N-fold tilings

N. Bédaride, Th. Fernique
I2M, université Aix Marseille

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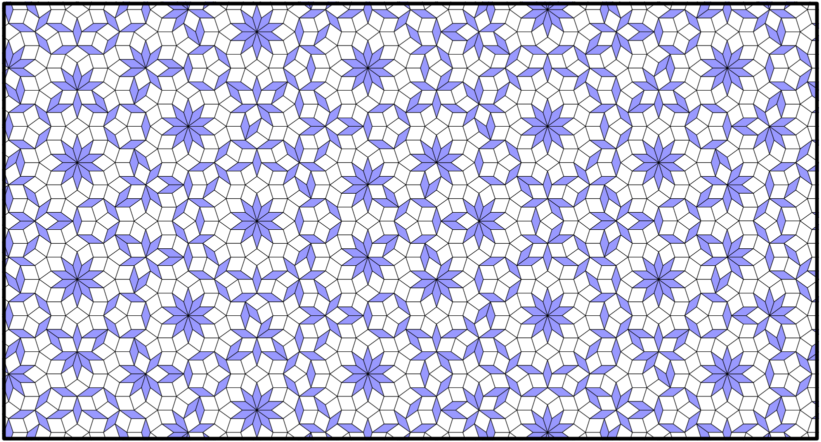
Tiling of the plane



Several classes of tilings

- Substitution.
- Cut and project.
- SFT (subshift of finite type).

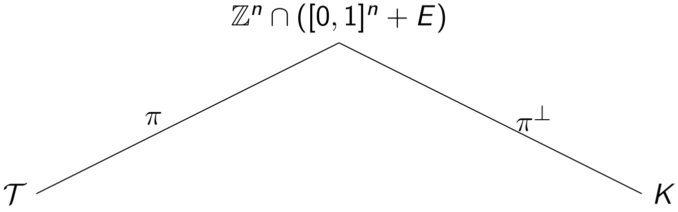
Cut and project

Let E be a 2-plane in $\mathbb{R}^n = E \oplus E'$.

π projection on E with respect to E'.

$\pi^\perp$ projection on E' with respect to E.

Strip: $[0,1]^n + E$



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Planar tilings : lift in $E + [0, t]^n$, where E is a 2-plane called the *slope* and t is the *thickness*.

Local rules

Definition

A slope E has **local rules** (LR) if there is a finite set of *patches* s. t. any rhombus tiling without any such patch is planar with slope E.

Question:

Find some LR for a given plane E ?

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Other possibility:

Question:

Find some SFT such that every tiling in the subshift is **planar**.

Subject of the talk

Study of the plan E_n given by

- If $n = 2p + 1$ the 2-plane of $\mathbb{R}^{2p+1}$ generated by

$$\begin{pmatrix} 1 \\ \cos \frac{2\pi}{2p+1} \\ \vdots \\ \cos \frac{4p\pi}{2p+1} \end{pmatrix}, \begin{pmatrix} 0 \\ \sin \frac{2\pi}{2p+1} \\ \vdots \\ \sin \frac{4p\pi}{2p+1} \end{pmatrix}.$$

- If $n = 2p$ the 2-plane of $\mathbb{R}^p$ generated by

$$\begin{pmatrix} 1 \\ \cos \frac{\pi}{p} \\ \vdots \\ \cos \frac{(p-1)\pi}{p} \end{pmatrix}, \begin{pmatrix} 0 \\ \sin \frac{\pi}{p} \\ \vdots \\ \sin \frac{(p-1)\pi}{p} \end{pmatrix}.$$

These planes define n **fold tilings**.

History

Tiling	local rules	
5,10-fold	yes	Penrose
8-fold	none	Burkov 88
(4k+i)-fold, $i \neq 0$	yes	Socolar 90
non algebraic slope	none	Le

Result

Theorem (B-Fernique)

The plane E_n admits local rules if and only if $n \neq 0[4]$.

Proof I: Subperiod

How do you find the good SFT ?

Grassmann-Plücker coordinates

Definition
The plane $\mathbb{R}\vec{u} + \mathbb{R}\vec{v}$ has GP-coordinates $(G_{ij})_{i<j} = (u_i v_j - u_j v_i)_{i<j}$.

Proposition (Grassmann-Plücker)
GP-coordinates satisfy all the relations $G_{ij} G_{kl} = G_{ik} G_{jl} - G_{il} G_{jk}$.

Definition
A **subperiod** of the plane is a vector $p\vec{e}_i + q\vec{e}_j + r\vec{e}_k$ such that we have a relation of the form: $pG_{jk} - qG_{ik} + rG_{ij} = 0$ where p, q, r are integers.

Subperiod and SFT

Lemma

There exists a subshift of finite type in which planar tilings all have the same subperiods.

Lift of the tiling: Surface $\mathcal{S}$

Tiling by rhombi

Subperiod

Plane E

Background

Theorem (B-Fernique 13)

Let E be a plane in $\mathbb{R}^4$.
We find an iff condition such that every tilling in the SFT is planar.
We find a sufficient condition such that E admits local rules.

Application to some planes in $\mathbb{R}^n$.

Proposition (B-Fernique 13)

For every n there exists a SFT such that every tiling in the SFT is planar and one tiling is planar with slope E_{4n}.

Proposition (Socolar 90, B-Fernique 13)

The plane $E_{4n+i}, i \neq 0$ admits local rules.

Example $n = 8$. Amman-Beenker tilings

$$\begin{cases} G_{12} = G_{23} = G_{34} = G_{14} \\ 2G_{12}^2 = G_{13}G_{24} \end{cases}$$

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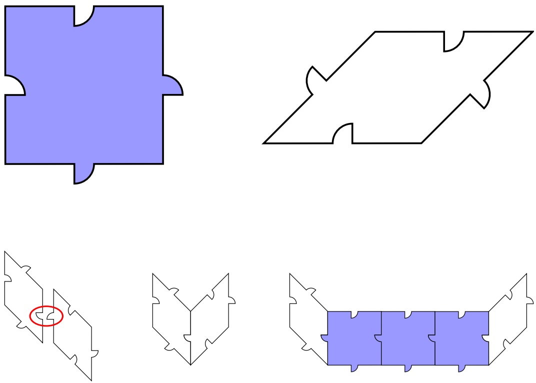
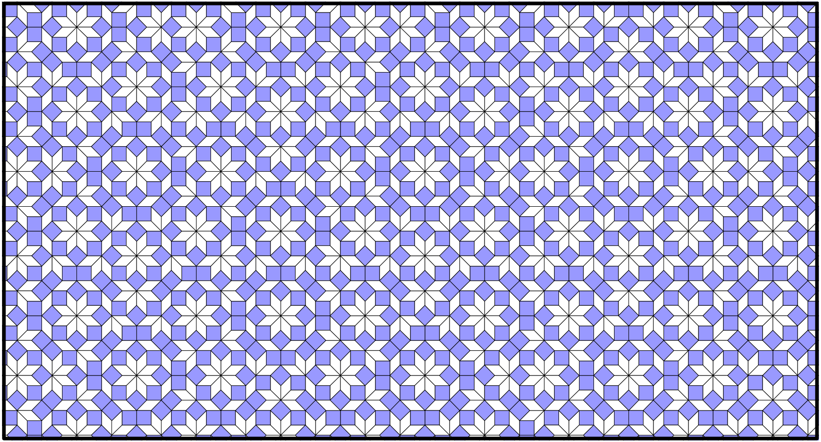
$$\begin{cases} G_{12} = G_{23} = G_{34} = G_{14} = 1 \\ 2 = G_{13}G_{24} \end{cases}$$

Infinity of planes solutions.
Subperiods thus characterize all the slopes $(1, t, 1, 1, 2/t, 1)$, $t > 0$.
First theorem implies the planarity of these planes.
The AB tilings are those maximizing the rhombus frequencies.

SFT and E_8



Ammann-Beenker tilings



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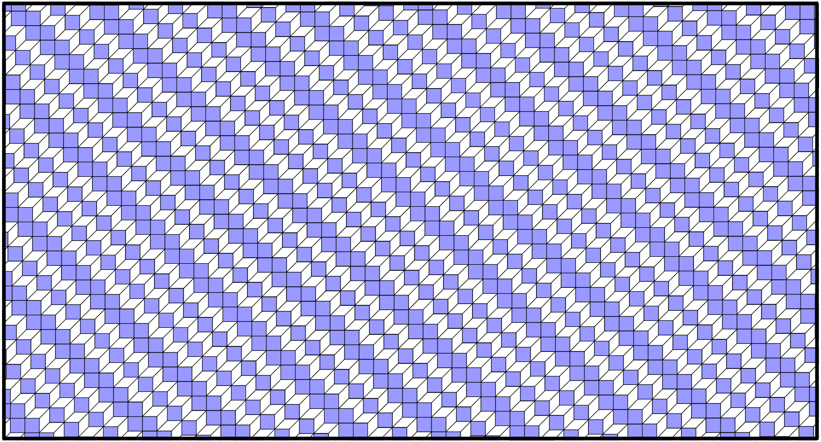

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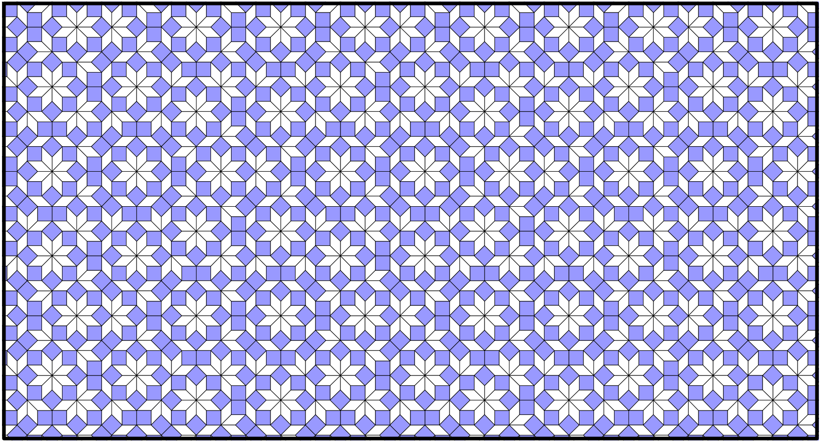

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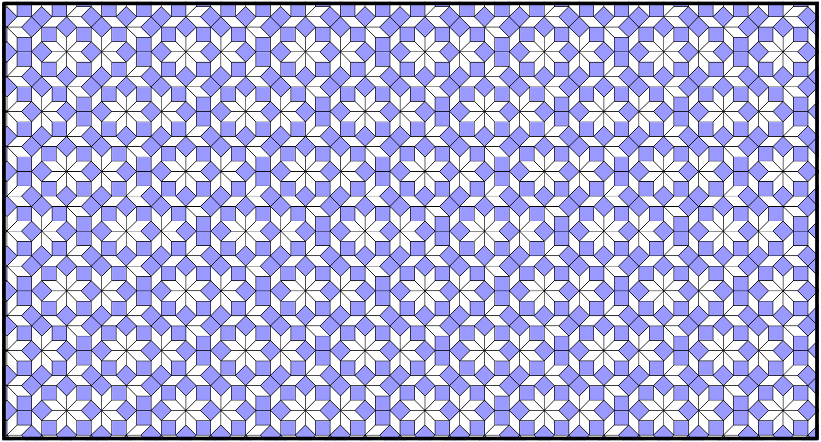

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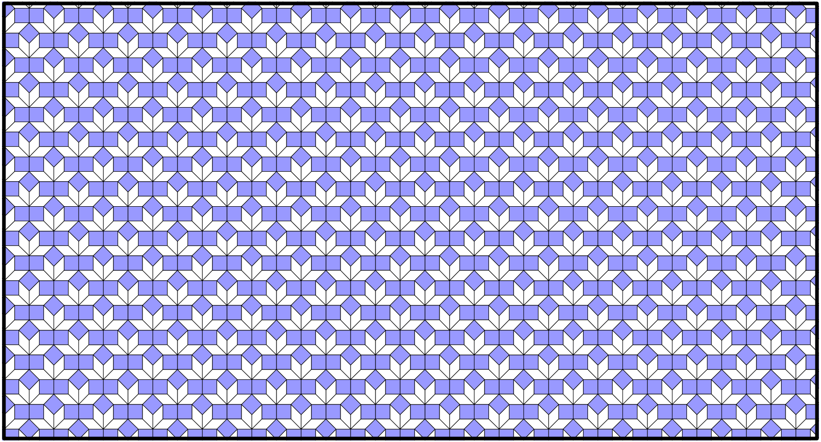


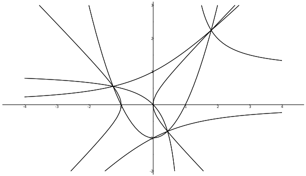
Example E_7

$$\begin{cases} G_{12} = G_{23} = G_{34} = G_{45} = G_{56} = G_{67} = G_{71} \\ G_{13} = G_{35} = G_{57} = G_{72} = G_{24} = G_{46} = G_{61} \\ G_{14} = G_{47} = G_{73} = G_{36} = G_{62} = G_{25} = G_{51} \end{cases}$$

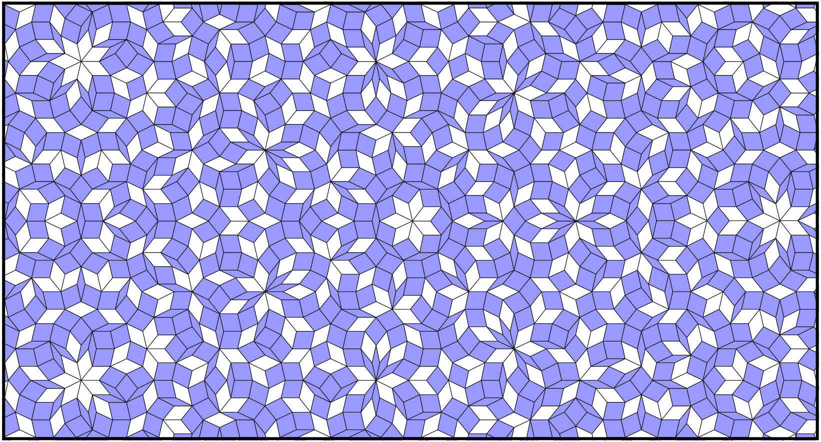
$$\begin{cases} G_{12}^2 = G_{13}^2 - G_{14}G_{12} \\ G_{12}G_{13} = G_{12}G_{14} + G_{13}G_{14} \\ G_{12}^2 = G_{14}^2 - G_{13}G_{14} \\ G_{14}^2 = G_{13}^2 - G_{13}G_{12} \end{cases}$$

$$X^3 - X^2 - 2X + 1 = 0.$$

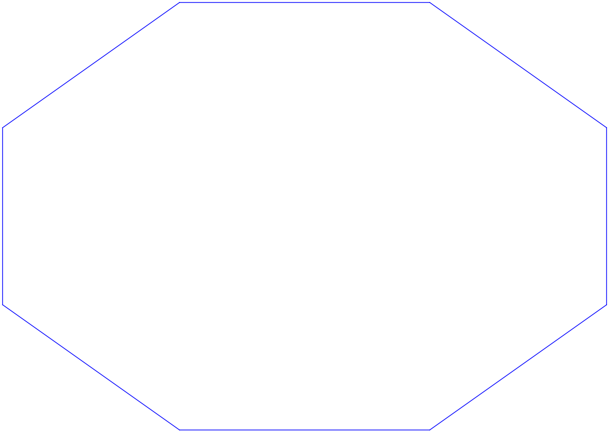
Finite number of solutions.



The plane E_7 admits local rules and $7 \neq 0[4]$.



Proof II: Window



Projection of $[0,1]^n$ on E'

Patch and window

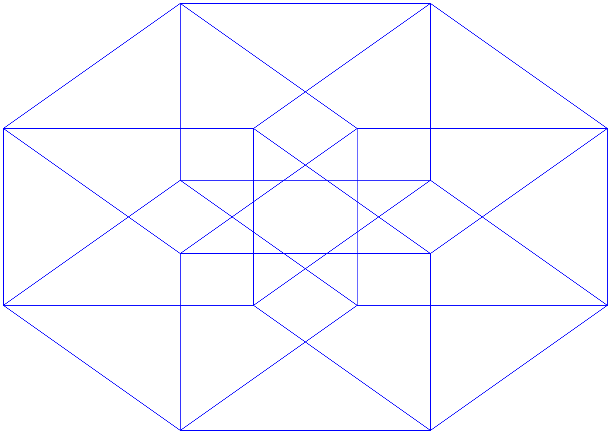
Let E be a 2 plane in $\mathbb{R}^n$.

The window is a polytope in $\mathbb{R}^{n-2}$.

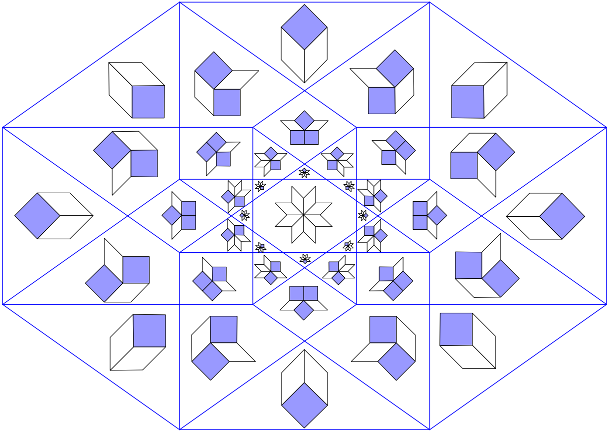
Partition of the window in polytopes with the projections of the faces of dimension $n-3$ of $[0,k]^n$ for $k \in \mathbb{N}$.

Lemma (Julien)

Bijection of the set of polytopes with the size k patches of the planar tiling of slope E and thickness 1.



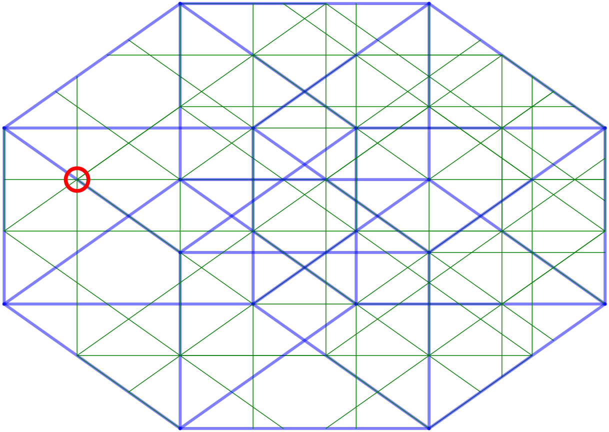
Partition of the window by the faces of dimension $n-3$ of the cube $[0,k]^n$



Bijection with the size k patches of the planar tiling of slope E and thickness 1

Coincidence

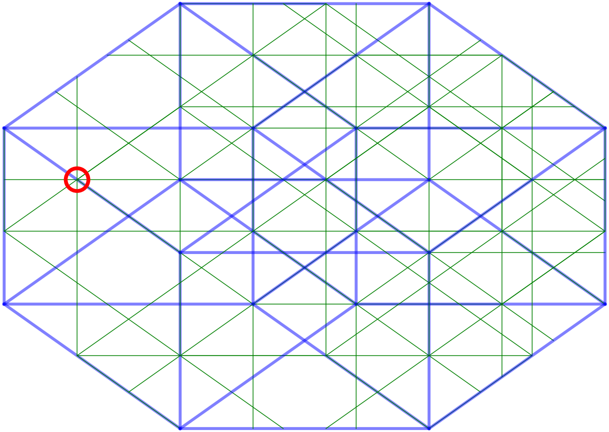
Intersection of at least $n-1$ projected faces of dimension $n-3$.



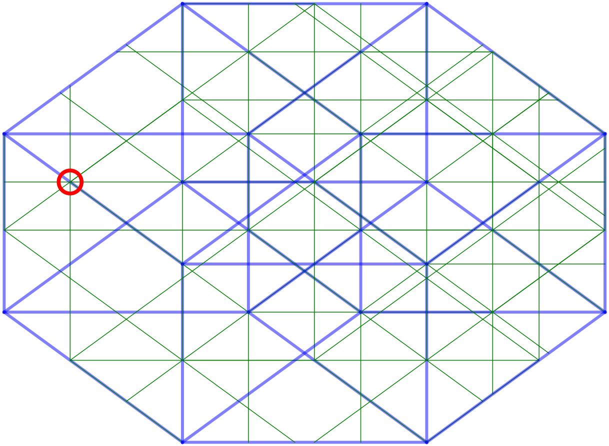
Now consider the SFT associated to E_n.

The plane moves inside this SFT.

Coincidence

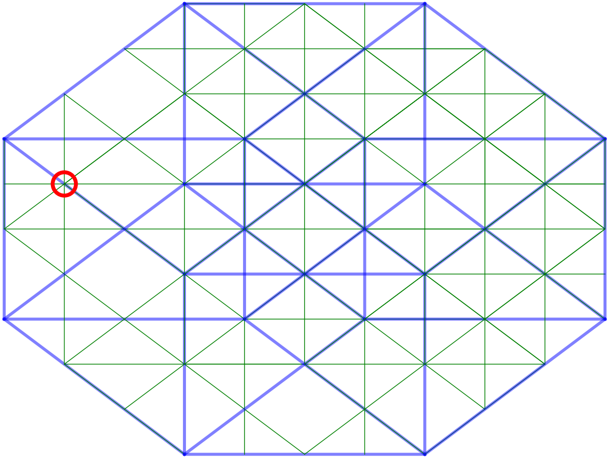


Coincidence



Coincidences preserved along the family

Coincidence



Coincidences preserved along the family

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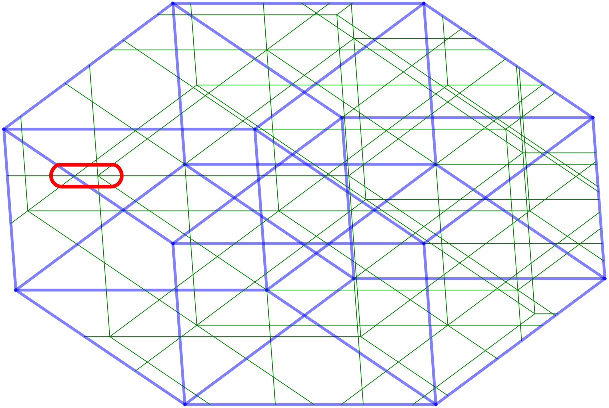

Coincidences not preserved outside the family

The proof is done...

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Lemma

Assume that E belongs to a curve such that any coincidence of E is also a coincidence of the points of this curve which are close enough to E. Then E does not admit weak local rules.

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Assume that E belongs to a curve such that any coincidence of E is also a coincidence of the points of this curve which are close enough to E. Then E does not admit weak local rules.

If a pattern of size k appear in E_t and not in E, then a new coincidence appears for $t = 0$: impossible. Any pattern of size k of E also appears in E_t. Then E does not admit weak local rules.

Questions

- Consider the SFT associated to E_8. Entropy ?
- Thickness of this SFT ?
- ...