its factor Y with multiple MMEs such that there are Climenhaga-Thompson decompositions for X and Y, respectively

$$\mathcal{B}(X) = \mathcal{C}^{\mathsf{p}}_{\mathsf{X}} \cdot \mathcal{G}_X \cdot \mathcal{C}^{\mathsf{s}}_{\mathsf{X}}$$
$$\mathcal{B}(Y) = \mathcal{C}^{\mathsf{p}}_{\mathsf{Y}} \cdot \mathcal{G}_Y \cdot \mathcal{C}^{\mathsf{s}}_{\mathsf{Y}}$$

with $h(\mathcal{G}_X) > h(\mathcal{C}^{\mathsf{p}}_{\mathsf{X}} \cup \mathcal{C}^{\mathsf{s}}_{\mathsf{X}})$ and $h(\mathcal{G}_Y) < h(\mathcal{C}^{\mathsf{p}}_{\mathsf{Y}} \cup \mathcal{C}^{\mathsf{s}}_{\mathsf{Y}})$.

[1]Proved independently by Ronnie Pavlov.

Example(s)

There exists a family of shift spaces which contains:

- A shift space with almost specification and multiple MMEs[1].
- A shift space with weak specification and multiple MMEs[1].
- A shift space with almost specification but without weak specification.
- An intrinsically ergodic shift space X and its factor Y with multiple MMEs such that there are Climenhaga-Thompson decompositions for X and Y, respectively

$$\mathcal{B}(X) = \mathcal{C}_{\mathsf{X}}^{\mathsf{p}} \cdot \mathcal{G}_X \cdot \mathcal{C}_{\mathsf{X}}^{\mathsf{s}}$$
$$\mathcal{B}(Y) = \mathcal{C}_{\mathsf{Y}}^{\mathsf{p}} \cdot \mathcal{G}_Y \cdot \mathcal{C}_{\mathsf{Y}}^{\mathsf{s}}$$

  with $h(\mathcal{G}_X) > h(\mathcal{C}_{\mathsf{X}}^{\mathsf{p}} \cup \mathcal{C}_{\mathsf{X}}^{\mathsf{s}})$ and $h(\mathcal{G}_Y) < h(\mathcal{C}_{\mathsf{Y}}^{\mathsf{p}} \cup \mathcal{C}_{\mathsf{Y}}^{\mathsf{s}})$.

[1] Proved independently by Ronnie Pavlov.

Example(s)

There exists a family of shift spaces which contains:

- A shift space with almost specification and multiple MMEs[1].
- A shift space with weak specification and multiple MMEs[1].
- A shift space with almost specification but without weak specification.
- An intrinsically ergodic shift space X and its factor Y with multiple MMEs such that there are Climenhaga-Thompson decompositions for X and Y, respectively

$$\mathcal{B}(X) = \mathcal{C}_{\mathsf{X}}^{\mathsf{p}} \cdot \mathcal{G}_X \cdot \mathcal{C}_{\mathsf{X}}^{\mathsf{s}}$$
$$\mathcal{B}(Y) = \mathcal{C}_{\mathsf{Y}}^{\mathsf{p}} \cdot \mathcal{G}_Y \cdot \mathcal{C}_{\mathsf{Y}}^{\mathsf{s}}$$

with $h(\mathcal{G}_X) > h(\mathcal{C}_{\mathsf{X}}^{\mathsf{p}} \cup \mathcal{C}_{\mathsf{X}}^{\mathsf{s}})$ and $h(\mathcal{G}_Y) < h(\mathcal{C}_{\mathsf{Y}}^{\mathsf{p}} \cup \mathcal{C}_{\mathsf{Y}}^{\mathsf{s}})$.

[1] Proved independently by Ronnie Pavlov.

Example(s)

There exists a family of shift spaces which contains:

- A shift space with almost specification and multiple MMEs[1].
- A shift space with weak specification and multiple MMEs[1].
- A shift space with almost specification but without weak specification.
- An intrinsically ergodic shift space X and its factor Y with multiple MMEs such that there are Climenhaga-Thompson decompositions for X and Y, respectively

$$\mathcal{B}(X) = \mathcal{C}^{\mathsf{p}}_{\mathsf{X}} \cdot \mathcal{G}_X \cdot \mathcal{C}^{\mathsf{s}}_{\mathsf{X}}$$
$$\mathcal{B}(Y) = \mathcal{C}^{\mathsf{p}}_{\mathsf{Y}} \cdot \mathcal{G}_Y \cdot \mathcal{C}^{\mathsf{s}}_{\mathsf{Y}}$$

  with $h(\mathcal{G}_X) > h(\mathcal{C}^{\mathsf{p}}_{\mathsf{X}} \cup \mathcal{C}^{\mathsf{s}}_{\mathsf{X}})$ and $h(\mathcal{G}_Y) < h(\mathcal{C}^{\mathsf{p}}_{\mathsf{Y}} \cup \mathcal{C}^{\mathsf{s}}_{\mathsf{Y}})$.

[1]Proved independently by Ronnie Pavlov.

Definition(s)

A shift space X has the

Definition(s)

A shift space X has the

- *almost specification property* if there exists $\theta\colon \mathbb{N} \to \mathbb{N}$ with $\theta \in o(n)$ such that for every $n \in \mathbb{N}$ and $w_1, \ldots, w_n \in \mathcal{B}(X)$, there exist words $v_1, \ldots, v_n \in \mathcal{B}(X)$ with $|v_i| = |w_i|$ such that $v_1 v_2 \ldots v_n \in \mathcal{B}(X)$ and each v_i differs from w_i in at most $\theta(|v_i|)$ places.

Definition(s)

A shift space X has the

- *almost specification property* if there exists $\theta\colon \mathbb{N} \to \mathbb{N}$ with $\theta \in o(n)$ such that for every $n \in \mathbb{N}$ and $w_1, \dots, w_n \in \mathcal{B}(X)$, there exist words $v_1, \dots, v_n \in \mathcal{B}(X)$ with $|v_i| = |w_i|$ such that $v_1 v_2 \dots v_n \in \mathcal{B}(X)$ and each v_i differs from w_i in at most $\theta(|v_i|)$ places.
- *weak specification property* if there exists $t\colon \mathbb{N} \to \mathbb{N}$ with $t \in o(n)$ such that for any words $u \in \mathcal{B}(X)$, $w \in \mathcal{B}_n(X)$ there exists a word $v \in \mathcal{B}_{t(n)}(X)$ such that $x = uvw \in \mathcal{B}(X)$.

Climenhaga-Thompson decompositions

We say that the language $\mathcal{B}(X)$ of a shift space X *admits Climenhaga-Thompson decomposition* if there are $\mathcal{C}^{\mathsf{p}}, \mathcal{G}, \mathcal{C}^{\mathsf{s}} \subset \mathcal{B}(X)$ such that:

Climenhaga-Thompson decompositions

We say that the language $\mathcal{B}(X)$ of a shift space X *admits Climenhaga-Thompson decomposition* if there are $\mathcal{C}^{\mathsf{p}}, \mathcal{G}, \mathcal{C}^{\mathsf{s}} \subset \mathcal{B}(X)$ such that:

Climenhaga-Thompson decompositions

We say that the language $\mathcal{B}(X)$ of a shift space X *admits Climenhaga-Thompson decomposition* if there are $\mathcal{C}^{\mathsf{p}}, \mathcal{G}, \mathcal{C}^{\mathsf{s}} \subset \mathcal{B}(X)$ such that:

(I) for every $w \in \mathcal{B}(X)$ there are $u_p \in \mathcal{C}^{\mathsf{p}}$, $v \in \mathcal{G}$, $u_s \in \mathcal{C}^{\mathsf{s}}$ such that $w = u_p v u_s$.

Climenhaga-Thompson decompositions

We say that the language $\mathcal{B}(X)$ of a shift space X *admits Climenhaga-Thompson decomposition* if there are $\mathcal{C}^{\mathsf{p}}, \mathcal{G}, \mathcal{C}^{\mathsf{s}} \subset \mathcal{B}(X)$ such that:

(I) for every $w \in \mathcal{B}(X)$ there are $u_p \in \mathcal{C}^{\mathsf{p}}$, $v \in \mathcal{G}$, $u_s \in \mathcal{C}^{\mathsf{s}}$ such that $w = u_p v u_s$.

(II) there exists $t \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ and $w_1, \ldots, w_n \in \mathcal{G}$, there exist $v_1, \ldots, v_{n-1} \in \mathcal{B}(X)$ such that $x = w_1 v_1 w_2 v_2 \ldots v_{n-1} w_n \in \mathcal{B}(X)$ and $|v_i| = t$ for $i = 1, \ldots, n-1$.

Climenhaga-Thompson decompositions

We say that the language $\mathcal{B}(X)$ of a shift space X *admits Climenhaga-Thompson decomposition* if there are $\mathcal{C}^{\mathrm{p}}, \mathcal{G}, \mathcal{C}^{\mathrm{s}} \subset \mathcal{B}(X)$ such that:

(I) for every $w \in \mathcal{B}(X)$ there are $u_p \in \mathcal{C}^{\mathrm{p}}$, $v \in \mathcal{G}$, $u_s \in \mathcal{C}^{\mathrm{s}}$ such that $w = u_p v u_s$.

(II) there exists $t \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ and $w_1, \ldots, w_n \in \mathcal{G}$, there exist $v_1, \ldots, v_{n-1} \in \mathcal{B}(X)$ such that $x = w_1 v_1 w_2 v_2 \ldots v_{n-1} w_n \in \mathcal{B}(X)$ and $|v_i| = t$ for $i = 1, \ldots, n-1$.

(III) For every $M \in \mathbb{N}$, there exists $\tau \in \mathbb{N}$ such that given $w \in \mathcal{B}(X)$ satisfying $w = u_p v u_s$ for some $u_p \in \mathcal{C}^{\mathrm{p}}$, $v \in \mathcal{G}$, $u_s \in \mathcal{C}^{\mathrm{s}}$, with $|u_p| \leqslant M$ and $|u_s| \leqslant M$, there exist words u', u'' with $|u'| \leqslant \tau$, $|u''| \leqslant \tau$ for which $u' w u'' \in \mathcal{G}$.

Results of Climenhaga & Thompson

Theorem (Climenhaga & Thompson, 2012)

If a shift space X has the CT decomposition given by the sets $\mathcal{C}^{\mathsf{p}}$, $\mathcal{G}$, $\mathcal{C}^{\mathsf{s}}$, and

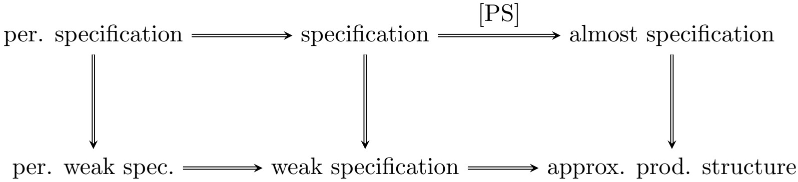
$$h(\mathcal{G}) > h(\mathcal{C}^{\mathsf{p}} \cup \mathcal{C}^{\mathsf{s}}),$$

then X is intrinsically ergodic. Furthermore, if

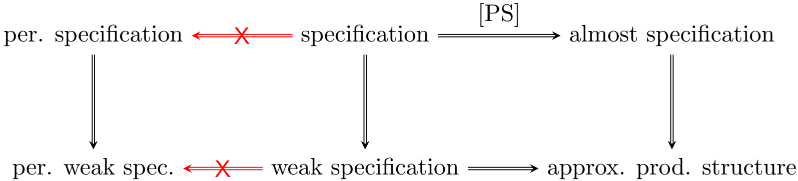
$$h(\mathcal{C}^{\mathsf{p}} \cup \mathcal{C}^{\mathsf{s}}) = 0,$$

then every shift factor of X is intrinsically ergodic.

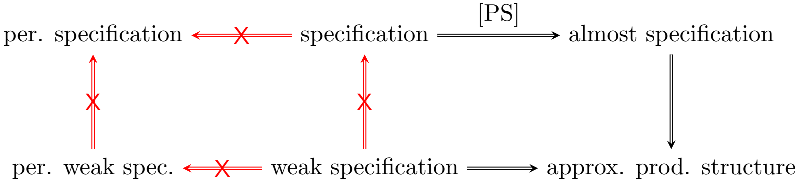
Specification-like notions



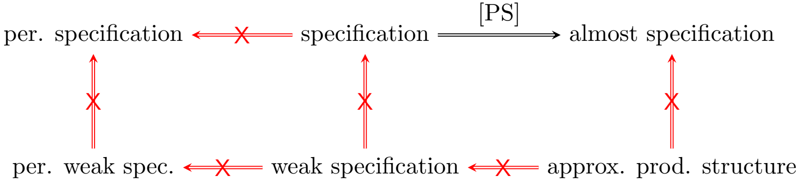
Specification-like notions



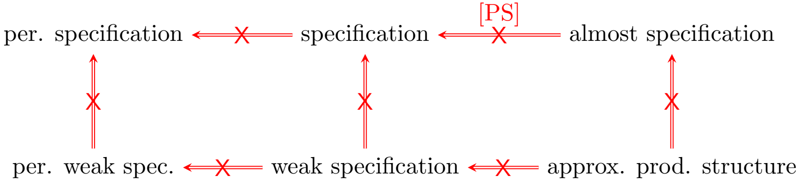
Specification-like notions



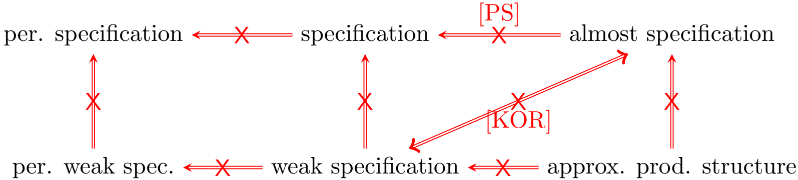
Specification-like notions



Specification-like notions



Specification-like notions



Positive results

Positive results

Theorem ($\Leftarrow$Kulczycki, DK, Oprocha, $\Rightarrow$Chen, Oprocha, Wu)
A dynamical system has the almost specification property if and only if its restriction to the measure center does.

Positive results

Theorem (⇐Kulczycki, DK, Oprocha, ⇒Chen, Oprocha, Wu)
A dynamical system has the almost specification property if and only if its restriction to the measure center does.

Theorem (DK, Oprocha, Rams)
The restriction of a dynamical system with the almost specification property to its measure center is a topological K system with dense set of minimal points. In particular, this restriction is topologically weakly mixing and non-minimal provided that the measure center is non-trivial.

Construction of $X_{\mathbf{R}}$

Construction of $X_{\mathbf{R}}$

- Fix $p, q \in \mathbb{N}$, $p, q \geqslant 2$.

Construction of $X_{\mathbf{R}}$

- Fix $p, q \in \mathbb{N}$, $p, q \geqslant 2$.
- Pick $\mathbf{R} = \{R_n\}_{n=1}^{\infty}$ such that

$$\{1\} = R_1 \subset R_2 \subset R_3 \subset \dots \text{ and } R_n \subset \{1, \dots, n\}.$$

  Assume also that for each k there is n such that $\{1, \dots, n\} \setminus R_n$ contains k consecutive integers.

Construction of $X_{\mathbf{R}}$

- Fix $p, q \in \mathbb{N}$, $p, q \geqslant 2$.
- Pick $\mathbf{R} = \{R_n\}_{n=1}^{\infty}$ such that

$$\{1\} = R_1 \subset R_2 \subset R_3 \subset \ldots \text{ and } R_n \subset \{1, \ldots, n\}.$$

Assume also that for each k there is n such that $\{1, \ldots, n\} \setminus R_n$ contains k consecutive integers.

Construction of $X_{\mathbf{R}}$

- Fix $p, q \in \mathbb{N}$, $p, q \geqslant 2$.
- Pick $\mathbf{R} = \{R_n\}_{n=1}^{\infty}$ such that

$$\{1\} = R_1 \subset R_2 \subset R_3 \subset \dots \text{ and } R_n \subset \{1, \dots, n\}.$$

Assume also that for each k there is n such that $\{1, \dots, n\} \setminus R_n$ contains k consecutive integers.

Construction of $X_{\mathbf{R}}$

- Fix $p, q \in \mathbb{N}$, $p, q \geqslant 2$.
- Pick $\mathbf{R} = \{R_n\}_{n=1}^{\infty}$ such that

  $$\{1\} = R_1 \subset R_2 \subset R_3 \subset \dots \text{ and } R_n \subset \{1, \dots, n\}.$$

  Assume also that for each k there is n such that $\{1, \dots, n\} \setminus R_n$ contains k consecutive integers.
- Let $r \colon \mathbb{N}_0 \to \mathbb{N}_0$ by $r(0) = 0$ and

  $$r(n) = |R_n| \quad \text{for } n \in \mathbb{N}.$$

Construction of $X_{\mathbf{R}}$

- Fix $p, q \in \mathbb{N}$, $p, q \geqslant 2$.
- Pick $\mathbf{R} = \{R_n\}_{n=1}^{\infty}$ such that

  $$\{1\} = R_1 \subset R_2 \subset R_3 \subset \dots \text{ and } R_n \subset \{1, \dots, n\}.$$

  Assume also that for each k there is n such that $\{1, \dots, n\} \setminus R_n$ contains k consecutive integers.
- Let $r \colon \mathbb{N}_0 \to \mathbb{N}_0$ by $r(0) = 0$ and

  $$r(n) = |R_n| \quad \text{for } n \in \mathbb{N}.$$

- Let $\mathcal{A} = \{1, \dots, p\} \times \{0, 1, \dots, q-1\} \cup \{(0, 0)\}$.

Construction of $X_{\mathbf{R}}$

- Fix $p, q \in \mathbb{N}$, $p, q \geqslant 2$.
- Pick $\mathbf{R} = \{R_n\}_{n=1}^{\infty}$ such that

$$\{1\} = R_1 \subset R_2 \subset R_3 \subset \dots \text{ and } R_n \subset \{1, \dots, n\}.$$

  Assume also that for each k there is n such that $\{1, \dots, n\} \setminus R_n$ contains k consecutive integers.
- Let $r \colon \mathbb{N}_0 \to \mathbb{N}_0$ by $r(0) = 0$ and

$$r(n) = |R_n| \quad \text{for } n \in \mathbb{N}.$$

- Let $\mathcal{A} = \{1, \dots, p\} \times \{0, 1, \dots, q-1\} \cup \{(0, 0)\}$.
- Identify $(a, b) \in \mathcal{A}$ with $\left\lfloor\begin{smallmatrix} a \\ b \end{smallmatrix}\right\rceil$ and regard $a \in \{0, 1, \dots, p\}$ as the *color*.

Construction of $X_{\mathbf{R}}$

- Fix $p, q \in \mathbb{N}$, $p, q \geqslant 2$.
- Pick $\mathbf{R} = \{R_n\}_{n=1}^{\infty}$ such that

$$\{1\} = R_1 \subset R_2 \subset R_3 \subset \dots \text{ and } R_n \subset \{1, \dots, n\}.$$

  Assume also that for each k there is n such that $\{1, \dots, n\} \setminus R_n$ contains k consecutive integers.
- Let $r \colon \mathbb{N}_0 \to \mathbb{N}_0$ by $r(0) = 0$ and

$$r(n) = |R_n| \quad \text{for } n \in \mathbb{N}.$$

- Let $\mathcal{A} = \{1, \dots, p\} \times \{0, 1, \dots, q-1\} \cup \{(0, 0)\}$.
- Identify $(a, b) \in \mathcal{A}$ with $\left\lfloor \begin{smallmatrix} a \\ b \end{smallmatrix} \right\rceil$ and regard $a \in \{0, 1, \dots, p\}$ as the *color*.

Construction of $X_{\mathbf{R}}$

- Fix $p, q \in \mathbb{N}$, $p, q \geqslant 2$.
- Pick $\mathbf{R} = \{R_n\}_{n=1}^{\infty}$ such that

$$\{1\} = R_1 \subset R_2 \subset R_3 \subset \dots \text{ and } R_n \subset \{1, \dots, n\}.$$

  Assume also that for each k there is n such that $\{1, \dots, n\} \setminus R_n$ contains k consecutive integers.
- Let $r \colon \mathbb{N}_0 \to \mathbb{N}_0$ by $r(0) = 0$ and

$$r(n) = |R_n| \quad \text{for } n \in \mathbb{N}.$$

- Let $\mathcal{A} = \{1, \dots, p\} \times \{0, 1, \dots, q-1\} \cup \{(0,0)\}$.
- Identify $(a, b) \in \mathcal{A}$ with $\left\lfloor\begin{smallmatrix} a \\ b \end{smallmatrix}\right\rceil$ and regard $a \in \{0, 1, \dots, p\}$ as the *color*.
- The symbol $\mathbf{0} = \left\lfloor\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right\rceil$ is the *marker*.

Construction of $X_{\mathbf{R}}$ cont.

Construction of $X_{\mathbf{R}}$ cont.

- A word $\begin{bmatrix} a_1 \dots a_n \\ b_1 \dots b_n \end{bmatrix} \in \mathcal{A}^*$ is *monochromatic* or *of color* $a \in \{0, 1, \dots, p\}$ if $a = a_1 = \dots = a_n$, and *polychromatic* otherwise.

Construction of $X_{\mathbf{R}}$ cont.

- A word $\begin{bmatrix} a_1 \ldots a_n \\ b_1 \ldots b_n \end{bmatrix} \in \mathcal{A}^*$ is *monochromatic* or *of color* $a \in \{0, 1, \ldots, p\}$ if $a = a_1 = \ldots = a_n$, and *polychromatic* otherwise.

Construction of $X_{\mathbf{R}}$ cont.

- A word $\left[\begin{smallmatrix} a_1 \dots a_n \\ b_1 \dots b_n \end{smallmatrix}\right] \in \mathcal{A}^*$ is *monochromatic* or *of color* $a \in \{0, 1, \dots, p\}$ if $a = a_1 = \dots = a_n$, and *polychromatic* otherwise.
- A monochromatic block $W = \left[\begin{smallmatrix} a_1 \dots a_n \\ b_1 \dots b_n \end{smallmatrix}\right] \in \mathcal{A}^*$ is *restricted* if $b_j = 0$ for each $j \in R_k$.

Construction of $X_{\mathbf{R}}$ cont.

- A word $\begin{bmatrix} a_1 \ldots a_n \\ b_1 \ldots b_n \end{bmatrix} \in \mathcal{A}^*$ is *monochromatic* or *of color* $a \in \{0, 1, \ldots, p\}$ if $a = a_1 = \ldots = a_n$, and *polychromatic* otherwise.
- A monochromatic block $W = \begin{bmatrix} a_1 \ldots a_n \\ b_1 \ldots b_n \end{bmatrix} \in \mathcal{A}^*$ is *restricted* if $b_j = 0$ for each $j \in R_k$.
- A block W is *free* if $W = \mathbf{0}V$, where $V \in \mathcal{R}$.

Construction of $X_{\mathbf{R}}$ cont.

- A word $\left\lfloor {a_1 \ldots a_n \atop b_1 \ldots b_n} \right\rceil \in \mathcal{A}^*$ is *monochromatic* or *of color* $a \in \{0, 1, \ldots, p\}$ if $a = a_1 = \ldots = a_n$, and *polychromatic* otherwise.
- A monochromatic block $W = \left\lfloor {a_1 \ldots a_n \atop b_1 \ldots b_n} \right\rceil \in \mathcal{A}^*$ is *restricted* if $b_j = 0$ for each $j \in R_k$.
- A block W is *free* if $W = \mathbf{0}V$, where $V \in \mathcal{R}$.
- A *good word* is finite concatenation of free blocks.

Construction of $X_{\mathbf{R}}$ cont.

- A word $\left[\begin{smallmatrix} a_1 \dots a_n \\ b_1 \dots b_n \end{smallmatrix}\right] \in \mathcal{A}^*$ is *monochromatic* or *of color* $a \in \{0, 1, \dots, p\}$ if $a = a_1 = \dots = a_n$, and *polychromatic* otherwise.
- A monochromatic block $W = \left[\begin{smallmatrix} a_1 \dots a_n \\ b_1 \dots b_n \end{smallmatrix}\right] \in \mathcal{A}^*$ is *restricted* if $b_j = 0$ for each $j \in R_k$.
- A block W is *free* if $W = \mathbf{0}V$, where $V \in \mathcal{R}$.
- A *good word* is finite concatenation of free blocks.
- A word W is *allowed* if there exist $U \in \mathcal{M}$ and $V_1, \dots, V_k \in \mathcal{F}$ such that $W = UV_1 \dots V_k$. The set of all allowed words is the language of $X_{\mathbf{R}}$.

Construction of $X_{\mathbf{R}}$ cont.

- A word $\begin{bmatrix} a_1 \dots a_n \\ b_1 \dots b_n \end{bmatrix} \in \mathcal{A}^*$ is *monochromatic* or *of color* $a \in \{0, 1, \dots, p\}$ if $a = a_1 = \dots = a_n$, and *polychromatic* otherwise.
- A monochromatic block $W = \begin{bmatrix} a_1 \dots a_n \\ b_1 \dots b_n \end{bmatrix} \in \mathcal{A}^*$ is *restricted* if $b_j = 0$ for each $j \in R_k$.
- A block W is *free* if $W = \mathbf{0}V$, where $V \in \mathcal{R}$.
- A *good word* is finite concatenation of free blocks.
- A word W is *allowed* if there exist $U \in \mathcal{M}$ and $V_1, \dots, V_k \in \mathcal{F}$ such that $W = UV_1 \dots V_k$. The set of all allowed words is the language of $X_{\mathbf{R}}$.

Almost (weak) specification of $X_{\mathbf{R}}$

Almost (weak) specification of $X_{\mathbf{R}}$

Lemma

Almost (weak) specification of $X_{\mathbf{R}}$

Lemma

1. *If $r(n)/n \to 0$ as $n \to \infty$, then the shift space $X_{\mathbf{R}}$ has the almost specification property.*

Almost (weak) specification of $X_{\mathbf{R}}$

Lemma

1. *If $r(n)/n \to 0$ as $n \to \infty$, then the shift space $X_{\mathbf{R}}$ has the almost specification property.*
2. *Let N_k denote the smallest n such that $\{1, \ldots, n\} \setminus R_n$ contains k consecutive integers. The shift space $X_{\mathbf{R}}$ has the weak specification property if and only if $k/N_k \to 1$ as $k \to \infty$.*

A technical lemma

Lemma

If $r(n) > 0$ *for every* n *and* $\liminf_{n\to\infty} \frac{r(n)}{\ln n} > 0$, *then there is* $Q \geqslant 2$ *such that the series* $\sum_{n=1}^{\infty} q^{-r(n)}$ *converges for all integers* $q \geqslant Q$ *and its sum tends to* 0 *as* $q \to \infty$.

A technical lemma

Lemma

If $r(n) > 0$ *for every* n *and* $\liminf_{n\to\infty} \frac{r(n)}{\ln n} > 0$*, then there is* $Q \geqslant 2$ *such that the series* $\sum_{n=1}^{\infty} q^{-r(n)}$ *converges for all integers* $q \geqslant Q$ *and its sum tends to* 0 *as* $q \to \infty$*. In particular, for every* $p \geqslant 2$ *there is* $q \geqslant Q$ *such that*

$$1 + p \sum_{j=1}^{\infty} q^{-r(j)} \leqslant q. \tag{1}$$

Counting words

Counting words

Lemma

Let $\mathcal{F}_n$ ($\mathcal{G}_n$) be the number of n-blocks in $\mathcal{F}$ ($\mathcal{G}$). Then

Counting words

Lemma

Let $\mathcal{F}_n$ ($\mathcal{G}_n$) be the number of n-blocks in $\mathcal{F}$ ($\mathcal{G}$). Then

1. $\mathcal{F}_0 = \mathcal{F}_1 = 1$ *and* $\mathcal{F}_n = p \cdot q^{(n-1)-r(n-1)}$ *for all* $n > 1$;

Counting words

Lemma

Let $\mathcal{F}_n$ ($\mathcal{G}_n$) be the number of n-blocks in $\mathcal{F}$ ($\mathcal{G}$). Then

1. $\mathcal{F}_0 = \mathcal{F}_1 = 1$ *and* $\mathcal{F}_n = p \cdot q^{(n-1)-r(n-1)}$ *for all* $n > 1$*;*
2. $\mathcal{G}_0 = \mathcal{G}_1 = 1$ *and*
$$\mathcal{G}_n = \sum_{i=1}^{n} \mathcal{F}_i \mathcal{G}_{n-i} = \mathcal{G}_{n-1} + \sum_{j=1}^{n-1} pq^{j-r(j)} \mathcal{G}_{n-1-j} \text{ for } n > 1.$$

Counting words

Lemma

Let $\mathcal{F}_n$ ($\mathcal{G}_n$) be the number of n-blocks in $\mathcal{F}$ ($\mathcal{G}$). Then

1. $\mathcal{F}_0 = \mathcal{F}_1 = 1$ *and* $\mathcal{F}_n = p \cdot q^{(n-1)-r(n-1)}$ *for all* $n > 1$*;*
2. $\mathcal{G}_0 = \mathcal{G}_1 = 1$ *and*
$$\mathcal{G}_n = \sum_{i=1}^{n} \mathcal{F}_i \mathcal{G}_{n-i} = \mathcal{G}_{n-1} + \sum_{j=1}^{n-1} pq^{j-r(j)} \mathcal{G}_{n-1-j} \text{ for } n > 1.$$

Lemma

1. *If* (1) *holds, then* $\mathcal{G}_n \leqslant q^n$ *for every* $n \geqslant 0$*. In particular,* $h(X_{\mathbf{R}}) = \log q$ *and* $X_{\mathbf{R}}$ *has at least* p *MME's.*

Counting words

Lemma

Let $\mathcal{F}_n$ ($\mathcal{G}_n$) be the number of n-blocks in $\mathcal{F}$ ($\mathcal{G}$). Then

1. $\mathcal{F}_0 = \mathcal{F}_1 = 1$ *and* $\mathcal{F}_n = p \cdot q^{(n-1)-r(n-1)}$ *for all* $n > 1$;
2. $\mathcal{G}_0 = \mathcal{G}_1 = 1$ *and*
   $$\mathcal{G}_n = \sum_{i=1}^{n} \mathcal{F}_i \mathcal{G}_{n-i} = \mathcal{G}_{n-1} + \sum_{j=1}^{n-1} pq^{j-r(j)} \mathcal{G}_{n-1-j} \text{ for } n > 1.$$

Lemma

1. *If* (1) *holds, then* $\mathcal{G}_n \leqslant q^n$ *for every* $n \geqslant 0$. *In particular,* $h(X_{\mathbf{R}}) = \log q$ *and* $X_{\mathbf{R}}$ *has at least* p *MME's.*
2. *If* (1) *does not hold, then*

$$\liminf_{n\to\infty} \frac{\log \mathcal{G}_n}{n} > \log q.$$

CT decomposition and intrinsic ergodicity of $X_{\mathbf{R}}$

Lemma

Let $\mathcal{C}^{\mathsf{p}} = \mathcal{M}$, $\mathcal{C}^{\mathsf{s}} = \emptyset$. Then $\mathcal{C}^{\mathsf{s}}, \mathcal{G}, \mathcal{C}^{\mathsf{p}}$ is a Climenhaga-Thompson decomposition for $X_{\mathbf{R}}$ and (1) does not hold if and only if

$$h(\mathcal{G}) > h(\mathcal{C}^{\mathsf{s}} \cup \mathcal{C}^{\mathsf{p}}) = h(\mathcal{C}^{\mathsf{p}}).$$

Therefore $X_{\mathbf{R}}$ s intrinsically ergodic if and only if (1) does not hold.

Orbit segments and specifications

Orbit segments and specifications

- Let (X, ρ) be a compact metric space.

Orbit segments and specifications

- Let (X, ρ) be a compact metric space.
- Let $T\colon X \to X$ be continuous.

Orbit segments and specifications

- Let (X, ρ) be a compact metric space.
- Let $T\colon X \to X$ be continuous.
- Let $a, b \in \mathbb{N} \cup \{0\}$, $a \leqslant b$. The orbit segment of $x \in X$ over $[a, b]$ is the sequence

$$T^{[a,b]}(x) = (T^a(x), T^{a+1}(x), \ldots, T^b(x)).$$

Orbit segments and specifications

- Let (X, ρ) be a compact metric space.
- Let $T \colon X \to X$ be continuous.
- Let $a, b \in \mathbb{N} \cup \{0\}$, $a \leqslant b$. The orbit segment of $x \in X$ over $[a, b]$ is the sequence

$$T^{[a,b]}(x) = (T^a(x), T^{a+1}(x), \ldots, T^b(x)).$$

- A specification is a family of orbit segments

$$\xi = \{T^{[a_j,b_j]}(x_j)\}_{j=1}^{n}$$

such that $n \in \mathbb{N} \cup \{\infty\}$ and $b_j < a_{j+1}$ for all $1 \leqslant j < n$.

Specification-like properties

Specification-like properties

- A "generic" specification property:

Specification-like properties

- A "generic" specification property:
  There exists a family Ξ of "good" specifications

Specification-like properties

- A “generic” specification property:
  There exists a family Ξ *of “good” specifications*

Specification-like properties

- A "generic" specification property:
  There exists a family Ξ of "good" specifications

Specification-like properties

- A “generic” specification property:
  There exists a family Ξ of “good” specifications such that every specification from Ξ can approximated by an orbit of a single point in X.

Specification-like properties

- A “generic” specification property:
  There exists a family Ξ of “good” specifications such that every specification from Ξ can approximated by an orbit of a single point in X.
- Varying the notion of “good specification” and “approximation” we obtain various *specification-like* properties.

Specification property

Specification property

- Let $N \in \mathbb{N}$.

Specification property

- Let $N \in \mathbb{N}$. A family of orbit segments $\xi = \{T^{[a_j,b_j]}(x_j)\}_{j=1}^{n}$ is an N-spaced specification if $a_i - b_{i-1} \geqslant N$ for $2 \leqslant i \leqslant n$.

Specification property

- Let $N \in \mathbb{N}$. A family of orbit segments $\xi = \{T^{[a_j,b_j]}(x_j)\}_{j=1}^n$ is an N-spaced specification if $a_i - b_{i-1} \geqslant N$ for $2 \leqslant i \leqslant n$.

Specification property

- Let $N \in \mathbb{N}$. A family of orbit segments $\xi = \{T^{[a_j,b_j]}(x_j)\}_{j=1}^n$ is an N-spaced specification if $a_i - b_{i-1} \geqslant N$ for $2 \leqslant i \leqslant n$.
- We say that a specification $\xi = \{T^{[a_j,b_j]}(x_j)\}_{j=1}^n$ is ε-traced by $y \in X$ if

$$\rho(T^k(y), T^k(x_i)) \leqslant \varepsilon \quad \text{for } a_i \leqslant k \leqslant b_i \text{ and } 1 \leqslant i \leqslant n.$$

Specification property

- Let $N \in \mathbb{N}$. A family of orbit segments $\xi = \{T^{[a_j, b_j]}(x_j)\}_{j=1}^{n}$ is an N-spaced specification if $a_i - b_{i-1} \geqslant N$ for $2 \leqslant i \leqslant n$.
- We say that a specification $\xi = \{T^{[a_j, b_j]}(x_j)\}_{j=1}^{n}$ is ε-traced by $y \in X$ if

$$\rho(T^k(y), T^k(x_i)) \leqslant \varepsilon \quad \text{for } a_i \leqslant k \leqslant b_i \text{ and } 1 \leqslant i \leqslant n.$$

- We say that T has the specification property if for any $\varepsilon > 0$ there is $N = N(\varepsilon) \in \mathbb{N}$ such that any $N(\varepsilon)$-spaced specification is ε-traced by some $y \in X$.

Weak specification

Weak specification

- Let $L\colon \mathbb{N} \to \mathbb{N}$ be a function.

Weak specification

- Let $L\colon \mathbb{N} \to \mathbb{N}$ be a function. A family of orbit segments $\{T^{[a_j,b_j]}(x_j)\}_{j=1}^n$ is an L-spaced specification if

$$a_i - b_{i-1} \geqslant L(b_i - a_i) \quad \text{for } 2 \leqslant i \leqslant n.$$

Weak specification

- Let $L \colon \mathbb{N} \to \mathbb{N}$ be a function. A family of orbit segments $\{T^{[a_j,b_j]}(x_j)\}_{j=1}^{n}$ is an L-spaced specification if

$$a_i - b_{i-1} \geqslant L(b_i - a_i) \quad \text{for } 2 \leqslant i \leqslant n.$$

Weak specification

- Let $L\colon \mathbb{N} \to \mathbb{N}$ be a function. A family of orbit segments $\{T^{[a_j,b_j]}(x_j)\}_{j=1}^{n}$ is an L-spaced specification if

$$a_i - b_{i-1} \geqslant L(b_i - a_i) \quad \text{for } 2 \leqslant i \leqslant n.$$

- We say that T has the weak specification property if for any $\varepsilon > 0$ there is a function $L_\varepsilon\colon \mathbb{N} \to \mathbb{N}$ with $L_\varepsilon(m)/m \to 0$ as $(m \to \infty)$ such that any L_ε-spaced specification is ε-traced by some $y \in X$.

Weak specification

- Let $L\colon \mathbb{N} \to \mathbb{N}$ be a function. A family of orbit segments $\{T^{[a_j,b_j]}(x_j)\}_{j=1}^{n}$ is an L-spaced specification if

$$a_i - b_{i-1} \geqslant L(b_i - a_i) \quad \text{for } 2 \leqslant i \leqslant n.$$

- We say that T has the weak specification property if for any $\varepsilon > 0$ there is a function $L_\varepsilon\colon \mathbb{N} \to \mathbb{N}$ with $L_\varepsilon(m)/m \to 0$ as $(m \to \infty)$ such that any L_ε-spaced specification is ε-traced by some $y \in X$.

Bowen balls with mistakes

Bowen balls with mistakes

- A nondecreasing map $g\colon \mathbb{N} \to \mathbb{N}$ is called a blowup function if for all $n \in \mathbb{N}$ one has $g(n) < n$ and

$$\lim_{n\to\infty} g(n)/n = 0.$$

Bowen balls with mistakes

- A nondecreasing map $g\colon \mathbb{N} \to \mathbb{N}$ is called a blowup function if for all $n \in \mathbb{N}$ one has $g(n) < n$ and

$$\lim_{n\to\infty} g(n)/n = 0.$$

- Given a blowup function g we define the function $k_g\colon \mathbb{R}_+ \to \mathbb{N}$ such that $k_g(\varepsilon)$ is the smallest integer n such that $g(m) < \varepsilon m$ for all $m \geqslant n$ and we define the set

$$\mathcal{P}(n,g) = \{\Lambda \subset \langle n\rangle \,:\, |\langle n\rangle \setminus \Lambda| \leqslant g(n)\}.$$

Bowen balls with mistakes

- A nondecreasing map $g\colon \mathbb{N} \to \mathbb{N}$ is called a blowup function if for all $n \in \mathbb{N}$ one has $g(n) < n$ and
$$\lim_{n\to\infty} g(n)/n = 0.$$
- Given a blowup function g we define the function $k_g\colon \mathbb{R}_+ \to \mathbb{N}$ such that $k_g(\varepsilon)$ is the smallest integer n such that $g(m) < \varepsilon m$ for all $m \geqslant n$ and we define the set
$$\mathcal{P}(n,g) = \{\Lambda \subset \langle n \rangle \,:\, |\langle n \rangle \setminus \Lambda| \leqslant g(n)\}.$$
- For $\Lambda \subset \{0,1,2,\ldots\}$ and $x, y \in X$ we define
$$\rho_\Lambda(x,y) = \max_{j\in\Lambda}\{\rho(T^j(x), T^j(y))\}.$$

Bowen balls with mistakes

- A nondecreasing map $g\colon \mathbb{N} \to \mathbb{N}$ is called a blowup function if for all $n \in \mathbb{N}$ one has $g(n) < n$ and

$$\lim_{n\to\infty} g(n)/n = 0.$$

- Given a blowup function g we define the function $k_g\colon \mathbb{R}_+ \to \mathbb{N}$ such that $k_g(\varepsilon)$ is the smallest integer n such that $g(m) < \varepsilon m$ for all $m \geqslant n$ and we define the set

$$\mathcal{P}(n,g) = \{\Lambda \subset \langle n\rangle \ :\ |\langle n\rangle \setminus \Lambda| \leqslant g(n)\}.$$

- For $\Lambda \subset \{0,1,2,\dots\}$ and $x, y \in X$ we define

$$\rho_\Lambda(x,y) = \max_{j\in\Lambda}\{\rho(T^j(x), T^j(y))\}.$$

- For every $x \in X$ and $\varepsilon > 0$ let

$$B_n(g,x,\varepsilon) = \{y \in X \ :\ \min\{\rho_\Lambda(x,y) : \Lambda \in \mathcal{P}(n,g)\} \leqslant \varepsilon\}.$$

Almost specification property

Almost specification property

Definition (Pfister & Sullivan)

A dynamical system (X, T) satisfies the almost specification property if there exists a blowup function g such that for any $k \in \mathbb{N}$, $\varepsilon_1, \ldots, \varepsilon_k > 0$, any $n_1 \geqslant k_g(\varepsilon_1), \ldots, n_k \geqslant k_g(\varepsilon_k)$, and every specification $\xi = \{T^{[a_{j-1}, a_j)}(x_j)\}_{j=1}^{k}$ such that $a_0 = 0$ and $a_j - a_{j-1} = n_j$ one can find a point y in

$$\bigcap_{j=1}^{k} T^{-a_{j-1}} B_{n_j}(g, x_j, \varepsilon_j) \neq \emptyset.$$

We say that y $(g; n_j, \varepsilon_j)$-traces x_j over $[a_{j-1}, a_j)$.

Approximate product structure

Approximate product structure

Definition (Pfister & Sullivan)

We say that T has the approximate product structure if for any $\varepsilon > 0$, $\delta_1 > 0$ and $\delta_2 > 0$ there exists an integer $N > 0$ such that for any $n \geqslant N$ and any sequence $\{x_i\}_{i=1}^{\infty}$ of X there exist a sequence of integers $\{h_i\}_{i=1}^{\infty}$ and a point $y \in X$ satisfying $h_1 = 0$, $n \leqslant h_{i+1} - h_i \leqslant n(1 + \delta_2)$ and

$$\left|\{0 \leqslant j < n : \rho(T^{h_i+j}(y), T^j(x_i)) > \varepsilon\}\right| \leqslant \delta_1 n \text{ for any } i \in \mathbb{N}.$$

Some references I

- Marcin Kulczycki, Dominik Kwietniak, Piotr Oprocha
  On almost specification and average shadowing properties
  Fund. Math. **224** (2014), 241–278 doi:10.4064/fm224-3-4

- Dominik Kwietniak, Piotr Oprocha, Michał Rams
  On entropy of dynamical systems with almost specification
  arXiv:1411.1989[math.DS]

- Ronnie Pavlov
  On intrinsic ergodicity and weakenings of the specification property
  arXiv:1411.2077[math.DS]

- V. Climenhaga, D. Thompson,
  Intrinsic ergodicity beyond specification: β-shifts, S-gap shifts, and their factors
  Israel J. Math. **192** (2012), 785–817.