Nowhere-zero 5-flows and $(1, 2)$-factors

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Abstract

A graph $G = (V, E)$ has a nowhere-zero k-flow if there exists an orientation $H = (V, A)$ of G and an integer flow $\varphi : A \to \mathbb{Z}$ such that for all $a \in A, 0 < |\varphi(a)| < k$. A $(1, 2)$-factor of G is a set $F \subseteq E$ such that the degree of any vertex v in the subgraph induced by F, denoted by $d_F(v)$, is 1 or 2. The main result of this work is the following. A bridgeless cubic graph G has a nowhere-zero 5-flow if and only if there is a $(1, 2)$-factor F such that, the cardinality of the set $\{uv \in E(C) : uv \in F \text{ or } d_F(u) = d_F(v)\}$ is even, for every cycle C in a basis of cycles associated to an spanning tree.

Keywords: nowhere-zero flows, factors.

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1 Introduction

Let $G = (V, E)$ be a bridgeless undirected graph. We say that G has a nowhere-zero k-flow if there exists an orientation $H = (V, A)$ of G and a function $\varphi : A \to \mathbb{Z}$ such that for all $a \in A, 0 < |\varphi(a)| < k$ and for every $v \in V$, $\sum_{a \in v^+} \varphi(a) = \sum_{a \in v^-} \varphi(a)$, where v^+, v^- are the sets of outgoing and ingoing arcs incident with v, respectively. For an Abelian group Γ we have the following analogous definition. A graph G admits a nowhere-zero $\Gamma-$flow if there is an orientation $H = (V, A)$ of G and a function $\varphi : A \to \Gamma$ satisfying the following.

(i) For all $a \in A, \varphi(a) \neq 0$.

(ii) For every $v \in V$, $\sum_{a\in v^+} \varphi(a) = \sum_{a\in v^-} \varphi(a)$.

Arithmetic operations are carried out in Γ. Usually, a function φ satisfying previous conditions is called a nowhere-zero $\Gamma-$flow of G.

The concept of nowhere-zero k-flow was introduced by Tutte [6] as a refinement and a generalization for face coloring problems in planar graphs. The four-color Theorem is equivalent to say that every bridgeless planar graph has a nowhere-zero 4-flow. This result can not be extended to arbitrary bridgeless graph since the Petersen graph has no a nowhere-zero 4-flow. However, in [7] Tutte formulated his famous 5-flow conjecture which still is open:

Conjecture 1.1 *Every bridgeless graph admits a nowhere-zero 5-flow.*

Work about Conjecture 1.1 have focused on properties of a minimal counterexample (see [3],[2],[1]) and into the study of structural properties of graphs having a nowhere-zero 5-flow (see [5]).

The best approximation for Conjecture 1.1 is a result of Seymour [4] where he proved that every bridgeless graph has a nowhere-zero 6-flow.

The motivation for studying this conjecture in cubic graphs has two sources. First, it is known that for Conjecture 1.1 to be true, it is enough to prove it for cubic graphs. Second, for cubic graphs there exist well known characterizations of the existence of nowhere-zero k-flow for $k = 3, 4$. These characterizations provide some intuition about the structural properties of cubic graphs admitting nowhere-zero $k-$flow.

On one hand, Tutte gave the following characterization.

Theorem 1.2 *[6] Let G be a cubic graph. G admits a nowhere-zero 3-flow if and only if G is bipartite.*

This result can be seen as a parity condition that must satisfy all cycles of a cubic graph to admit a nowhere-zero 3$-$flow. The parity condition being that each cycle has even length.

On the other hand, for nowhere-zero 4$-$flow we only need to check this parity property in the complement of a perfect matching.

Theorem 1.3 *Let $G = (V, E)$ be a cubic graph. G has a nowhere-zero 4-flow if and only if G has a perfect matching M such that all cycles in $G - M$ have even length.*

Our main result is somehow similar to Theorem 1.3. It characterizes the existence of nowhere-zero $5-$flows in a bridgeless cubic graph G in terms of a *parity* condition that must satisfy a family of cycles of G. In our case, the family of cycles $\mathcal{C}$ to be checked is a *basis of cycles* associated to a tree $T = (V, E')$. That is, $\mathcal{C} = \{C_e : e \notin E'\}$, where C_e is the cycle in $T \cup \{e\}$. The role of the perfect matching in Theorem 1.3 is played by a subset of edges F such that each vertex of G is incident with 1 or 2 edges of F, and called $(1, 2)$-factors. Our main result is the following.

Theorem 1.4 *For an undirected cubic graph G the following statements are equivalents.*

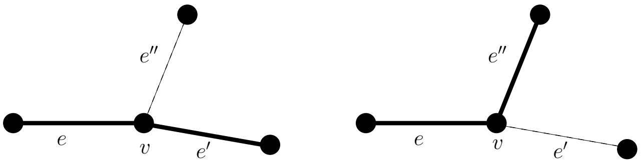
(i) *The graph G admits a nowhere-zero* 5*-flow.*

(ii) *There exists a* $(1, 2)$*-factor F such that, the cardinality of the set*

$$C_F := \{uv \in E(C) : uv \in F \text{ or } d_F(u) = d_F(v)\}$$

is even, for every cycle C in a basis of cycles associated to a tree.

In Figure 1, edges in C_F are pointed out by an arrow.



$p_F(e, e') = 2$ $p_F(e, e') = 1$ $p_F(e, e') = 3$

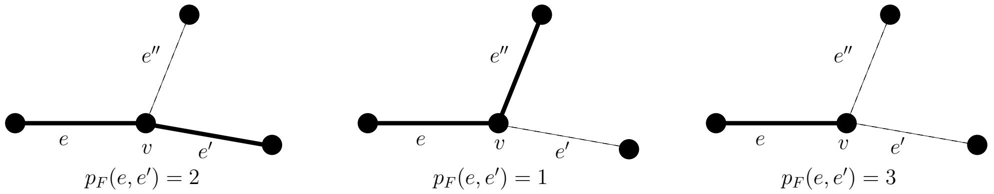
Fig. 1. Example of C_F. Solid edges are in F. Dashed edges are in $E \setminus F$. Edges in C_F are indicated by an arrow.

2 Preliminaries

It is known by ([7]) that a graph admits a nowhere-zero k-flow if and only if it admits a nowhere-zero Γ-flow, where Γ is any Abelian group of cardinality k. Hence, (i) in Theorem 1.4 can be replaced by

(i') The graph G admits a nowhere-zero $\mathbb{Z}_5$-flow.

Let F be any $(1, 2)-$factor of a cubic graph G. Let v be a vertex of G and let u, w, r its three neighbors. We say that the edges uv and vw are $F-$related if $uv, vw \in F$ or $uv, vw \notin F$. We define the $F-$*parity* of the tuple (u, v, w), denoted by $F(u, v, w)$, by 2 when uv and wv are $F-$related. It is 1 when uv and vr are $F-$related; It is 3 when vw and vr are $F-$related. (see Figure 2).



$p_F(e,e') = 2$ $p_F(e,e') = 1$ $p_F(e,e') = 3$

Fig. 2. Definition of $F(u,v,w)$. Solid edges are in F. Dashed edges are in $E \setminus F$. In the first diagram uv and vw are $F-$related; In the second diagram uv and vr are $F-$related; In the third diagram vr and vw are $F-$related.

We extend the definition of $F-$parity to cycles. Let $C = (u_0, \dots, u_{n-1}, u_n = u_0)$ a cycle of length $n \geq 3$ in G. We define the $F-$parity of C, denoted by $F(C)$, as the sum of the $F-$parities of tuples (u_i, u_{i+1}, u_{i+2}), for $i = 0, \dots, n-2$, plus the $F-$parity of (u_{n-1}, u_0, u_1), where arithmetic operations are carried out in $\mathbb{Z}_4$.

Lemma 2.1 *Let G be a bridgeless cubic graph. For each $(1,2)-$factor F and each cycle C of G we have the following.*

$$F(C) = 2|C_F| \ mod \ 4,$$

where

$$C_F := \{uv \in E(C) : uv \in F \ or \ d_F(u) = d_F(v)\}.$$

Proof. Let $C = (u_0, \dots, u_{n-1}, u_0)$. The relation is clear when $E(C) \subseteq F$ since, in this case, $F(C) = 2|E(C)|$. Let Q be a non trivial (at least one edge) connected component of $C \setminus F$. Let us assume that $Q = (u_1, \dots, u_i)$, $i \geq 2$. Let $\alpha(Q)$ be defined by $\alpha(Q) := \sum_{j=1}^{i} F(u_{j-1}, u_j, u_{j+1})$. We prove that $\alpha(Q) = 2|E(Q) \cap C_F| + 2 \bmod 4$. Let us consider first the case $i \geq 3$. It is easy to see that $F(u_0, u_1, u_2) + F(u_{i-1}, u_i, u_{i+1}) = 0 \bmod 4$ when $u_1u_2, u_{i-1}u_i \in C_F$ or $u_1u_2, u_{i-1}u_i \notin C_F$. Moreover, $F(u_0, u_1, u_2) + F(u_{i-1}, u_i, u_{i+1}) = 2 \bmod 4$ when $|\{u_1u_2, u_{i-1}u_i\} \cap C_F| = 1$. Hence,

$$\begin{aligned}\alpha(Q) &= \sum_{j=1}^{i} F(u_{j-1}, u_j, u_{j+1}) \\ &= F(u_0, u_1, u_2) + 2(i-3) + 2 + F(u_{i-1}, u_i, u_{i+1}) \\ &= 2|E(Q) \cap C_F| + 2 \bmod 4.\end{aligned}$$

For $i = 2$ we have that $F(u_0, u_1, u_2) + F(u_1, u_2, u_3) = 4$ if $u_1u_2 \in C_F$, and it is 2 otherwise.

Let Q' be a non trivial connected component of $C \cap F$. Let us assume that $Q' = (u_1, \dots, u_i)$, $i \geq 2$. Let $\alpha(Q')$ be defined as follows. For $i \geq 3$, we set

$\alpha(Q') = \sum_{j=2}^{i-1} F(u_{j-1}, u_j, u_{j+1})$. For $i = 2$, we set $\alpha(Q') = 0$. Clearly, with this definition $\alpha(Q') = 2|E(Q') \cap C_F| + 2$, when $i = 2$. When $i \geq 3$ we have

$$\begin{aligned}\sum_{j=2}^{i-1} F(u_{j-1}, u_j, u_{j+1}) &= 2(i-2)\\ &= 2|E(Q') \cap C_F| + 2 \bmod 4\end{aligned}$$

We can now compute $F(C)$ in terms of $\alpha(Q)$, where Q ranges over non trivial connected components of $C \setminus F$ and over non trivial connected components of $C \cap F$. As for each of them $\alpha(Q) = 2|E(Q) \cap C_F| + 2$, and the connected components of $C \setminus F$ and $C \cap F$ alternate, we get the conclusion. □

When F is an $(1,2)-$factor and C is a cycle such that $F(C) = 0 \bmod 4$, we say that F *reduces* C or that C is reduced by F. By using Lemma 2.1, our result can be stated as follows: A cubic graph G has a nowhere-zero $\mathbb{Z}_5-$flow if and only if there is a $(1,2)-$factor F reducing every cycle in a basis of cycles of G associated with a tree.

3 The proof

We split the proof of Theorem 1.4 in two parts associated with the forward and the backward implications.

Proposition 3.1 *Let $G = (V, E)$ be an undirected cubic graph. If G admits a nowhere-zero $\mathbb{Z}_5$-flow, then there exists a $(1,2)$-factor F reducing every cycle of G.*

Proof. Let us assume that G has a nowhere-zero $\mathbb{Z}_5$-flow associated with an orientation H and a function φ. Let F be defined as follows.

$$F = F_\varphi := \{uv \in E : \varphi(u,v) \in \{1,4\} \text{ or } \varphi(v,u) \in \{1,4\}\}$$

In $\mathbb{Z}_5$, the equation $x + y + z = 0$ has exactly 5 distinct solutions given by $\{\{a, a, 3a\} : a \in \mathbb{Z}_5\}$. Then, for each vertex v, at least one arc incident with v has flow in the set $\{1,4\}$ and at least one arc incident with v has flow, in the set $\{2,3\}$. Therefore, F is a $(1,2)-$factor of G.

Let $a \in A$ and let $A' = A \backslash \{a\} \cup \{-a\}$ and $\varphi' : A' \to \mathbb{Z}_5$, where $\varphi'(b) = \varphi(b)$, if $b \neq -a$ and $\varphi'(-a) = -\varphi(a)$. Then, $F_{\varphi'} = F$. Hence, by modifying the orientation assigned for H to any subgraph of G and, accordingly, the associated flow the set F remains the same.

Let $C = (u_0, \ldots, u_{n-1}, u_0)$ be a cycle of length n in G. To ease the notation, let us define $a_i = (u_i, u_{i+1})$, for $i = 0, \ldots, n-1$. w.l.o.g, we can assume that

$a_i \in A$, for $i = 0, \ldots, n-1$. From the choice of F and the definition of $F(u, v, w)$, it is not difficult to see that $\varphi(v, w) = \varphi(u, v)2^{F(u,v,w)}$ mod 5, for each path (u, v, w) in H. Hence, for each $i = 0, \ldots, n-2$ it holds:

$$\varphi(a_{i+1}) = \varphi(a_i)2^{F(u_i,u_{i+1},u_{i+1})} \bmod 5. \tag{1}$$

Moreover, $\varphi(a_0) = \varphi(a_{n-1})2^{F(u_{n-1},u_0,u_1)}$ mod 5. We now prove that $\varphi(a_0) = \varphi(a_0)2^{F(C)}$ mod 5, from which we get $F(C) = 0$ mod 4, that is F reduces C. By starting with a_0, and by iteratively applying 1, we have

$$\begin{aligned}
\varphi(a_0) &= \varphi(a_0)\prod_{i=0}^{n-2} 2^{F(u_i,u_{i+1},u_{i+2})}2^{F(u_{n-1},u_0,u_1)} \\
&= \varphi(a_0)2^{\sum_{i=0}^{n-2} F(u_i,u_{i+1},u_{i+2})+F(u_{n-1},u_0,u_1)} \\
&= \varphi(a_0)2^{F(C)}.
\end{aligned}$$

□

Theorem 3.2 *Let $G = (V, E)$ be an undirected cubic graph. If there exists a $(1, 2)$-factor F reducing every cycle in a basis of cycles associated with a tree T, then G admits a nowhere-zero $\mathbb{Z}_5$-flow*

Proof. Let $\hat{G} = (V, D)$ be the directed graph obtained from G by replacing each edge uv by two arcs (u, v) and (v, u). In $\hat{G}$ the outdegree (resp. indegree) of each vertex is three. A *$F-$valid local solution* for a vertex $v \in \hat{G}$ is any mapping f from v^+ to $\mathbb{Z}_5 \setminus \{0\}$ such that

$$\sum_{a \in v^+} f(a) = 0 \bmod 5.$$

and such that $f(v, u) = f(v, w)$ if and only if vu and vw are $F-$related. For each vertex v, there are exactly four *$F-$valid local solutions.*

Let us root the tree T at a vertex r. We define f_r as any $F-$valid local solution for r. We assign $F-$valid local solutions to the remaining vertices so as, for each edge $uv \in E(T)$, $f_v(v, u) = -f_u(u, v)$. These $F-$valid local solutions are completely determined by F, T and f_r.

Let $\varphi : D \to \mathbb{Z}_5 \setminus \{0\}$ be defined by

$$\varphi(v, u) = f_v(v, u).$$

Clearly, for each arc $a \in D$, $\varphi(a) \neq 0$, and for each vertex v we have that

$$\sum_{a \in v^+} \varphi(a) = \sum_{a \in v^+} f_v(a) = 0 \bmod 5.$$

Moreover, for every edge $uv \in E(T)$ it holds that $\varphi(u,v) = -\varphi(v,u)$, and for each edge $vw \in E$

$$\varphi(v,w) = \varphi(u,v)2^{F(u,v,w)}. \tag{2}$$

We now prove that equality $\varphi(w,v) = -\varphi(v,w)$ holds for every edge $vw \notin E(T)$. Let $w = u_0, u_1, \ldots, u_n = v$ be the path in T between w and v. By using equation 2, it can be proved that

$$\varphi(w,v) = \varphi(v,u_{n-1})2^{F(v,u_{n-1},u_{n-2})+\cdots+F(u_1,w,v)}.$$

Moreover, $\varphi(v,u_{n-1}) = -\varphi(u_{n-1},v)$ and $\varphi(v,w) = \varphi(u_{n-1},v)2^{F(u_{n-1},v,w)}$. Then,

$$\begin{aligned}\varphi(w,v) &= -\varphi(v,w)2^{F(w,v,u_{n-1})+F(v,u_{n-1},u_{n-2})+\cdots+F(u_1,w,v)}\\ &= -\varphi(v,w)2^{F(C)},\end{aligned}$$

where C is the cycle in $T \cup \{vw\}$. As F reduces C, we conclude that $\varphi(w,v) = -\varphi(v,w)$. By choosing any orientation $H = (V,A)$, it follows that the function φ restricted to A is a nowhere-zero $\mathbb{Z}_5-$flow of G.

□

The proof of Theorem 1.4 is now easy. The forward direction is included in Proposition 3.1, while Theorem 3.2 corresponds to the backward implication.

It is worth to notice the following consequence of Theorems 3.1 and 3.2.

Corollary 3.3 *To decide whether a given $(1,2)-$factor F reduces all cycles of G can be done in polynomial time.*

Proof. Given F and an spanning tree $T = (V,E')$ we first attempt to construct a nowhere-zero $\mathbb{Z}_5-$flow as it was done in the proof of Theorem 3.2. If this construction fails, then F does not reduce a cycle C_e, for some $e \notin E'$. Otherwise, a nowhere-zero $\mathbb{Z}_5-$flow φ is constructed such that $F = F_\varphi$. From the proof of Proposition 3.1, we conclude that in this case, F reduces every cycle of G. □

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