Communication Complexity for Multidimensional subshifts

Towards Characterizing Soficness

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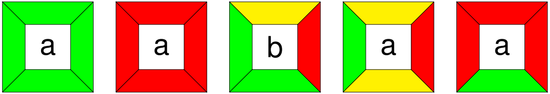
Main idea

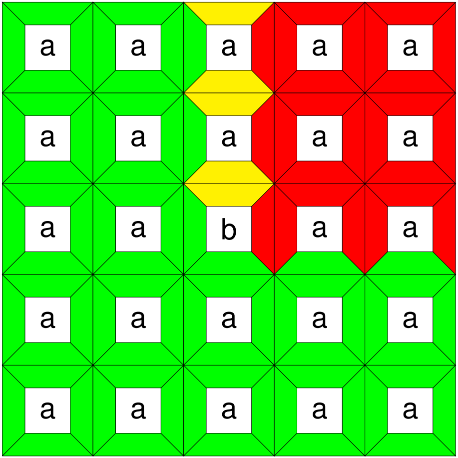
Use tools from Theoretical Computer Science to give insights into sofic shifts in $\mathbb{Z}^2$.

- Can be used possibly on more general groups G

Joint work with Pierre Guillon, Work in Progress.

Sofic shifts in 2D





Sofic shifts in 2D

- Sofic shifts are given by labelled Wang tiles.

Main question: when is a shift sofic ?

The idea

Let S be a subshift.

$$S \subseteq A^{\mathbb{Z}^2} \subseteq A^{\mathbb{Z}\times\mathbb{Z}^+} \times A^{\mathbb{Z}\times\mathbb{Z}^-}$$

- The set of lower half planes form a $\mathbb{Z}$-dynamical system S_1
- The set of upper half planes form a $\mathbb{Z}$-dynamical system S_2
- When is $(x, y) \in S_1 \times S_2$ an element of S ?

Intuitively, in a sofic shift, the “information” “needed” “to” “know” “whether” “x” and “y” “can” “be” “put” “together” “is” “small” (finite entropy).

Fancy words on a slide

- S is a joining of S_1 and S_2
- We search to measure how independent S_1 and S_2 are inside this joining.

The idea

- Divide the plane into two halfs.
- Give the first half to Almighty Alice, the second one to Almighty Bob.

How much information should they exchange to decide whether they would obtain a valid picture by putting the two halfs together ?

Example: At most one symbol b on the entire plane.

a	a	a	a	a	a	a	a	a	a		a	a	a	a	a	a	a	a
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a	a	a	a	a	a	a	a	a	a		a	a	a	a	a	a	a	a
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a	a	a	a	a	a	a	a	a	a		a	a	a	a	a	a	a	a
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a	a	a	a	a	a	a	a
a	a	a	a	a	a	a	a

Protocol

If S is sofic, there is a protocol that exchanges few bits:

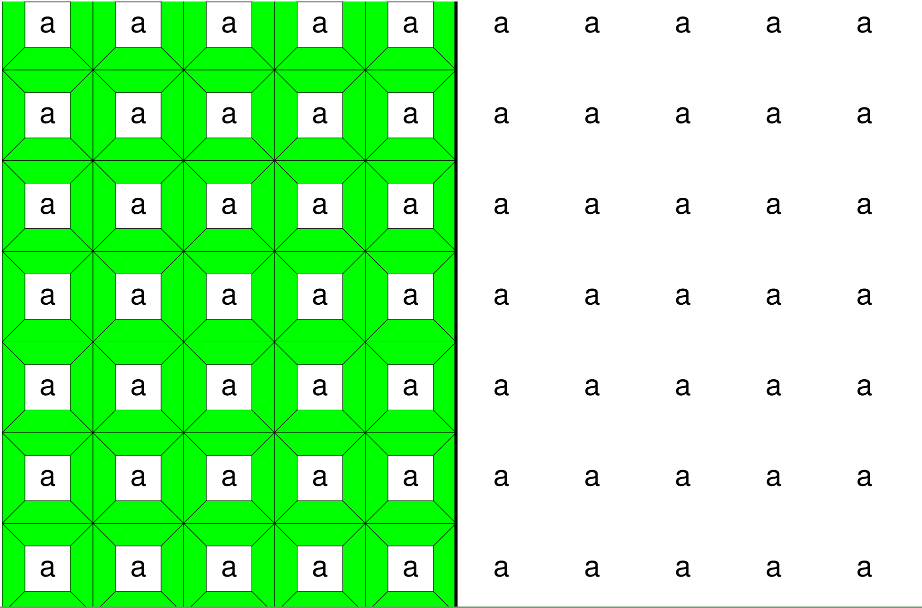
- Alice decides on how to tile its part of the plane.
- Alice sends the boundary to Bob
- Bob checks if it can tile its part of the plane with the same boundary as Alice.

If Alice makes the good choice, this protocol will succeed (non deterministic protocol).

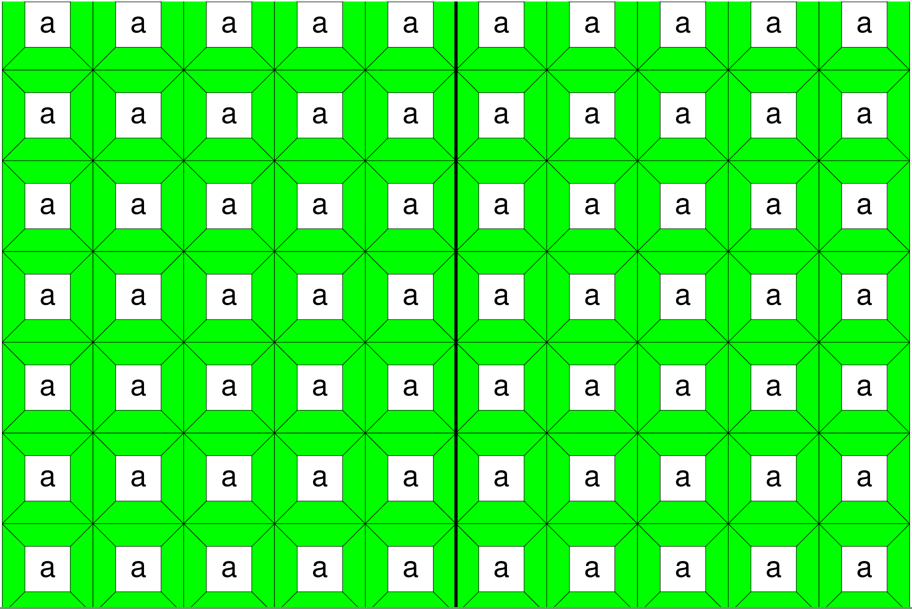
First example

a	a	a	a	a	a	a	a	a	a
a	a	a	a	a	a	a	a	a	a
a	a	a	a	a	a	a	a	a	a
a	a	a	a	a	a	a	a	a	a
a	a	a	a	a	a	a	a	a	a
a	a	a	a	a	a	a	a	a	a
a	a	a	a	a	a	a	a	a	a

First example



First example



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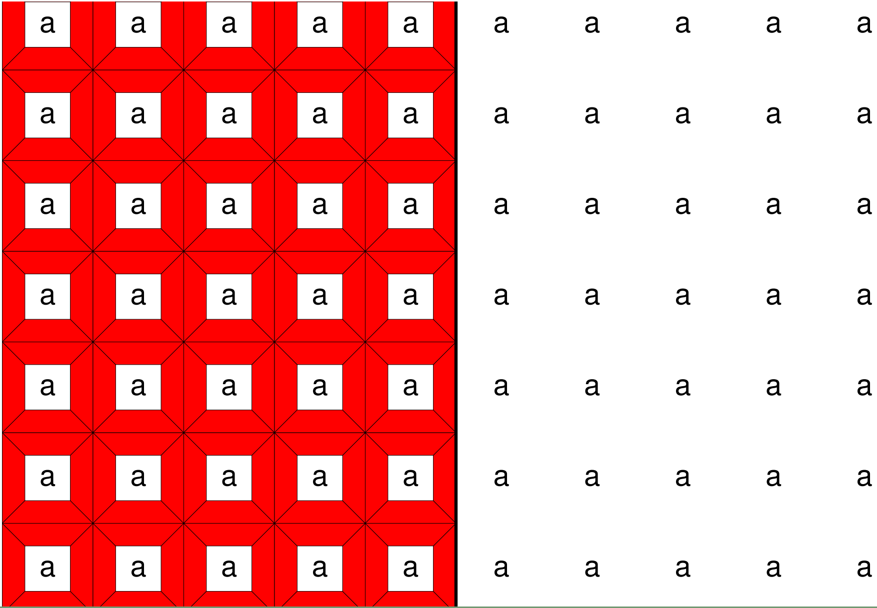
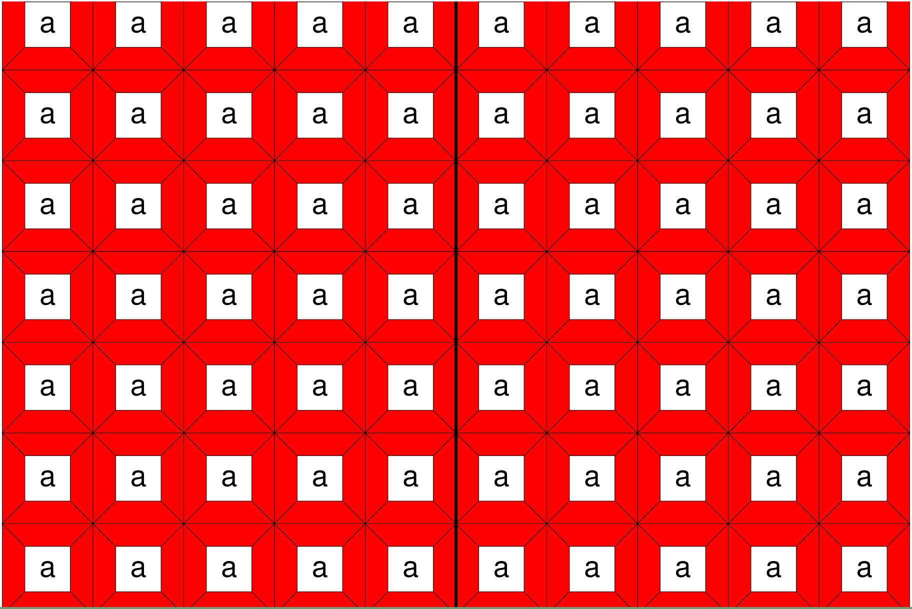


First example



Second example

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a	a	a	a	a	a	b	a	a	a
a	a	a	a	a	a	a	a	a	a
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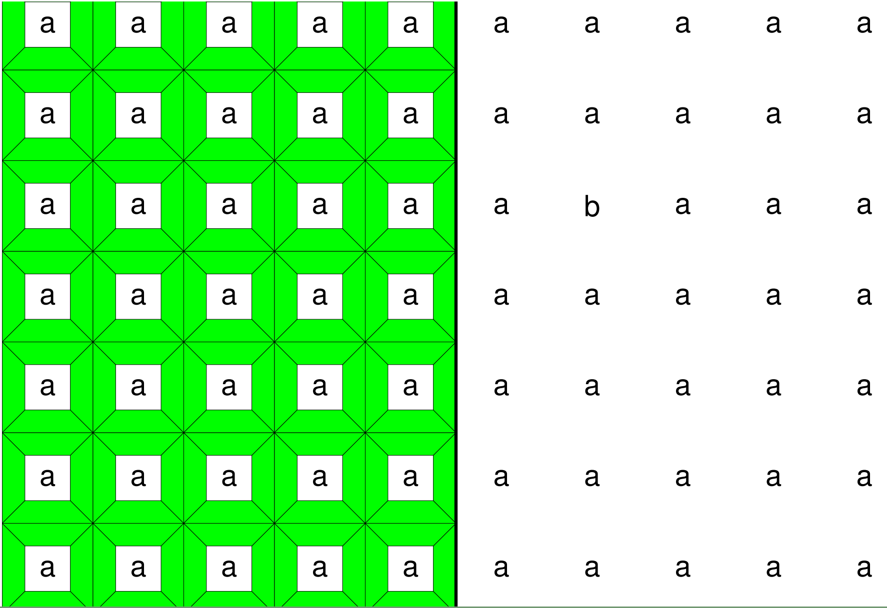

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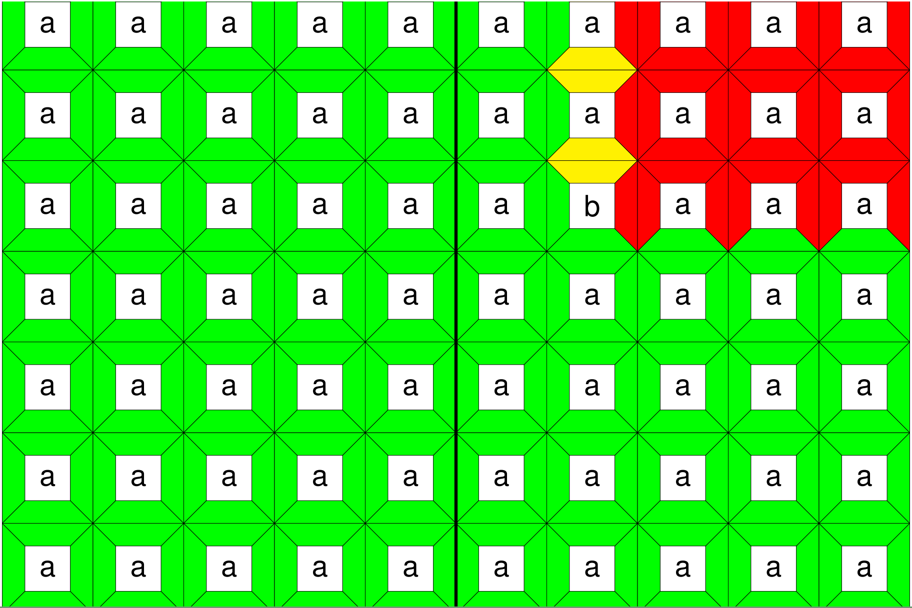

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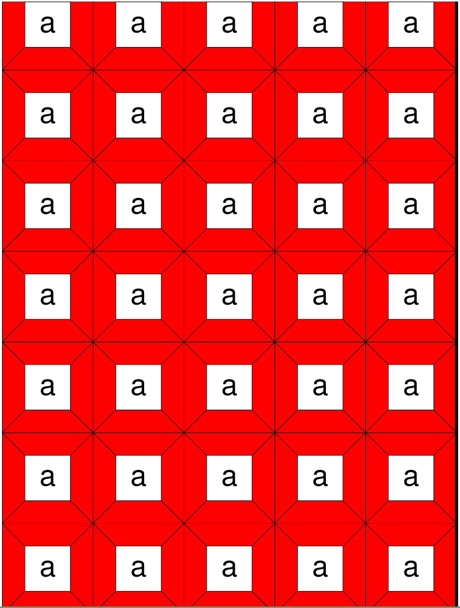

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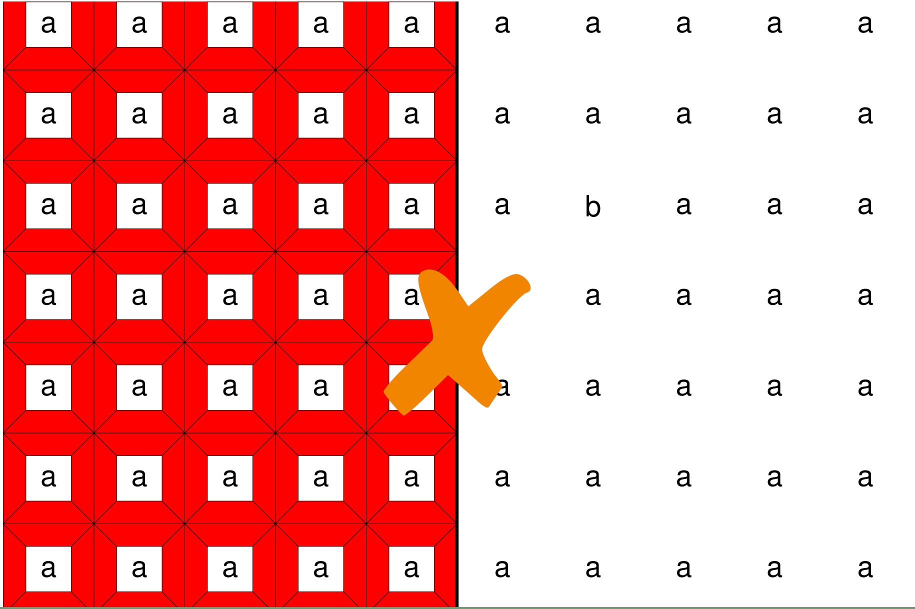

Third example

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a	a	a	a	a	a	b	a	a	a
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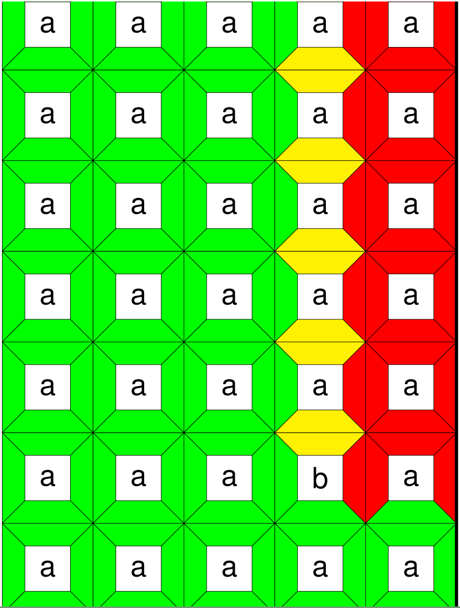
Third example



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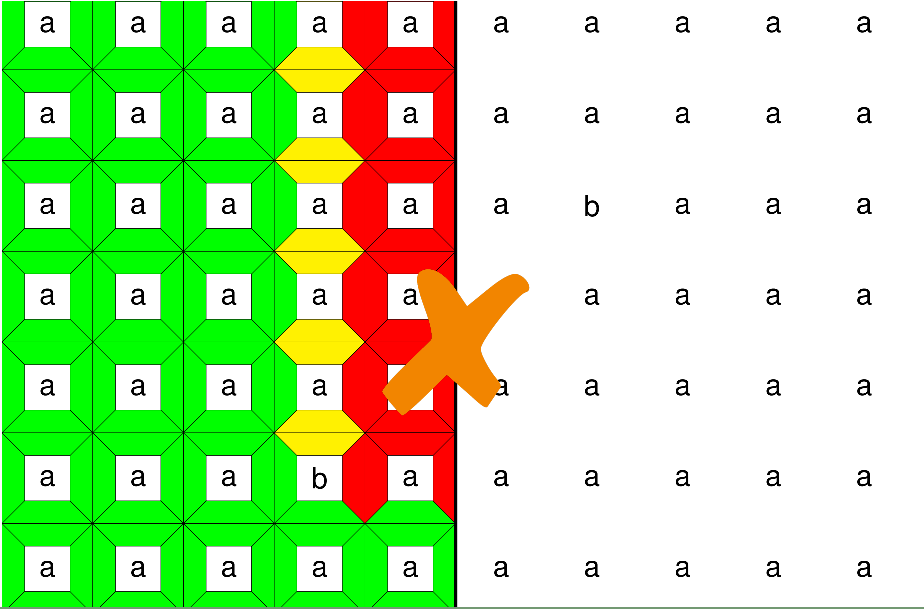


Communication Complexity

We now give formal definitions.

- We also symmetrize the protocol. Both Alice and Bob are given some boundary x, and they each verify that they can tile their half of the plane.
- To simplify things, we will only give to Alice and Bob the first n columns of their half, and not the whole half.
- This means that Alice and Bob both have an element in some subshift
  - Instead of a zero-dimensional dynamical system.

Communication Complexity

Definition

Let $S \subset X \times Y$ be a subshift (X and Y are also subshifts)
A *protocol* for S is three subshifts Z, S_X, S_Y so that:

$$(x, y) \in S \iff \exists z \in Z, (x, z) \in S_X \wedge (y, z) \in S_Y$$

- Alice has $x \in X$, obtains z and tests whether $(x, z) \in S_A$
- Bob has $y \in Y$, obtains z and tests whether $(y, z) \in S_B$

Communication Complexity

Definition

The communication complexity $N(S)$ of a subshift S is the infimum of $H(Z)$ for a protocol (Z, S_A, S_B) for S.

$H(Z)$ is the entropy of Z.

Once More With Feeling

- Alice has $x \in X$
- Bob has $y \in y$
- They want to decide if $(x, y) \in S$

To accomplish this, we give them the same hint $z \in Z$, and the objective is to minimise the set of possible z (the entropy of Z)

Some trivial facts

- $N(S) \leq H(X)$ (We can always send Alice's input to Bob)
- $N(X \times Y) = 0$ (Nothing to transmit)

Let T be any subshift and $EQ_T = \{(x, x) | x \in T\}$

$$N(EQ_T) = H(T)$$

Some proofs of this later on.

Plan

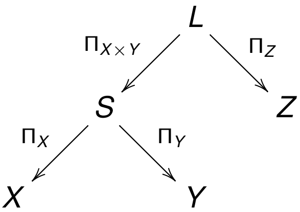
Notation

If (Z, S_X, S_Y) is a protocol for S, let

$$L = \{(x, y, z) | z \text{ is a protocol for } (x, y)\}$$

$$L = (X \times Y \times Z) \cap (S_X \times Y) \cap (S_Y \times X)$$

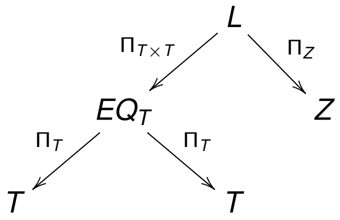
with a slight abuse of notation



wlog, all the maps involved in this diagram are factor maps.

Proof for EQ

$$N(EQ_T) \geq H(T)$$



$$L = \{(x, y, z) | z \text{ is a protocol for } (x, y)\}$$

Π_Z must be one-to-one.

Fooling sets

Definition

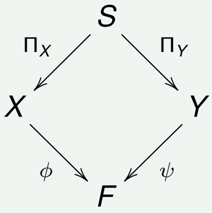
$Fo \subseteq S$ is a fooling set if for any $(x, y) \in Fo$ there are at most countably many pairs (x', y') so that $(x, y') \in S$ and $(x', y) \in S'$

If Fo is a fooling set for S, then $N(S) \geq H(Fo)$.

(Π_Z must be countable-to-one when the inputs are restricted to Fo, hence preserves entropy)

Factors

If F is a common factor of X and Y, then $N(S) \geq H(F)$



The communication complexity is bigger than the common information that can be extracted from X and Y.

The protocol induces a map from L to F which depends only on z, hence is a (continuous) map from Z to F.

Conditional entropy

$$N(S) \geq H(Y) - H_S(Y|X)$$

(Conditional entropy measures how many different completions *y* a given word *x* can have)

Plan

SFT

When S is a shift of finite type, we can say more.

If S is a SFT, then S admits near optimal protocols where S_X, S_Y and Z are SFTs.

Idea: If (Z, S_X, S_Y) is a protocol, then upper approximations of Z, S_X and S_Y by SFTs gives an upper approximation of S.
For SFTs, such an approximation will be eventually equal to S.

Classical Communication Complexity

Definition

Let $R \subseteq X_f \times Y_f$ be a *finite* relation.
If we change subshift into finite set and $H(Z)$ into $\log |Z|$ into the previous definition, we obtain the communication complexity $N_f(R)$ of a relation.

Theorem

For a SFT S,

$$N(S) = \lim_{n\to\infty} N_f(S_n)/n$$

Where $S_n \subseteq X_n \times Y_n$ is the set of words of size n of S (resp X, Y).

Rectangle Spanning Entropy

Work in Progress Work in Progress

A set $A \subset S$ is a rectangle if $A = B \times C$.
A set $\mathcal{F}$ of rectangles is $(\epsilon, n) -$ *spanning* if for every $a \in S$, there exists $b \in \cup\mathcal{F}$ so that $d(\sigma^i(a), \sigma^i(b)) \leq \epsilon$ for all $i \leq n$
Let $r(n, \epsilon)$ be the minimum size of a $(\epsilon, n) -$ *spanning* family of rectangles
The rectangle spanning entropy is

$$R(S) = \lim_{\epsilon} \limsup_{n} \frac{1}{n} \log r(n, \epsilon)$$

For a SFT, $R(S) = N(S)$

Work in Progress Work in Progress

Plan

Multidimensional subshifts

Proposition

Let S be a two-dimensional subshift.
Let C_n be the shift of n consecutive columns of S.

$$S_{n,m} = \{(a, b) \in C_n \times C_m | ab \in C_{n+m}\}$$

If S is sofic, then $N(S_{n,m}) = O(1)$.

- This is “tight”, in the sense that a similar proposition for 1D subshift characterize sofic subshifts.

Application

S_1 a 1D shift. S a 2D shift where all lines are in S_1.

Does S sofic implies S_1 sofic ?

Application (2/2)

What is C_n (the set of n columns of S) ?

By definition $C_n = L_n^{\mathbb{Z}}$, where L_n is the set of words of size n of S_1.

In particular, it is a SFT.

Application (3/3)

Theorem

Let $R_n = \{(x, y) \in L_n | xy \in L_{2n}\}$
Then $N(S_{n,n}) \geq N_f(R_n) - \log\log L_n + O(1)$
In particular, if $N_f(R_n) - \log\log L_n \neq O(1)$*, then S is not sofic.*

Direct translation of a known result about asymptotic communication complexity (Feder et al 91)

- If $N_f(R_n) > \log\log L_n + O(1)$, S is not sofic.
- If $N_f(R_n) = O(1)$, S_1 is sofic.
- It remains to fill the gap.

Implies the result by Pavlov that if S_1 has no synchronizing word, then S is not sofic.

Plan

Strange example

$$EQ_{:/} = (\{0,1\} \times \{0,1\})^{\mathbb{Z}} \cup \{(0,0),(1,1)\}^{-\omega} 2 \{(0,0),(1,1)\}^{\omega}$$

If Alice and Bob both have a 2, they should have the same word.

$$CC(EQ_{:/}) = 0$$

Strange example

- If Alice has a 2, she sends all her information to Bob in a sparse way
  - Alice has
$$\cdots 0100101010210001010 \cdots$$
  - She sends
$$\cdots 0\sharp\sharp\sharp\sharp 1\sharp\sharp\sharp 0\sharp\sharp 1\sharp 021\sharp 0\sharp\sharp 0\sharp\sharp\sharp 0\sharp\sharp\sharp\sharp 1 \cdots$$
- Otherwise she sends ${}^{\omega}\sharp^{\omega}$ (possibly with one $0/1$ symbol at some place)

Should this example be forbidden somehow ?

More about this example

Let $S \subseteq X \times Y$.

- In general, the knowledge about n bits of x gives a knowledge about $f(n, x)$ bits of y, where f can be any function with $\lim_n f(n, x) = +\infty$
- If S is an SFT, $f(n, x) = n - O(1)$
- If S is functional, it is a block map, and $f(n, x) = n - O(1)$
- If $f(n, x) = n - O(1)$ most of the theorems about SFTs remain valid.

Question

Open question

Let S be a subshift and α so that

$$\forall x, x' \in X, d_H(Sx, Sx') \leq \alpha d(x, x')$$

$$\forall y, y' \in Y, d_H(Sy, Sy') \leq \alpha d(y, y')$$

What does S look like ?

$Sx = \{y | (x, y) \in S\}$ d_H is Hausdorff distance.

Open questions

- Find a measurable version
- Define it like an entropy (like the rectangle spanning entropy)
- Is $N(S)$ always achieved by some protocol ?
- Is $N(S)$ upper semicontinuous ? (If S_i is a decreasing family of subshifts, $N(\cap S_i) \leq \liminf N(S_i)$)

An example

Theorem

$N_f(R) = \max_\mu \min_{R_1 \times R_2 \subseteq R} - \log \mu(R_1 \times R_2)$