Sofic (and Effective) Subshifts on f.g. Groups

Lecture 3: Domino Problem, Part II: f.g. groups.

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Introduction

Mini-course divided into 4 lectures

- Lecture 1: SD on f.g. groups: a computational approach.
- Lecture 2: Domino Problem, Part I: Wang tiles.
- **Lecture 3: Domino Problem, Part II: f.g. groups.**
- Lecture 4: Effective subshifts.

Previously on Lecture 2

- Domino Problem is decidable on $\mathbb{Z}$.
- Encode Turing machines inside $\mathbb{Z}^2$-SFT, Robinson tiling.
- Domino Problem is undecidable on $\mathbb{Z}^2$.

Reminder: Domino Problem for f.g. groups

Fix G a f.g. group and S a generating set for G.

Domino Problem for G-SFTs

Input: F a finite set of forbidden patterns on S.
Output: **Yes** if there exists a configuration in X_F, **No** otherwise.

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Input: F a finite set of forbidden patterns on S.
Output: **Yes** if there exists a configuration in X_F, **No** otherwise.

Question

Which f.g. groups have decidable Domino Problem ?

Lecture 3: Domino Problem, Part II: f.g. groups.

1. Basic facts about DP for f.g. groups
   - Domino Problem and subgroups
   - Word Problem vs. Domino Problem
   - Domino Problem as a Markov property
   - Toward a characterization
2. Recent advances
   - Kari-Culik aperiodic tileset
   - DP on the hyperbolic plane $\mathbb{H}^2$
   - Baumslag-Solitar groups
   - Virtually nilpotent groups
3. How to go further ?
   - Aperiodic SFT and DP
   - How to go further ?

Domino Problem and subgroups (I)

Proposition

Let H and G be two f.g. groups s.t. H is a subgroup of G. If G has decidable Domino Problem, then so has H.

Sketch of the proof:

- Let X be an H-SFT given by $X = X_{\mathcal{F}}$, $\mathcal{F}$ finite.
- Consider H-patterns in $\mathcal{F}$ as G-patterns.
- Define X' the G-SFT given by $X' = X_{\mathcal{F}}$.
- Then $X = \emptyset \Leftrightarrow X' = \emptyset$.

Domino Problem and subgroups (II)

Proposition

Let H and G be two f.g. groups s.t. H is a subgroup of G of finite index. If H has decidable Domino Problem, then so has G.

Sketch of the proof: Let τ be a finite set of Wang tiles on G.

- Since $[G : H] < \infty$ there are finitely many left cosets $g_1 H, \ldots, g_k H$ (choose g_i of minimal length).
- Construct $\tau' \subset \tau^k$ the finite set of Wang tiles on H *compatible* with the choice of the g_i.
- $X_\tau = \emptyset \Leftrightarrow X_{\tau'} = \emptyset$.

Reminder: Word Problem

Does there exist an algorithm that decides whether two words w_1 and w_2 on the generators and their inverses represent the same element in G ($w_1 =_G w_2$)?

$$WP(G) = \left\{ w \in \left(S \cup S^{-1}\right)^* \mid w =_G 1_G \right\}.$$

Definition

A f.g. group G has **decidable WP** if there exists an algorithm that takes two words w_1 and w_2 as input and outputs **Yes** if $w_1 =_G w_2$ and **No** if $w_1 \neq_G w_2$.

Remark: Decidability of WP does not depend on the choice of S.

Word Problem vs. Domino Problem

Property

Let G be a finitely generated group with decidable domino problem, then G has decidable word problem.

Sketch of the proof:

- Suppose that S generates G.
- Consider a word $w \in \left(S \cup S^{-1}\right)^*$ s.t. $w =_G g$.
- Define the SFT $X_\mathcal{F}$ on A ($|A| \geq 3$) by forbidden patterns

$$\mathcal{F} = \{p_a\}_{a \in A}$$

  where p_a has support $\{1_G, g\}$ s.t. $(p_a)_{1_G} = (p_a)_g = a$.
- Lemma: $w =_G 1_G \Leftrightarrow X_\mathcal{F} = \emptyset$.

Domino Problem as a Markov property

A property of f.p. groups is a **Markov property** if

(i) there exists a f.p. group with this property,

(ii) there exists a f.p. group that cannot be embedded in any f.p. group with the property.

Examples: being trivial, abelian, nilpotent, solvable, free, torsion-free... are Markov properties.

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Theorem (Adian & Rabin, 1955-1958)

If $\mathcal{P}$ is a Markov property, the problem of deciding whether a f.p. group has property $\mathcal{P}$ is undecidable.

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Proposition

The group property *G has decidable domino problem* is a Markov property.

What do we know ? (in 2012)

Domino Problem is

- decidable on $\mathbb{Z}$, $\mathbb{F}_k$, VF groups.
- undecidable on $\mathbb{Z}^d$ $(d \geq 2)$, all f.g. groups having $\mathbb{Z}^2$ as subgroup.

What do we know ? (in 2012)

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Conjecture (Ballier)

A f.g. group G has decidable Domino Problem iff G is virtually free.

Lecture 3: Domino Problem, Part II: f.g. groups.

KC aperiodic tileset: Principle

Encode a small ***aperiodic dynamical system*** $\mathbf{T}$ inside a ***finite set of Wang tiles***.

$$\mathbf{T}^3(x)$$

$$\mathbf{T}^2(x)$$

$$\mathbf{T}^1(x)$$

$$x$$

$$\mathbf{T}^{-1}(x)$$

$$\mathbf{T}^{-2}(x)$$

$$\mathbf{T}^{-3}(x)$$

Representation of reals numbers

Given x a real number, a **representation of x** is a sequence of integers $(x_k)_{k\in\mathbb{Z}}$ such that :

- $\forall k \in \mathbb{Z}, x_k \in \{\lfloor x \rfloor, \lfloor x \rfloor + 1\}$;
- $\forall k \in \mathbb{Z}$,

$$\lim_{n\to\infty} \frac{x_{k-n} + x_{k+1} + \cdots + x_{k+n}}{2n+1} = x.$$

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$$\lim_{n\to\infty} \frac{x_{k-n} + x_{k+1} + \cdots + x_{k+n}}{2n+1} = x.$$

Remark: If $(x_k)_{k\in\mathbb{Z}}$ is a representation of x, then so is $\forall \ell \in \mathbb{Z}$, $(x_{k+\ell})_{k\in\mathbb{Z}}$.

Balanced representation of reals numbers

Let $x \in \mathbb{R}$ be arbitrary. For every $k \in \mathbb{Z}$, let

$$B_k = \lfloor kx \rfloor - \lfloor (k-1)x \rfloor.$$

The bi-infinite sequence $(B_k)_{k \in \mathbb{Z}}$ is a **balanced representation of x**.

Balanced representation of reals numbers

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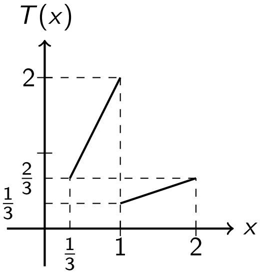
The bi-infinite sequence $(B_k)_{k\in\mathbb{Z}}$ is a **balanced representation of** x.

- $(B_k)_{k\in\mathbb{Z}}$ is a representation of x in the sense defined before.
- Balanced representations of irrational x are sturmian sequences, while for rational x the sequence is periodic.

A piecewise linear affine map T

Let $T : [\frac{1}{3}; 2] \to [\frac{1}{3}; 2]$ be the piecewise linear map defined by

$$T : x \mapsto \begin{cases} 2x \text{ if } x \in [\frac{1}{3}; 1] \\ \frac{1}{3}x \text{ if } x \in]1; 2] \end{cases}$$



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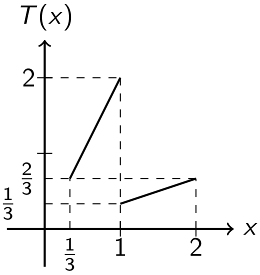


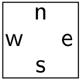
Proposition

The dynamical system **T** is aperiodic.

Encoding multiplications inside Wang tiles

A λ-multiplication tile is a Wang tile such that

$$\lambda \cdot s + w = n + e.$$



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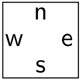
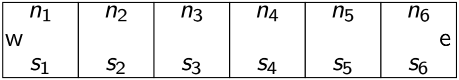


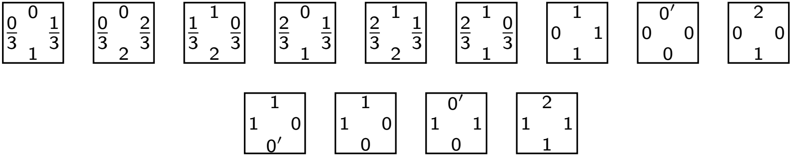
Such tiles perform multiplication by λ with some errors that propagate.



$$\lambda \cdot \frac{s_1 + \cdots + s_6}{6} + \frac{w}{6} = \frac{n_1 + \cdots + n_6}{6} + \frac{e}{6}.$$

Kari-Culik aperiodic set of 13 Wang tiles

All tiles are λ-multiplication tiles, with $\lambda = 2$ or $\frac{1}{3}$.



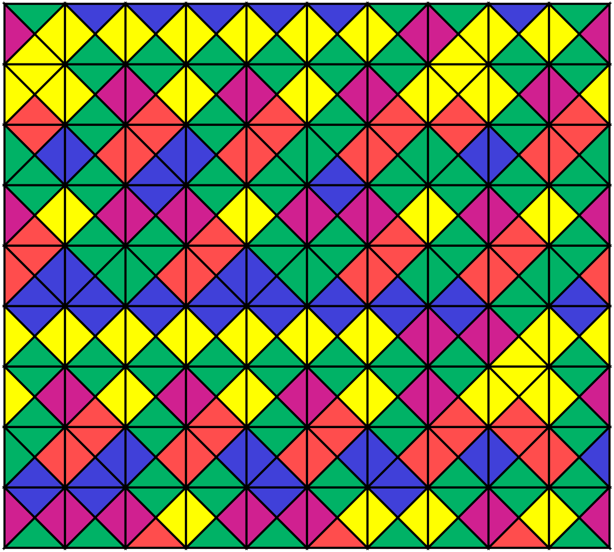
Theorem (Kari-Culik, 1996)

KC tileset is aperiodic.

An example of tiling

1 / 1 0 / 0′	2 / 0 0 / 1	2 / 0 0 / 1	2 / 0 0 / 1	2 / 0 0 / 1	2 / 0 0 / 1	1 / 0 1 / 1	1 / 1 0 / 0′	2 / 0 0 / 1	1 / 0 1 / 1
0′ / 0 0 / 0	1 / 0 1 / 1	1 / 1 0 / 0	1 / 0 1 / 1	1 / 1 0 / 0	1 / 0 1 / 1	1 / 1 0 / 0	0′ / 0 0 / 0	1 / 0 1 / 1	1 / 1 0 / 0
0 / $\frac{1}{3}$ $\frac{2}{3}$ / 1	1 / $\frac{2}{3}$ $\frac{0}{3}$ / 1	0 / $\frac{0}{3}$ $\frac{2}{3}$ / 2	1 / $\frac{2}{3}$ $\frac{0}{3}$ / 1	0 / $\frac{0}{3}$ $\frac{1}{3}$ / 1	1 / $\frac{1}{3}$ $\frac{0}{3}$ / 2	0 / $\frac{0}{3}$ $\frac{1}{3}$ / 1	0 / $\frac{1}{3}$ $\frac{2}{3}$ / 1	1 / $\frac{2}{3}$ $\frac{0}{3}$ / 1	0 / $\frac{0}{3}$ $\frac{1}{3}$ / 1
1 / 1 0 / 0	1 / 0 1 / 1	2 / 1 1 / 1	1 / 1 0 / 0	1 / 0 1 / 1	2 / 1 1 / 1	1 / 1 0 / 0	1 / 0 1 / 1	1 / 1 0 / 0	1 / 0 1 / 1
0 / $\frac{0}{3}$ $\frac{2}{3}$ / 2	1 / $\frac{2}{3}$ $\frac{1}{3}$ / 2	1 / $\frac{1}{3}$ $\frac{0}{3}$ / 2	0 / $\frac{0}{3}$ $\frac{2}{3}$ / 2	1 / $\frac{2}{3}$ $\frac{1}{3}$ / 2	1 / $\frac{1}{3}$ $\frac{0}{3}$ / 2	0 / $\frac{0}{3}$ $\frac{1}{3}$ / 2	1 / $\frac{1}{3}$ $\frac{0}{3}$ / 2	0 / $\frac{0}{3}$ $\frac{1}{3}$ / 1	1 / $\frac{1}{3}$ $\frac{0}{3}$ / 2
2 / 0 0 / 1	2 / 0 0 / 1	2 / 0 0 / 1	2 / 0 0 / 1	2 / 0 0 / 1	2 / 0 0 / 1	2 / 0 1 / 1	2 / 1 1 / 1	1 / 1 0 / 0′	2 / 0 0 / 1
1 / 0 1 / 1	1 / 1 0 / 0	1 / 0 1 / 1	1 / 1 0 / 0	1 / 0 1 / 1	1 / 1 0 / 0	1 / 0 1 / 1	1 / 1 0 / 0	0′ / 0 0 / 0	1 / 0 1 / 1
1 / $\frac{1}{3}$ $\frac{0}{3}$ / 2	0 / $\frac{0}{3}$ $\frac{2}{3}$ / 2	1 / $\frac{2}{3}$ $\frac{0}{3}$ / 1	0 / $\frac{0}{3}$ $\frac{2}{3}$ / 1	1 / $\frac{2}{3}$ $\frac{0}{3}$ / 2	0 / $\frac{0}{3}$ $\frac{2}{3}$ / 1	1 / $\frac{2}{3}$ $\frac{0}{3}$ / 2	0 / $\frac{0}{3}$ $\frac{2}{3}$ / 1	0 / $\frac{2}{3}$ $\frac{0}{3}$ / 1	1 / $\frac{0}{3}$ $\frac{2}{3}$ / 1
2 / 1 1 / 1	2 / 1 1 / 1	1 / 1 0 / 0	1 / 0 1 / 1	2 / 1 1 / 1	1 / 1 0 / 0	2 / 0 0 / 1	1 / 0 1 / 1	1 / 1 0 / 0	1 / 0 1 / 1

An example of tiling



Kari's proof of undecidability of DP on $\mathbb{Z}^2$

Theorem (Kari, 2007)

The mortality problem for rational piecewise affine maps is undecidable.

Given any piecewise affine map f with rational coefficients, there exists a finite tileset τ s.t.

$$f \text{ has an immortal point} \Leftrightarrow X_\tau \neq \emptyset.$$

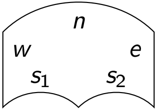
Theorem (Kari, 2007)

Domino Problem is undecidable on $\mathbb{Z}^2$.

DP on the hyperbolic plane $\mathbb{H}^2$

A **λ-multiplication**  **-tile** is a  -tile such that

$$\lambda \cdot n + w = \frac{s_1 + s_2}{2} + e.$$



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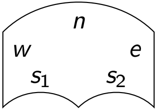

Such tiles perform multiplication by λ with some errors that propagate.

$$\lambda \cdot \frac{n_1 + \cdots + n_k}{k} + \frac{w}{k} = \frac{s_1 + \cdots + s_{2k}}{2k} + \frac{e}{k}.$$

Theorem (Kari,2007)

Domino Problem is undecidable on $\mathbb{H}^2$.

Baumslag-Solitar groups (I)

Baumslag-Solitar group: $BS(m, n) =< a, b | a^m b = ba^n >$

Properties

- Decidable WP (Magnus, 1932)
- Not VF (Baumslag-Solitar, 1962)
- Does not contain $\mathbb{Z}^2$ as a subgroup (but contains arbitrarily large finite grids) for $m \wedge n = 1$.

Baumslag-Solitar groups (I)

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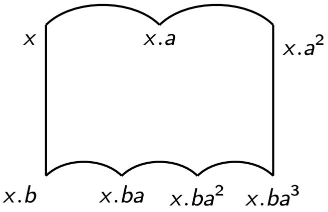
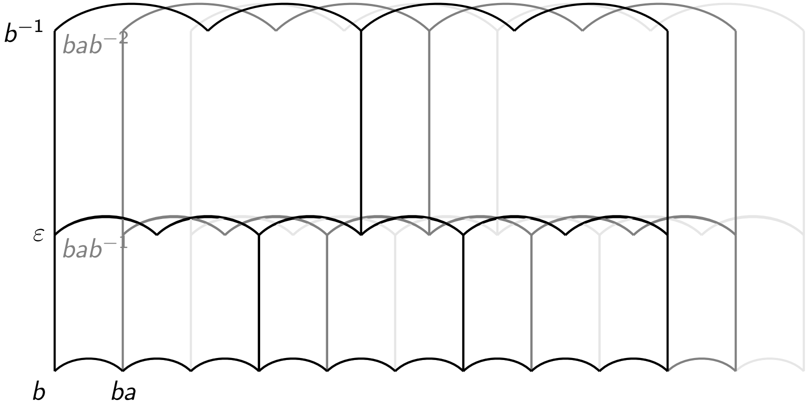
In the sequel: $BS(2,3) = < a, b | a^2 b = b a^3 >$



Structure



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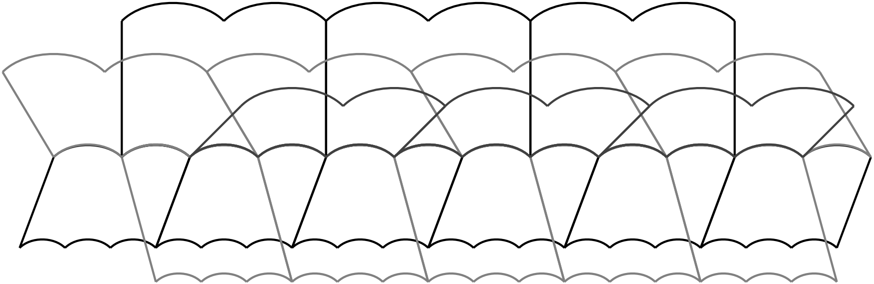
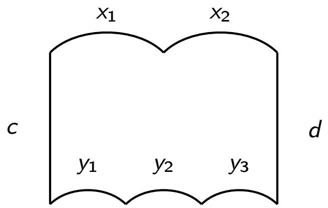


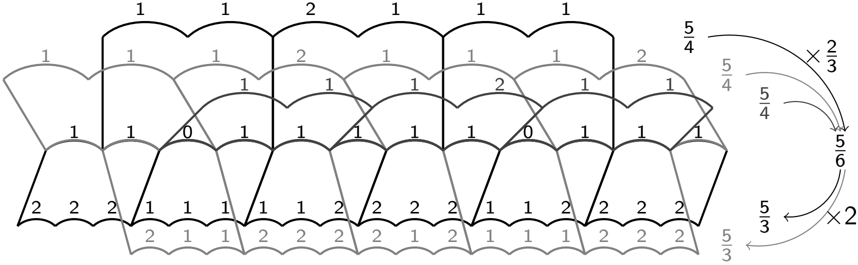
λ-multiplication tiles in BS(2, 3))

A **λ-multiplication BS(2, 3)-tile** is a BS(2, 3)-tile such that



$$\lambda \frac{x_1 + x_2}{2} + c = \frac{y_1 + y_2 + y_3}{3} + d$$

An example of tiling



Baumslag-Solitar groups (II)

Theorem (A.& Kari, 2013)

There exist weakly aperiodic SFT on $BS(m, n)$ for every $m, n > 0$.

But the SFT constructed is not strongly aperiodic (cannot avoid period like $bab^{-1}a^{m-1}ba^{-1}a^{-(m-1)}$).

Theorem (A.& Kari, 2013)

The domino problem is undecidable on $BS(m, n)$.

Question

Does $BS(m, n)$ admits strongly aperiodic SFT ?

Virtually nilpotent groups (I)

Theorem (Ballier & Stein, 2014)

Let G be a f.g. and virtually nilpotent group. Then the following are equivalent

(i) G is virtually free,

(ii) G has decidable domino problem.

Virtually nilpotent groups (I)

Theorem (Ballier & Stein, 2014)

Let G be a f.g. and virtually nilpotent group. Then the following are equivalent

(i) G is virtually free,

(ii) G has decidable domino problem.

Theorem (Kuske & Lorhey, Muller & Schupp)

Let G be a f.g. group. Then the following conditions are equivalent

(i) G is virtually free.

(ii) G has finite tree-width.

(iii) MSO is decidable on G.

(iv) G has context-free WP.

Virtually nilpotent groups (II)

Theorem (Ballier & Stein, 2014)

If G is a f.g. virtually nilpotent group with infinite tree-width, then DP is undecidable on G.

Sketch of the proof:

- If G has infinite tree-width, then G has a **thick end**.
- If $[G : H] < \infty$, then G has a thick end iff H has a thick end.
- A nilpotent group G has a torsion free subgroup H of finite index.
- If H is a f.g. torsion-free nilpotent group with a thick end, then H contains a $\mathbb{N} \times \mathbb{Z}$ structure
- Reduction from the DP on $\mathbb{Z}^2 \Rightarrow$ DP is undecidable on H.
- $\Rightarrow$ DP is undecidable on G.

Lecture 3: Domino Problem, Part II: f.g. groups.

Domino Problem and Aperiodicity

$e(G) \geq 2$? Cohen's Conjecture $e(G) = 1$?

	$\nexists$ weakly aperiodic SFT	$\nexists$ strongly aperiodic SFT $\exists$ weakly aperiodic SFT	$\exists$ strongly aperiodic SFT
Decidable DP	$\mathbb{Z}$	Free groups	?
Undecidable DP	?	?	$\mathbb{Z}^2, \mathbb{Z}^3$ Heisenberg

BS(m,n) ?

Groups with Undecidable WP ?

Remark

Proposition

Let H and G be two f.g. groups s.t. H is a subgroup of G of finite index. Then H has decidable DP iff G has decidable DP.

Proposition

Let H and G be two f.g. groups s.t. H is a subgroup of G of finite index. Let τ be a finite tileset on H.

- X_τ is weakly aperiodic on H iff X_τ is weakly aperiodic on G.
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Domino Problem and Aperiodicity

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Groups with Undecidable WP ?

Domino Problem and Aperiodicity

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	$\nexists$ weakly aperiodic SFT	$\nexists$ strongly aperiodic SFT $\exists$ weakly aperiodic SFT	$\exists$ strongly aperiodic SFT
Decidable DP	$e(G) = 2$	VF groups	?
Undecidable DP	?	?	$\mathbb{Z}^2, \mathbb{Z}^3$ Heisenberg

BS(m,n) ?

Groups with Undecidable WP ?

Some questions and conjectures (I)

Conjecture

Let G be a f.g. group. Then $e(G) = 2$ iff G has decidable DP and no weakly aperiodic SFT.

Conjecture (generalizes Wang's original idea)

If G is f.g. and does not admit weakly aperiodic SFT, then G has decidable DP.

Domino Problem and Aperiodicity

$e(G) \geq 2$? Cohen's Conjecture $e(G) = 1$?

	$\nexists$ weakly aperiodic SFT	$\nexists$ strongly aperiodic SFT $\exists$ weakly aperiodic SFT	$\exists$ strongly aperiodic SFT
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BS(m,n) ?

Groups with Undecidable WP ?

Some questions and conjectures (II)

Questions

- Does BS(m,n) admits strongly aperiodic SFTs ?
- Which groups admits weakly but not strongly aperiodic SFTs ?

Question

Does there exists a f.g. group with decidable Domino Problem that admits a strongly aperiodic SFT ?

Question

What about weakly/aperiodic SFTs and f.g. groups with undecidable WP ?

How to go further ?

- Use different characterizations of VF groups (decidable MSO logic, finite tree-width, context-free WP,...) ?
- Encode TM computation inside G ? or another computational model (which one) ?
- Construct (weakly/strongly) aperiodic SFT ?

Thank you for your attention !!