Automorphism Groups of Low Complexity Minimal Subshifts

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Basic topological notion

Definition

Let (X, T) be a topological dynamical system, X a compact metric space. An automorphism $\phi\colon X \to X$ is an homeomorphsim s.t.

$$\phi \circ T = T \circ \phi.$$

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$$\langle T \rangle \subset \mathrm{Aut}(X, T)$$

Basic topological notion

Let A be a finite alphabet.
Let $X \subset A^{\mathbb{Z}}$ be a subshift invariant by the shift

$$\begin{aligned} \sigma : X &\rightarrow X \\ (x_n)_{n\in\mathbb{Z}} &\mapsto (x_{n+1})_{n\in\mathbb{Z}} \end{aligned}$$

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Theorem (Curtis-Hedlund-Lyndon)

Let ϕ be an automorphism of (X, σ)
There exists a local map $\hat{\phi}\colon A^{2r+1} \to A$ *s.t.*

$$\phi(x)_n = \hat{\phi}(x_{n-r} \ldots x_{n+r}) \textit{ for any } n \in \mathbb{Z}.$$

ϕ is a cellular automata.

Main theorem

The complexity $p_X : \mathbb{N} \to \mathbb{N}$, $p_X(n) = \sharp$ words of length n in X.

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Theorem (DDMP)

Let (X, σ) be a minimal subshift. If

$$\liminf_n \frac{p_X(n)}{n} < +\infty,$$

then $Aut(X, \sigma)/\langle \sigma \rangle$ is finite.

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Example. Primitive substitutive subshifts:
e.g. Tribonacci substitution

$$\tau(1) \mapsto 12,\ \tau(2) \mapsto 13,\ \text{and}\ \tau(3) \mapsto 1.$$

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Example. Primitive substitutive subshifts:
Generalizes results of V. Salo-I. Törmä.
Similar result by V. Cyr-B. Kra

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Example. This includes also

- Subshifts of polynomial complexity of arbitrarily high degree.
- Subshifts with subexponential complexity $p_X(n) \geq g(n)$ i.o. where $\lim_n g(n)/\alpha^n = 0$ for any $\alpha \in \mathbb{R}$.

Previous results: in the measurable setting

Centralizer group: for a measurable dynamical system $(X, \mathcal{B}, \mu, T)$,

$$C(T) = \{\phi\colon X \to X;\ \text{bi-measurable},\ \phi \circ T = T \circ \phi\}$$

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- A. Del Junco (78): same is true for the Chacon subshift.
- J. King, J.-P. Thouvenot (91): mixing system of finite rank

$$C(T)/\langle T \rangle \text{ is finite.}$$

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- M. Boyle, D. Lind, R. Rudolph (88): mixing subshift of finite type contains various subgroup.
- M. Hochman (2010): any SFT with positive entropy admits any finite group in $\text{Aut}(X, T)$.

From the measurable to the topological setting

For zero-entropy system:

- B. Host, F. Parreau (89): for a family of substitutive systems

$$C(\sigma) = \mathrm{Aut}(X, \sigma) \text{ and } C(\sigma)/\langle \sigma \rangle \text{ is finite.}$$

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- M. Lemánczyk, M. Mentzen (89): any finite group can be realized as $C(\sigma)/\langle\sigma\rangle$.

Main Ideas

Lemma

Let (X, T) be a minimal aperiodic dynamical system. The action of $Aut(X, T)$ on X

$$\begin{aligned} Aut(X,T) \times X &\rightarrow X \\ (\phi, x) &\mapsto \phi(x), \end{aligned}$$

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Proof. For any automorphism ϕ, the set

$$\{x; \phi(x) = x\}$$

is closed and T invariant.

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Two points $x, y \in (X, T)$ are asymptotic if

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Any automorphism ϕ maps an asymptotic pair to an asymptotic pair

Any automorphism ϕ induces a permutation on the collection of asymptotic pair.

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Corollary

For an infinite minimal t.d.s. (X, T), with an asymptotic pair, we have

$$\{1\} \longrightarrow \langle T \rangle \longrightarrow \ Aut(X,T) \ \stackrel{j}{\longrightarrow} \ Per\mathcal{A}_{/\sim},$$

where :

- $\mathcal{A}$ *denote the collection of asymptotic unordered pairs*
- $\{x, y\} \sim \{x', y'\}$ *if* x *and* x' *are in the same* T*-orbit.*
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$j(\phi)$ has a fixed point $\Leftrightarrow \phi \in \langle T \rangle$

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If $\sharp\mathcal{A}_{/\sim} = 1$, then $Aut(X, T) = \langle T \rangle$.
e.g. for Sturmian sequences

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If $\sharp\mathcal{A}_{/\sim} < +\infty$, then $\sharp Aut(X, T)/\langle T \rangle$ divides $\sharp\mathcal{A}_{/\sim}$.

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Proposition

Let (X, σ) be a subshift with $\liminf_n p_X(n)/n < \infty$, then there is a finite number of asymptotic pair, i.e.

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By contradiction: $\forall n \geq m$ big enough

$$\begin{aligned} p_X(n) - p_X(m) &= \sum_{i=m}^{n-1} p_X(i+1) - p_X(i) > (n-m)(K+1) \\ p_X(n) &> (n-m)(K+1) + p_X(m) \end{aligned}$$

More generally

In the same way: $x, y \in X$ are proximal if

$$\liminf_n dist(T^n x, T^n y) = 0.$$

$\phi \in \mathrm{Aut}(X, T)$ maps proximal points to proximal points.

Commutator in a group G: $[g, h] = ghg^{-1}h^{-1}$

$$G_0 = G, \qquad G_j = [G_{j-1}, G] = \langle [a, b];\ a \in G_{j-1}, b \in G \rangle.$$

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A group G is d-step nilpotent if $G_d = \{e\}$.

Example. If $d = 1$, G is Abelian.

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Theorem (DDMP)

If $\pi\colon (X,T) \to (G/\Gamma, L_g)$ is a proximal extension of a minimal d-step nil system,
then $Aut(X,T)$ is a d-step nilpotent group.
Moreover, $\hat{\pi}\colon Aut(X,T) \to Aut(G/\Gamma, L_g)$ is injective.

Theorem (DDMP)

If (X, T) is a minimal proximal extension of its maximal non trivial d-step nilfactor (X_d, T_d). Then $Aut(X, T)$ embeds into $Aut(X_d, T_d)$, and $Aut(X, T)$ is a d-step nilpotent group.

Example. Toeplitz subshifts are proximal extension of their maximal equicontinuous factor ($d = 1$). Their automorphism group is Abelian.

Bound the step of nilpotency

For (X, σ) a transitive subshift, $n \geq 1$, Local recurrence time:

$$N_X(n) = \inf\{|w|; w \in \mathcal{L}(X) \text{ contains all the words of length } n\}.$$

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Theorem (DDMP)

(X, σ) transitive subshift, with $\sup_{n\geq 1} \frac{N_X(n)}{n^d} < +\infty$.
Then, there exists C, depending only on d, such that any f.g. group of $Aut(X, \sigma)$ is virtually nilpotent of step at most C.

Open questions

Question

Given a countable group G. Does it exists a minimal subshift such that $\mathrm{Aut}(X, \sigma)/\langle\sigma\rangle$ is isomorphic to G ?

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Relation with the complexity ?

Cyr and Kra: if $p_X(n)/n^2 \to 0$ then $\mathrm{Aut}(X,\sigma)$ is periodic