$(M+1)$-step shift spaces that are not conjugate to M-step shift spaces

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- **Not so nice:** Shift map is continuous everywhere except at one point.

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- **Known:** Conjugacy classes of shifts of finite type, edge shifts, and M-step shifts are distinct.
- **Conjecture:** For each $M \in \mathbb{N} \cup \{0\}$ there exist an $(M+1)$-step shift space that is not conjugate to any M-step shift.

Shift spaces over infinite alphabets

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For all $x \in \Sigma_A$ the lenght of x is denoted by $l(x)$.

The topology.

Definition 2.2.

Given $x = (x_1, \ldots, x_k) \in \Sigma_A^{fin}$, $x \neq \emptyset$, *and a finite set* $F \subset A$, *we define a* generalized cylinder set *of* Σ_A *as*

$$Z(x, F) := \{y \in \Sigma_A : \; y_i = x_i \; \forall i = 1 \ldots, k, \; y_{k+1} \notin F\}.$$

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Σ_A *with the topology generated by the generalized cylinder sets above is zero dimensional (that is, it has a basis of clopen sets) and compact.*

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Remark 2.4.

Notice that a sequence (x^i) *converges to* y*, with* $l(y) < \infty$ *iff given a finite subset* $F \subset A$*, there exists a* K *such that, for all* $i > K$*,* $l(x^i) \geq l(y)$*,* $x^i_{l(y)+1} \notin F$ *and* $x^i_l = y_l$ *for* $l = 1, \ldots, l(y)$*.*

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Remark 2.6.

Property 3 ensures that there always exists infinite sequences in a non-empty shift space. It fact, Λ^{inf} *is dense in* Λ*.*

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Remark 2.8.

Λ is a shift space iff $\Lambda = X_{\mathbf{F}}$ for some $\mathbf{F} \subset \bigcup_{k \geq 1} A^k$.

M-step shifts and conjugacy

Definition 2.9.

Λ is a **shift of finite type**, *SFT, if we can take* **F** *having only finitely elements and an M-***step shift** *if $l(x) = M + 1$ for all $x \in$* **F**.

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Definition 2.10.

A map $\phi : \Lambda \to Y$, where $\Lambda \subseteq \Sigma_A$, $Y \subseteq \Sigma_B$ are shift spaces over alphabets A and B respectively, is a conjugacy if, and only if, it is bijective, continuous, shift commuting and $l(\phi(x)) = l(x)$ for all $x \in \Lambda$.

Idea of the proof.

Theorem 1.

For every $M \in \mathbb{N} \cup \{0\}$ there exists a one-sided shift space over an infinite alphabet that is an $(M+1)$-step shift space and is not conjugate to an M-step shift space".

References I

D. Gonçalves, D. Royer, $(M+1)$*-step shift spaces that are not conjugate to M-step shift spaces*. To appear at Bulletin des Sciences Mathématiques.

W. Ott, M. Tomforde and P. Willis, *One-sided shift spaces over infinite alphabets*. New York Journal of Mathematics Monographs **(5)**, 2014.