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Constructibility of speed one signal on cellular automata ☆

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Abstract

This paper studies the capabilities of one dimensional, neighborhood of radius one cellular automata to construct speed one signals. We define the notions of constructibility, consistent constructibility and preserving complexity consistent constructibility. Our main results are the following.

- There exists a one dimensional neighborhood of radius one cellular automaton with four states which preserving complexity consistently constructs (E_*), the set of all speed one signals with a finite number of changes of direction.
- The class of speed one signal preserving complexity consistently constructed by any one dimensional neighborhood of radius one cellular automaton with three states is a proper subclass of (E_*).

Keywords: Cellular automata; Signal constructibility

1. Introduction

A one-dimensional cellular automaton with neighborhood of radius one (CA_1^1) is defined by a finite set of states Q and a local transition function $f: Q^3 \to Q$. The local transition function induces a global function $F: Q^{\mathbb{Z}} \to Q^{\mathbb{Z}}$ by the rule $F_i(c) = f(c_{i-1}, c_i,$

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c_{i+1}). Further, we assume that there exists a state $0 \in Q$ such that $f(0,0,0)=0$. This state is called the quiescent state. Let X denote the set $Q^{\mathbb{Z}}$. An element of X is called a configuration.

One-dimensional cellular automata (CA^1) were studied in [4] where several results about their capabilities as language recognizers were obtained. It is known that they are equivalent to Turing machines and universal CA^1 with small number of states have been built [1,2]. The smaller known universal CA^1 with neighborhood of radius one (CA^1_1) has seven states [2]. The only known lower bound is the trivial one and it is conjectured that two states are enough to get universal CA^1_1 [5].

The whole evolution of a CA^1_1 over an initial configuration can be represented in a time–space diagram. One main feature of low states universal CA^1_1 and two states universal CA^1_1 candidates, is the existence of *structures* or *patterns* in their time–space diagrams.

To distinguish a pattern in the time–space diagram of a CA^1_1 (Q,f) with initial configuration c we take a subset Q_0 of Q and consider the set of all pairs in the time–space diagram having a state in Q_0. The resulting pattern can be seen as a subset s of $\mathbb{N}\times\mathbb{Z}$ that we will call the filtered time–space diagram of (Q,f,Q_0,c).

Given a set $s\subseteq\mathbb{N}\times\mathbb{Z}$ we consider the problem of finding a CA^1_1 let us say (Q,f), a subset Q_0 of Q and an initial configuration c such that s is the filtered time–space diagram of (Q,f,Q_0,c). When this is possible we say that s is constructed by (Q,f,Q_0,c).

A restricted version of this problem was considered in [3] where the cellular automata are one-side bounded and the initial configurations have only one non-quiescent state at cell 0. Moreover, the class of subset considered is that of *signals*. A signal is a subset s of $\mathbb{N}\times\mathbb{Z}$ such that for every $t\in\mathbb{N}$ the set $\{i\colon (t,i)\in s\}$ has exactly one element and for every pair $i,j\in\mathbb{Z}$ if $(t,i)\in s$ and $(t+1,j)\in s$ then $|i-j|\leqslant 1$. Therefore, a signal s can be seen as a function from $\mathbb{N}$ to $\mathbb{Z}$. We denote by s_t the unique element of s that has its first coordinate equal to t. A further interpretation is that a signal s can be considered as the trajectory of an object moving in a one-dimensional space. In this sense s_t is the position visited by the object at time t. In this restricted version of the problem the number of signals constructed by a given CA^1_1 is finite since both the number of configurations and the number of subsets of Q are finites.

Here we consider another restricted version of the problem where the set Q_0 has exactly one element. With this restriction the set s is the filtered time–space diagram of $(Q,f,\{\bullet\},c)$ if s is the set of all pairs with the state $\bullet$ in the time–space diagram of (Q,f) with initial configuration c. A relevant aspect of this problem is that one CA^1_1 may construct a whole class of subsets by using different initial conditions. By instance if s is constructed by $(Q,f,\{\bullet\},c)$ then the following sets are constructed by $(Q,f,\{\bullet\},c')$ where the initial configuration c' is given below:

(1) $\sigma(s)$ defined by $(t,i)\in\sigma(s)$ if and only if $(t+1,i)\in s$ is constructed by $(Q,f,\{\bullet\}, F(c))$, where F is the global transition function induced by f.
(2) $m^r(s)$ defined by $(t,i)\in m^r(s)$ if and only if $(t,i-1)\in s$ is constructed by $(Q,f,\{\bullet\}, M^r(c))$, where $M^r_i(c)=c_{i-1}$.
(3) $m^l(s)$ defined by $(t,i)\in m^l(s)$ if and only if $(t,i+1)\in s$ is constructed by $(Q,f,\{\bullet\}, M^l(c))$, where $M^l_i(c)=c_{i+1}$.

We denote by $A = A(Q, f, \{\bullet\})$ the class of all subsets of $\mathbb{N} \times \mathbb{Z}$ constructed by $(Q, f, \{\bullet\}, c)$ when c varies over all configurations. From previous reasoning we see that the class A is closed under σ, m^l and m^r. Moreover, one could argue that if c constructs s the natural configurations constructing $\sigma(s)$, $m^r(s)$ and $m^l(s)$ are $F(c)$, $M^r(c)$ and $M^l(c)$, respectively. It is not difficult to see that two different configurations c and c' could construct the same set s. Let us denote by φ an injective function from A to X (the set of all configurations). We say that the function φ is consistent if for all $s \in A$ $\varphi(s)$ constructs s, $\varphi(\sigma(s)) = F(\varphi(s))$, $\varphi(m^r(s)) = M^r(\varphi(s))$ and $\varphi(m^l(s)) = M^l(\varphi(s))$. By extension we say that a class A of subsets of $\mathbb{N} \times \mathbb{Z}$ closed under σ, m^r and m^l is consistently constructed by $(Q, f, \{\bullet\})$ if there exists an injective consistent function φ from A to X.

In this work we consider a subclass of signals that we call *speed one signals* (s1-signals). A signal e has speed one if for every $t \in \mathbb{N}$ whenever (t, i) and $(t+1, j)$ belong to e we have that $|i - j| = 1$.

We denote by E the set of all s1-signals,

$$E = \{e : \mathbb{N} \to \mathbb{Z} : \forall t \in \mathbb{N},\ |e_t - e_{t+1}| = 1\}.$$

A s1-signal in E represents the trajectory of an object which moves from one position to a contiguous one and never stays at the same place. We denote by $\langle e \rangle$ the number of changes of direction of e. Let E_k be the set of all s1-signals whose associated trajectories change of direction at most k times and $E_* = \bigcup_{k \in \mathbb{N}} E_k$.

The last ingredient that we have to consider is a measure of complexity. For a configuration $c \in X$ it is natural to define its complexity as $d(c)$ the cardinality of its support, that is the number of non-quiescent states appearing in c. We define the complexity of a s1-signal e as its number of changes of direction. In this sense, more changes of direction a s1-signal has more complex it appears. Our intention is that the function φ that associates configurations to s1-signal does not increase the complexity too much. An injective consistent function $\varphi : A \to Q^{\mathbb{Z}}$ *preserves complexity* if the number of non-quiescent states in $\varphi(e)$ is bounded in terms of $\langle e \rangle$, that is there exists a function $h : \mathbb{N} \to \mathbb{N}$ such that $\forall e \in A$, $d(\varphi(e)) \leqslant h(\langle e \rangle)$.

We prove in Section 3 that there exist a CA_1^1 with four states (Q, f) such that $E_* \subseteq A(Q, f, \{\bullet\})$. Moreover, we prove that there is an injective preserving complexity consistent function that constructs E_*.

In Section 2 we analyze the capabilities of CA_1^1 with less than three states to construct E_*. We prove that two states CA_1^1 are only able to construct proper subclasses of E_0 and that three states are only able to preserving complexity consistently construct proper subclasses of E_2.

2. First properties

In this section we prove some basic results about classes of s1-signals.

The simplest s1-signals are those going always rightwards or always leftwards. We denote by R_0 the subset of E of all s1-signals going rightwards, i.e. a s1-signal r

belongs to R_0 if for every $t \geqslant 0$ we have that $r_{t+1} = r_t + 1$. It this case it follows that $r_t = r_0 + t$, for all $t \geqslant 0$. We denote by L_0 the subset of all s1-signals going leftwards.

A s1-signal e has a *leftwards* change of direction at time t if $e_{t-1} = e_{t+1} = e_t - 1$. A *rightwards* change of direction is defined analogously.

A s1-signal e representing the trajectory of an object has a leftwards change of direction at time t if at the tth step the object moves rightwards ($e_t = e_{t-1} + 1$) and at the $(t+1)$th step the object moves leftwards ($e_{t+1} = e_t - 1$).

Therefore, the trajectory of an object associated to a s1-signal in $R_0 \cup L_0$ does not have changes of direction.

We denote by $\langle e \rangle$ the number of changes of direction of a s1-signal e. Then the set E_* is the set of all s1-signals with a finite number of changes of direction ($\langle e \rangle < \infty$).

Let e be the s1-signal given by

$$e_t = \begin{cases} t & \text{if } t \leqslant 4, \\ t - 2 & \text{if } t > 4. \end{cases}$$

The s1-signal e has a leftwards change of direction at time 4, because $e_3 = e_5 = 3 = e_4 - 1$. Since $e_4 = e_6 = 4 = e_5 + 1$ it has a rightwards change of direction at time 5. It has no other change of direction. Thus $\langle e \rangle = 2$.

Let E_r be the set of all s1-signals e such that there exists an integer k satisfying $\sigma^k(e) \in R_0$ that is, $e \in E_r$ if it has a finite number of changes of direction and the last one is rightwards. We define E_l analogously.

Let $E_k = \{e \in E_* \colon \langle e \rangle \leqslant k\}$, $R_k = E_k \cap E_r$ and $L_k = E_k \cap E_l$. A s1-signal e in E_k has at most k changes of direction. If the s1-signal e also belongs to R_k its last change of direction is rightwards.

When we restrict the functions σ, m^r and m^l to E_* they are defined by $\sigma_t(e) = e_{t+1}$, $m^r_t(e) = e_t + 1$ and $m^l_t(e) = e_t - 1$ for all $t \in \mathbb{N}$.

Lemma 1. *We have the following relations*:

(1) $E_* = E_r \cup E_l$.
(2) $E_0 = R_0 \cup L_0$.
(3) $E_0 \subseteq E_1 \subseteq \cdots \subseteq E_* = \bigcup_{k \geqslant 0} E_k$.
(4) *For all* $e \in L_0$ *we have* $L_0 = \bigcup_{k \in \mathbb{Z}} (m^r)^k(e)$ *and* $L_0 = \bigcup_{k \in \mathbb{Z}} (m^l)^k(e)$. *The same properties hold for* R_0.
(5) σ *equals* m^r *on* R_0 *and it equals* m^l *on* L_0.

Proof. The proof can be done easily from the definitions. □

Let (Q, f) be a CA^1_1. We summarize the constructibility definitions associated to (Q, f) that we have introduced for a class of s1-signals B:

(1) We say that a (Q, f) constructs B if $B \subseteq A(Q, f, \{\bullet\})$.

(2) We say that a (Q, f) consistently constructs (c. constructs) B if there exists an injective function $\varphi : B \to Q^{\mathbb{Z}}$ such that

P1. For all $e \in B$, $\varphi(\sigma(e)) = F(\varphi(e))$.
P2. For all $e \in B$, $\{i\colon\ \varphi_i(e) = \bullet\} = \{e_0\}$.
P3. For all $e \in B$, $\varphi(m^r(e)) = M^r(\varphi(e))$ and $\varphi(m^l(e)) = M^l(\varphi(e))$.

(3) We say that (Q, f, φ) preserving complexity consistently (p.c.c.) constructs B if it consistently constructs B and the function φ preserves the complexity:

P4. There exists a function $h : \mathbb{N} \to \mathbb{N}$ such that for all $e \in B$, $d(\varphi(e)) \leqslant h(\langle e \rangle)$, that is, the number of non-quiescent states in $\varphi(e)$ is bounded in terms of the number of changes of direction of the s1-signal e.

In the sequel we will refer to previous properties as P1–P3 and P4. Let c^t be the configuration obtained by iterating t times the function F on the configuration $\varphi(e)$. From P1 and P2 we have for all $t \in \mathbb{N}$ that $\{i \in \mathbb{Z}\colon\ c_i^t = \bullet\} = \{e_t\}$ that is, $\varphi(e)$ constructs the s1-signal e.

3. Two and three states CA

The constructibility capabilities of two states CA_1^1 are delimited in the next proposition.

Proposition 2. *If (Q, f) constructs A with $E_0 \subseteq A$, then $|Q| \geqslant 3$.*

Proof. Let us assume that there exists a cellular automaton (Q, f) with two states 0 and $\bullet$ that constructs A. In order to construct s1-signals in R_0 the function f must satisfies $f(\bullet, 0, 0) = \bullet$, $f(0, \bullet, 0) = 0$ and $f(0, 0, \bullet) = 0$. The transition $f(0, 0, \bullet) = 0$ implies that (Q, f) cannot construct any s1-signal in L_0. $\square$

In the rest of the paper we use the shorthand $abc \to d$ for $f(a, b, c) = d$.

Using part 5 of Lemma 1, properties P1 and P3 it is not hard to prove the following lemma.

Lemma 3. (1) *If (Q, f, φ) c. constructs R_0 then for every $r \in R_0$ we have that $F(\varphi(r)) = M^r(\varphi(r))$ that is, for every $r \in R_0$ the function F shifts $\varphi(r)$ to the right.*

(2) *If (Q, f, φ) c. constructs L_0 then for every $l \in L_0$ we have that $F(\varphi(l)) = M^l(\varphi(l))$.*

Proposition 4. *If $(\{0, 1, \bullet\}, f, \varphi)$ p.c.c. constructs A, with $\sigma(A) \subseteq A$ and $E_1 \subseteq A$ then for every $r \in R_0$ and every $l \in L_0$ the configurations $\varphi(r)$ and $\varphi(l)$ are given by*

$\varphi(r)$: $\cdots 01x \bullet 0 \cdots$

and

$$\varphi(l)\colon\ \cdots 0\bullet y10\cdots,$$

where x and y belong to $\{0,1\}^*$ and neither x nor y have a 00 as a subword. Furthermore, the local transition function f satisfies $000\to 0$, $\bullet 00\to\bullet$, $00\bullet\to\bullet$, $001\to 0$ *and* $100\to 0$.

Proof. Let r and l be two s1-signals in R_0 and L_0, respectively. Using P2 and P4 we have that $\varphi(r)$ and $\varphi(l)$ are given by

$$\varphi(r)\colon\ \cdots 0u\bullet v0\cdots$$

and

$$\varphi(l)\colon\ \cdots 0u'\bullet v'0\cdots$$

with $u,u'\in 1(0+1)^*+\varepsilon$ and $v,v'\in(0+1)^*1+\varepsilon$, where ε denotes the empty word.

Claim 5. *We have the following properties*:

(1) $u\neq\varepsilon$ *if and only if* $u'=\varepsilon$.
(2) $v=\varepsilon$ *if and only if* $v'\neq\varepsilon$.

Proof. From Lemma 3 we know that $F(\varphi(r))=M^r(\varphi(r))$ and $F(\varphi(l))=M^l(\varphi(l))$. If $u\neq\varepsilon$ then $001\to 0$ which implies $u'=\varepsilon$. If $u'=\varepsilon$ then $00\bullet\to\bullet$ which implies that $u\neq\varepsilon$. The proof is similar in the second case. □

In what follows, we prove that $u\neq\varepsilon$ and $v=\varepsilon$.

Claim 6. $u\neq\varepsilon$.

Proof. Since $\sigma(A)\subseteq A$ and $E_1\subseteq A$ there exists $e\in R_1$ such that $r=\sigma(e)$. From P1 we have that $F(\varphi(e))=\varphi(r)=\cdots 0u\bullet v0\cdots$. Assume that $u=\varepsilon$.

Since $F(\varphi(r))=M^r(\varphi(r))$ we deduce that $00\bullet\to 0$. Since $F_{e_0-1}(\varphi(e))=\varphi_{e_0-1}(\sigma(e))=\bullet$ it follows that in the configuration $\varphi(e)$ there exists a pattern 001 to the left of the state $\bullet$, otherwise the transition $00\bullet\to 0$ would implies $F_{e_0-1}(\varphi(e))=0$.

Since $F(\varphi(l))=M^l(\varphi(l))$ we have that $001\to 1$, because according to Claim 5 $u'\neq\varepsilon$. Therefore, the configuration $F(\varphi(e))$ has a state 1 to the left of the state $\bullet$ which contradicts our assumption $u=\varepsilon$. □

Claim 7. $v=\varepsilon$.

Proof. This claim can be proved similarly to Claim 6 by taking a s1-signal $g\in L_1$. □

Using Claims 5–7 we deduce that $v=u'=\varepsilon$. Furthermore, we have the following transitions: $\bullet 00\rightarrow\bullet$, $00\bullet\rightarrow\bullet$, $100\rightarrow 0$ and $001\rightarrow 0$. The restrictions for x and y can be easily deduced from previous transitions. □

Lemma 8. *If* $(\{0,1,\bullet\},f,\varphi)$ *p.c.c. constructs* A, *with* $E_1\subseteq A$ *and* $\sigma(A)\subseteq A$ *then the local transition function* f *satisfies*:

0	0	1	$\bullet$
0	0	0	$\bullet$
1	0		$\bullet^{2}$
$\bullet$	$\bullet$	$\bullet^{1}$	-

1	0	1	$\bullet$
0		1^{10}	0^{8}
1	1^{11}		$\bullet^{7}$
$\bullet$	0^{9}	$\bullet^{6}$	-

$\bullet$	0	1	$\bullet$
0		1^{4}	-
1	1^{3}	0^{5}	-
$\bullet$	-	-	-

where the super-indices are references in the proof where the transitions are deduced.

Proof. Since $E_1\subseteq A$ from Proposition 4 we already know that the following transitions hold: $\bullet 00\rightarrow\bullet$, $00\bullet\rightarrow\bullet$, $100\rightarrow 0$ and $001\rightarrow 0$.

Claim 9. *Every configuration with states* 0 *and* 1 *and with at most* $h(1)$ *states* 1, *reaches the global steady state before time* $2^{2h(1)}+1$.

Proof. Since $001\rightarrow 0$ and $100\rightarrow 0$ each part of a configuration delimited by two consecutive patterns 00 evolves independently. Since the distance between two consecutive patterns 00 is at most $2h(1)$, each part reaches the global steady state before time $2^{2h(1)}+1$. □

Let $t_0=2^{2h(1)}+1$; let $e\in L_1$ be a s1-signal with $e_0=0$ and its change of direction at time t_1, with $t_1>t_0+2h(1)$, that is $e_t=t$ for $t=0,\ldots,t_1$ and $e_t=2t_1-t$ for $t\geqslant t_1+1$. Let g be the s1-signal in R_1 defined for all t by $g_t=-e_t$.

Let c^t and d^t be defined by $c^t=F^t(\varphi(e))$ and $d^t=F^t(\varphi(g))$, for $t\in\mathbb{N}$; and let $\alpha=c^t_0$.

Claim 10. $\alpha_{t_1+2}=1$ *or* $\alpha_{t_1+1}=1$.

Proof. Since the pattern 00 is stable under state 1 if $\alpha_{t_1+2}=\alpha_{t_1+1}=0$ then we would have $c^{t_1}_{\{t_1,t_1+1,t_1+2\}}=\bullet 00$ and then $c^{t_1+1}_{t_1+1}=\bullet$ which is a contradiction with the choice of the s1-signal e. □

Since $t_1-t_0>2h(1)$, in the configuration α from position t_0 to position t_1 there is at least one pattern 00. Let j_0 be the smaller integer $>t_0+1$ such that $\alpha_{j_0-1}=\alpha_{j_0}=0$ and $\alpha_{j_0+1}=1$. From Claim 10 it follows that $j_0\leqslant t_1+1$.

Claim 11. *We have that* $\bullet 01\rightarrow\bullet$, $10\bullet\rightarrow\bullet$, $1\bullet 0\rightarrow 1$, $0\bullet 1\rightarrow 1$, $1\bullet 1\rightarrow 0$ *and* $j_0+1=t_1$.

Proof. Since $t_0 > 2^{2h(1)}$ the part of α to the right of position j_0 has reached the steady state. Since the pattern 00 is stable with respect to the state 1, we have that $c^t_{\{j_0-1,j_0,j_0+1\}} = 001$ for every $t \in \{t_0,\ldots,j_0-2\}$. Then, in the configuration c^{j_0-2} we have $c^{j_0-2}_{\{j_0-2,j_0-1,j_0,j_0+1\}} = \bullet 001$. The evolution of this part is as follows:

	j_0-4	j_0-3	j_0-2	j_0-1	j_0	j_0+1	j_0+2	j_0+3	j_0+4
j_0-2		x_1	$\bullet$	0	0	1	a	c	
j_0-1			x_2	$\bullet$	0	1	b	d	
j_0				x_3	$\bullet$	1	y		
j_0+1					x_4	$\bullet$	z_1	z_2	
t_1+1					w				

From the second transition in previous evolution we deduce $\bullet 01 \to \bullet$[1]. We can perform a similar analysis with the s1-signal g and prove that $10\bullet \to \bullet$[2]. We already know that $00\bullet \to \bullet$, then for every $t \leqslant t_1$ we have that $d^t_{-t+1} = c^t_{t-1} = 1$, that is in order to avoid that the state $\bullet$ moves leftwards in c^t we have to put a *protection* in c^t_{t-1} ($x_1 = x_2 = x_3 = 1$). From the first transition in previous table we deduce that $1 \bullet 0 \to 1$[3]. Again from a similar analysis with the s1-signal g we get that $0 \bullet 1 \to 1$[4].

Since $c^{t_1} = \varphi(\sigma^{t_1}(e))$ and $\sigma^{t_1}(e) \in L_0$, we know from Proposition 4 that $c^{t_1}_{t_1-1} = 0$. If $x_4 = 1$, that is the transition $1 \bullet 1 \to 1$ holds, then for every $t \in \{j_0+1,\ldots,t_1\}$ we would have $c^t_{t-1} = 1$, because we already know that $1 \bullet 0 \to 1$. For $t = t_1$ we would obtain a contradiction. Therefore, we conclude that $1 \bullet 1 \to 0$[5].

Since, $c^{j_0+1}_{\{j_0-1,j_0,j_0+1\}} = x0\bullet \to \bullet$, $00\bullet \to \bullet$ and $10\bullet \to \bullet$ we get that $w = c^{j_0+2}_{j_0} = \bullet$. Therefore, we deduce that $j_0 + 1 = t_1$. □

Claim 12. *We have that* $y = c^{t_1-1}_{t_1+1} = 1$ *and that* $\bullet 11 \to \bullet$, $11\bullet \to \bullet$, $\bullet 10 \to 0$, $01\bullet \to 0$, $011 \to 1$ *and* $110 \to 1$.

Proof. Since $d^{t_1} = \varphi(\sigma^{t_1}(g))$ and $\sigma^{t_1}(g) \in R_0$, we know from Proposition 4 that $d^{t_1}_{-t_1+2} = 0$.

Since $\bullet 01 \to \bullet$ and $\bullet 00 \to \bullet$ if $y = 0$ we would have that $\bullet 10 \to \bullet$, then $d^t_{\{-t,-t+1,-t+2\}} = \bullet 11$ for every $t \leqslant t_1 - 1$ which would imply that $\bullet 11 \to 1$. In turn previous transition would imply that $d^{t_1}_{-t_1+2} = 1$ which is a contradiction.

Then $y = 1$ and we have that $\bullet 11 \to \bullet$[6] and by a similar argument applied to the s1-signal g we have that $11\bullet \to \bullet$[7]. Therefore, $c^t_{t-2} = d^t_{-t+2} = 0$ for every $t \leqslant t_1 - 1$, which implies that $01\bullet \to 0$[8] and that $\bullet 10 \to 0$[9].

Since $c^{t_1-1}_{t_1} = c^{t_1-2}_{t_1} = 1$ if $011 \to 0$ then $a = c^{t_1-2}_{t_1+1} = 0$ and $b = c^{t_1-3}_{t_1+1} = 0$. Since the pattern 00 is stable we deduce that $c = d = 1$. In this case we have two different values for transition $101 \to ()$ which is a contradiction. Therefore, $011 \to 1$[10] and by a similar argument using the s1-signal g we have that $110 \to 1$[11]. □

In previous claims we have proved that f satisfies the required transitions. □

Corollary 13. *There exists* $(\{0,1,\bullet\},f,\varphi)$ *that p.c.c. constructs* E_1.

Proof. Let e be a s1-signal in R_1 with its change of direction at time t_1. Define $\varphi(e)$ as zero in all the coordinates except in e_0, e_0+1, e_0-t_1-1 and e_0-t_1. In the coordinates e_0+1, e_0-t_1, e_0-t_1-1 set $\varphi(e)$ as 1. In e_0 set it as $\bullet$. If $e\in L_1$ we proceed similarly. Any cellular automaton satisfying the tables of Lemma 8 with the function φ described above p.c.c. constructs E_1 with $h(u)=2u+2$. □

We say that a change of direction of a s1-signal e *holds in position* j if the s1-signal has a change of direction at time t with $j=e_t$. Let T be the subset of E_* defined by: $e\in T$ if and only we have that $l_{i+1}\geqslant l_i+3$ and $r_{i+1}\leqslant r_i-3$, where l_i and r_i are the positions of the ith leftwards and of the ith rightwards change of direction, respectively. Let S be the subset of T defined by: $e\in S$ if and only if $e\in T$ and the first change of direction of e is leftwards and $r_1\leqslant e_0-3$ or it is rightwards and $e_0+3\leqslant l_1$.

Proposition 14. *If* $(\{0,1,\bullet\},f,\varphi)$ *p.c.c. constructs* A, *with* $\sigma(A)\subseteq A$ *and* $E_1\subseteq A$, *then* $A\subseteq T$.

Proof. From Lemma 8 we know some values of the local transition function f. Doing a case analysis (suggested by the numbering) we deduce that for a s1-signal $e\in A$ we have the following situation when the ith leftwards change of direction holds in position k:

$k-3$	$k-2$	$k-1$	k	$k+1$	$k+2$	$k+3$
0^1	1^1	$\bullet^0$	1^4	1^5	a	c
	0^2	0^3	$\bullet^0$	1^1	$b=0^1$	d
		$\bullet^0$	1^4	0^2	α	

Claim 15. $\alpha=0$.

Proof. We first prove that $a=0$. We know that $100\to 0$. If $a=1$ then $111\to 1$. Since $110\to 1$ we would get $b=1$ a contradiction.

If $d=0$ we obtain $\alpha=0$. If $d=1$ we have that $c=1$ and the transition $101\to 0$, then for any value of d we have $\alpha=0$. □

Since $000\to 0$, $\bullet 10\to 0$ and $100\to 0$ we deduce that while the state $\bullet$ moves leftwards it generates the configuration $\bullet 10\cdots 0$. When the state $\bullet$ comes back rightwards it will only find the state 0 until position $k+3$. Then, the $(i+1)$th leftwards change of direction would occur in position $l_{i+1}\geqslant k+3$. Analogous reasoning allows to prove the result for rightwards changes of direction. □

Proposition 16. *If* $(\{0,1,\bullet\}, f, \varphi)$ *p.c.c. constructs* A, *with* $\sigma(A) \subseteq A$ *and* $E_1 \subseteq A$, *then there exists* $e \in E_2$ *such that* $e \notin A$.

Proof. Consider the s1-signal defined by $e_i = -i+2$ for $i \geqslant 3$, $e_0 = e_2 = 0$ and $e_1 = -1$. We have $\langle e \rangle = 2$. A rightwards change of direction occurs at time 1 and a leftwards change of direction occurs at time 2. From the restrictions on f proved in Lemma 8 we deduce the following evolution for $\varphi(e)$:

	−3	−2	−1	0	1	2
0				$\bullet$	1^1	0^1
1	0^1	1^1	$\bullet$	0^3	0^2	
2		0^2	1^4	$\bullet$	0^4	

The states with super-index 1 are deduced from Lemma 8. The states with super-index 2 are deduced from those with super-index 1 and Lemma 8. From property P2 we know that at time 2 the state $\bullet$ must be in position 0. We need to put a state 0 as its right neighbor at time 1. At time 2 we get $01\bullet$ that produces a state 0 at time 3 at position -1 which is a contradiction. □

Corollary 17. *Let* A *be a class of s1-signals such that* $\sigma(A) \subseteq A$ *and* $E_2 \subseteq A$. *Then there is no* CA_1^1 $(\{0,1,\bullet\}, f)$ *p.c.c. constructing* A.

Proof. It follows directly from Proposition 16. □

Proposition 18. *There exists a* CA_1^1 $(\{0,1,\bullet\}, f)$ *which p.c.c. constructs* S.

Proof. We use the local transition function f as defined in Lemma 8. The function φ associates to each s1-signal $e \in S$ a configuration $\varphi(e)$ as follows: $\varphi(e)$: $\cdots 0110 \cdots 0 \bullet 10 \cdots 0110 \cdots 0110 \cdots$ where each pattern 0110 induces a change of direction. If a leftwards change of direction happens at position k, the coordinates k and $k+1$ of $\varphi(e)$ have the state 1. If a rightwards change of direction occurs in position l, the coordinates l and $l-1$ of $\varphi(e)$ have the state 1. We define $\varphi_{\{e_0,e_0+1\}}(e_0)$ depending on e_1: $\varphi_{\{e_0,e_0+1\}}(e_0) = \bullet 1$ if $e_1 = e_0 - 1$ and $\varphi_{\{e_0-1,e_0\}}(e_0) = 1\bullet$ if $e_1 = e_0 + 1$. Any other position is in the state 0 (see Fig. 1). □

4. Four states are enough to p.c.c. construct E_*

In previous section we have established that three-state cellular automata cannot p.c.c. construct arbitrary classes of s1-signals. This gives an impossibility result that in our knowledge is not known. Next result shows that E_* can be p.c.c. constructed by a four-states cellular automaton.

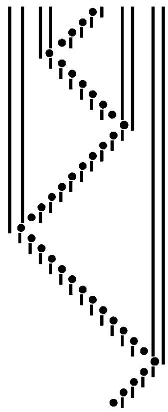

Fig. 1. $F^i(\varphi(e))$: $e \in S$, $i \in \mathbb{N}$.

We partition the set $\mathbb{Z}$ in the following sets: $\{e_0\}$,

$$E^+(e_0) = \{e_0 + 2i:\ i \geqslant 1\}, \quad O^+(e_0) = \{e_0 + 2i - 1:\ i \geqslant 1\}$$

and

$$E^-(e_0) = \{e_0 - 2i:\ i \geqslant 1\}, \quad O^-(e_0) = \{e_0 - 2i + 1:\ i \geqslant 1\}.$$

For $e \in E_*$ let $L(e)$ and $R(e)$ be sets of integers defined by

$$L(e) = \{e_t + t:\ e \text{ has a leftward change of direction at time } t\}$$

and

$$R(e) = \{e_t - t:\ e \text{ has a rightward change of direction at time } t\}.$$

Lemma 19. *The following inclusions hold*: $L(e) \subseteq E^+(e_0)$ *and* $R(e) \subseteq E^-(e_0)$.

Proof. We show by induction that for every $t \geqslant 0$, $e_t + t \in E^+(e_0)$ and $e_t - t \in E^-(e_0)$. The case $t = 0$ is direct. Since $|e_{t+1} - e_t| = 1$ we have that $e_{t+1} + t + 1 \in \{e_t + t, e_t + t + 2\}$ and $e_{t+1} - (t+1) \in \{e_t - t, e_t - t - 2\}$. □

Corollary 20. *Let e be a s1-signal with two leftwards changes of direction at time* t_1 *and* t_2, *with* $t_1 < t_2$. *Then* $e_{t_1} + t_1 < e_{t_2} + t_2$. *The analogous property holds for rightwards changes of direction.*

Proof. From the proof of Lemma 19 we know that the sequence $(e_t + t)$ is non-decreasing. The s1-signal e has a rightwards change of direction at some time $t \in \{t_1 + 1, \ldots, t_2 - 1\}$. Let t be the first time after t_1 when the s1-signal e has a rightwards change of direction. Then $e_t + t = e_{t_1} + t_1$ and $e_{t_2} + t_2 \geqslant e_{t+1} + t + 1 = e_t + t + 2 > e_{t_1} + t_1$. □

Let $Q = \{\bullet, 0, L, R\}$ be a set of states. Using the partition of $\mathbb{Z}$, Lemma 19 and its Corollary 20 we define the function $\varphi : E_* \to Q^{\mathbb{Z}}$ as follows.

For $i \in E^-(e_0) \cup E^+(e_0) \cup \{e_0\}$ the value $\varphi_i(e)$ is given by

$$\varphi_i(e) = \begin{cases} \bullet & \text{if } i = e_0, \\ R & \text{if } i \in R(e), \\ L & \text{if } i \in L(e), \\ 0 & \text{otherwise.} \end{cases}$$

For $i \in O^+(e_0) \cup O^-(e_0)$ we define $\varphi_i(e) = h_{a(e)}(\varphi_{i-1}(e), \varphi_{i+1}(e))$ where $a(e) = e_0 - e_1$ and the functions h_{-1} and h_1 are defined as follows:

h_{-1}	$\bullet$	0	R	L
$\bullet$		0		L
0	R	0	R	L
R	L	R	L	
L		L		R

h_1	$\bullet$	0	R	L
$\bullet$		L		R
0	0	0	R	L
R	R	R	L	
L		L		R

Lemma 21. *For every $e \in E_*$ we have the following statements*:

(1) *For every $i \leqslant e_0 - 3$ it follows $\varphi_i(\sigma(e)) = \varphi_{i-1}(e)$.*
(2) *For every $i \geqslant e_0 + 3$ it follows $\varphi_i(\sigma(e)) = \varphi_{i+1}(e)$.*

Proof. We consider only the case $i \leqslant e_0 - 3$. For $i \leqslant e_0 - 3$ we have that $i - 1 \in E^-(e_0)$ if and only if $i \in E^-(\sigma_0(e))$, $i - 1 \in O^-(e_0)$ if and only if $i \in O^-(\sigma_0(e))$. For every $t \geqslant 2$ a s1-signal e has a rightwards (leftwards) change of direction at time t if and only if $\sigma(e)$ has a rightwards (leftwards) change of direction at time $t - 1$.

For $i \leqslant e_0 - 3$ it follows that $i - 1 \in R(e)$ if and only if $i \in R(\sigma(e))$. Therefore, for every $i \in E^-(\sigma_0(e))$ with $i \leqslant e_0 - 3$ we have that $\varphi_i(\sigma(e)) = \varphi_{i-1}(e)$. Since $h_1(X, Y) = h_{-1}(X, Y)$ for every $X, Y \in \{0, R, L\}$ it follows that $\varphi_i(\sigma(e)) = \varphi_{i-1}(e)$ for every $i \leqslant e_0 - 3$. A similar argument shows that $\varphi_i(\sigma(e)) = \varphi_{i+1}(e)$ for every $i \geqslant e_0 + 3$. □

Let $(\{\bullet, 0, L, R\}, f)$ be a cellular automaton with f satisfying the following transitions:

$\bullet$	$\bullet$	0	R	L
$\bullet$				
0			R	L
R		R	L	0
L		L		R

0	$\bullet$	0	R	L
$\bullet$		$\bullet$		
0	$\bullet$	0	0	L
R		R	R	
L		0		L

R	$\bullet$	0	R	L
$\bullet$				L
0	0		0	
R	$\bullet$	R	R	R
L			L	L

L	$\bullet$	0	R	L
$\bullet$		0		$\bullet$
0				L
R	R		R	L
L		0	R	L

Lemma 22. *For the function F induced by f we have the following properties*:

(1) *For every $i \leqslant e_0 - 2$ we have that $F_i(\varphi(e)) = \varphi_{i-1}(e)$, that is the function F in the part of the configuration $\varphi(e)$ to the left of the state $\bullet$ acts as a rightwards shift*.

(2) *For every $i \geqslant e_0 + 2$ we have that $F_i(\varphi(e)) = \varphi_{i+1}(e)$.*

Proof. The set D_l of all local configurations that appear in $\varphi(e)$ to the left of the state $\bullet$ is given by

$$D_l = \{000, RRR, RR0, RRL, R00, RLR, R0R, 0RR, 00R, LRL, LRR\}.$$

The function f satisfies the following:

(1) $Rw \to R$ for all $w \in \{RR, 0R, R0, 00, RL, LR\}$.
(2) $0w \to 0$ for all $w \in \{0R, RR, 00\}$.
(3) $Lw \to L$ for all $w \in \{RL, RR\}$.

Then $f(x, y, z) = x$ for every $xyz \in D_l$. Thus,

$$F_i(\varphi(e)) = f(\varphi_{i-1}(e), \varphi_i(e), \varphi_{i+1}(e)) = \varphi_{i-1}(e)$$

for $i \leqslant e_0 - 2$. The symmetric result can be shown similarly. $\square$

Theorem 23. *The trio $(\{0, \bullet, L, R\}, f, \varphi)$ given above p.c.c. constructs E_*.*

Proof. We have to prove that φ satisfies properties P1–P4:

- Property P2 follows directly from the definition of φ.

- Since $R(M^r(e))=R(e)+1$, $L(M^r(e))=L(e)+1$, $R(M^l(e))=R(e)-1$, $L(M^l(e))=L(e)-1$ and φ depends only on e_0, e_1, $L(e)$ and $R(e)$ it follows that φ satisfies P3.
- Since $d(\varphi(e))\leqslant 3|R(e)\cup L(e)|$ and $|R(e)\cup L(e)|\leqslant\langle e\rangle$ the property P4 is satisfied with $h(u)=3u+1$.

We prove P1, that is for every $e\in E_*$ it follows that $F(\varphi(e))=\varphi(\sigma(e))$. From Lemmas 21 and 22 we only have to prove that $F_i(\varphi(e))=\varphi_i(\sigma(e))$ for $i\in\{e_0-2,e_0-1,e_0,e_0+1,e_0+2\}$. Let us denote

$$\varphi_{\{e_0-3,e_0-2,e_0-1,e_0,e_0+1,e_0+2,e_0+3\}}(e)=XAB\bullet DCY$$

with $A\in\{0,R\}$, $C\in\{0,L\}$ and $X,B,D,Y\in\{0,R,L\}$. We are going to prove that

$$F_{\{e_0-2,e_0-1,e_0,e_0+1,e_0+2\}}(\varphi(e))=X\,abcY=\varphi_{\{e_0-2,e_0-1,e_0,e_0+1,e_0+2\}}(\sigma(e)),$$

where the value abc depends on the values of $a(e)$ and $a(\sigma(e))$. We analyze the case $a(e)=-1$. The corresponding analysis for $a(e)=1$ can be done similarly.

Case 1: $a(e)=a(\sigma(e))=-1$. On the one hand, we have

- $e_0+2\notin L(e)$ which implies $C=0$ and then $D=h_{-1}(\bullet,0)=0$.
- Since $h_{-1}(0,V)=V$ with $V\in\{0,R,L\}$ and $Y=h_{-1}(0,\varphi_{e_0+4}(e))$ we have $Y=\varphi_{e_0+4}(e)$.

Then

$$\varphi_{\{e_0-3,e_0-2,e_0-1,e_0,e_0+1,e_0+2,e_0+3\}}(e)=XAB\bullet 00\varphi_{e_0+4}(e).$$

- It follows that $e_0-2\in R(e)$ if and only if $e_0-1\in R(\sigma(e))$ which implies $\varphi_{e_0-1}(\sigma(e))=\varphi_{e_0-2}(e)=A$. Therefore,

$$\varphi_{e_0-2}(\sigma(e))=h_{-1}(\varphi_{e_0-3}(\sigma(e)),A)=h_{-1}(\varphi_{e_0-4}(e),A)=\varphi_{e_0-3}(e)=X.$$

- Since $a(\sigma(e))=-1$ we have $\varphi_{e_0}(\sigma(e))=h_{-1}(A,\bullet)=B$.
- Since $h_{-1}(\bullet,Y)=Y$ we have that $\varphi_{e_0+2}(\sigma(e))=h_{-1}(\bullet,\varphi_{e_0+3}(\sigma(e)))=h_{-1}(\bullet,Y)=Y$.

Thus,

$$\varphi_{\{e_0-2,e_0-1,e_0,e_0+1,e_0+2\}}(\sigma(e))=XAB\bullet\varphi_{e_0+4}(e).$$

On the other hand, from Lemma 22 and the transitions $\bullet 00\to\bullet$, $Ah_{-1}(A,\bullet)\bullet\to A$ and $h_{-1}(A,\bullet)\bullet 0\to h_{-1}(A,\bullet)$ for $A\in\{0,R\}$ we have that

$$F_{\{e_0-2,e_0-1,e_0,e_0+1,e_0+2\}}(\varphi(e))=XAB\bullet Y.$$

Case 2: $a(e)=-1$ and $a(\sigma(e))=1$. On the one hand, we have

- $e_0+2\in L(e)$ which implies $C=L$ and then $D=h_{-1}(\bullet,L)=L$.
- $Y=h_{-1}(L,\varphi_{e_0+4}(e))$.

Then

$$\varphi_{\{e_0-3,e_0-2,e_0-1,e_0,e_0+1,e_0+2,e_0+3\}}(e)=XAh_{-1}(A,\bullet)\bullet LLY.$$

- It follows that $e_0-2\in R(e)$ if and only if $e_0-1\in R(\sigma(e))$ which implies $\varphi_{e_0-1}(\sigma(e))=\varphi_{e_0-2}(e)=A$ and

$$\begin{aligned}\varphi_{e_0-2}(\sigma(e)) &= h_1(\varphi_{e_0-3}(\sigma(e)),A)\\ &= h_1(\varphi_{e_0-4}(e),A)\\ &= h_{-1}(\varphi_{e_0-4}(e),A)\\ &= \varphi_{e_0-3}(e)=X.\end{aligned}$$

- $a(\sigma(e))=1$ implies that $\varphi_{e_0}(\sigma(e))=h_1(A,\bullet)=A$ and

$$\begin{aligned}\varphi_{e_0+2}(\sigma(e)) &= h_1(\bullet,\varphi_{e_0+3}(\sigma(e)))\\ &= h_1(\bullet,\varphi_{e_0+4}(e))\\ &= h_{-1}(L,\varphi_{e_0+4}(e))=Y.\end{aligned}$$

Thus,

$$\varphi_{\{e_0-2,e_0-1,e_0,e_0+1,e_0+2\}}(\sigma(e))=XAh_1(A,\bullet)\bullet Y=XAA\bullet Y.$$

On the other hand, from Lemma 22 and the transitions $Ah_{-1}(A,\bullet)\bullet\to A$, $h_{-1}(A,\bullet)\bullet L\to h_1(A,\bullet)=A$ and $\bullet LL\to\bullet$. Therefore,

$$F_{\{e_0-2,e_0-1,e_0,e_0+1,e_0+2\}}(\varphi(e))=XAA\bullet Y. \qquad \square$$

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