

Introduction to Probability

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To our wives
and in memory of
Reese T. Prosser

Preface

Probability theory began in seventeenth century France when the two great French mathematicians, Blaise Pascal and Pierre de Fermat, corresponded over two problems from games of chance. Problems like those Pascal and Fermat solved continued to influence such early researchers as Huygens, Bernoulli, and DeMoivre in establishing a mathematical theory of probability. Today, probability theory is a well-established branch of mathematics that finds applications in every area of scholarly activity from music to physics, and in daily experience from weather prediction to predicting the risks of new medical treatments.

This text is designed for an introductory probability course taken by sophomores, juniors, and seniors in mathematics, the physical and social sciences, engineering, and computer science. It presents a thorough treatment of probability ideas and techniques necessary for a firm understanding of the subject. The text can be used in a variety of course lengths, levels, and areas of emphasis.

For use in a standard one-term course, in which both discrete and continuous probability is covered, students should have taken as a prerequisite two terms of calculus, including an introduction to multiple integrals. In order to cover Chapter 11, which contains material on Markov chains, some knowledge of matrix theory is necessary.

The text can also be used in a discrete probability course. The material has been organized in such a way that the discrete and continuous probability discussions are presented in a separate, but parallel, manner. This organization dispels an overly rigorous or formal view of probability and offers some strong pedagogical value in that the discrete discussions can sometimes serve to motivate the more abstract continuous probability discussions. For use in a discrete probability course, students should have taken one term of calculus as a prerequisite.

Very little computing background is assumed or necessary in order to obtain full benefits from the use of the computing material and examples in the text. All of the programs that are used in the text have been written in each of the languages TrueBASIC, Maple, and Mathematica.

This book is on the Web at <http://www.geom.umn.edu/locate/chance>, and is part of the Chance project, which is devoted to providing materials for beginning courses in probability and statistics. The computer programs, solutions to the odd-numbered exercises, and current errata are also available at this site. Instructors may obtain all of the solutions by writing to either of the authors, at jlsnell@dartmouth.edu and cgrinst1@swarthmore.edu. It is our intention to place

items related to this book at this site, and we invite our readers to submit their contributions.

FEATURES

Level of rigor and emphasis: Probability is a wonderfully intuitive and applicable field of mathematics. We have tried not to spoil its beauty by presenting too much formal mathematics. Rather, we have tried to develop the key ideas in a somewhat leisurely style, to provide a variety of interesting applications to probability, and to show some of the nonintuitive examples that make probability such a lively subject.

Exercises: There are over 600 exercises in the text providing plenty of opportunity for practicing skills and developing a sound understanding of the ideas. In the exercise sets are routine exercises to be done with and without the use of a computer and more theoretical exercises to improve the understanding of basic concepts. More difficult exercises are indicated by an asterisk. A solution manual for all of the exercises is available to instructors.

Historical remarks: Introductory probability is a subject in which the fundamental ideas are still closely tied to those of the founders of the subject. For this reason, there are numerous historical comments in the text, especially as they deal with the development of discrete probability.

Pedagogical use of computer programs: Probability theory makes predictions about experiments whose outcomes depend upon chance. Consequently, it lends itself beautifully to the use of computers as a mathematical tool to simulate and analyze chance experiments.

In the text the computer is utilized in several ways. First, it provides a laboratory where chance experiments can be simulated and the students can get a feeling for the variety of such experiments. This use of the computer in probability has been already beautifully illustrated by William Feller in the second edition of his famous text *Introduction to Probability Theory and Its Applications* (New York: Wiley, 1950). In the preface, Feller wrote about his treatment of fluctuation in coin tossing: “The results are so amazing and so at variance with common intuition that even sophisticated colleagues doubted that coins actually misbehave as theory predicts. The record of a simulated experiment is therefore included.”

In addition to providing a laboratory for the student, the computer is a powerful aid in understanding basic results of probability theory. For example, the graphical illustration of the approximation of the standardized binomial distributions to the normal curve is a more convincing demonstration of the Central Limit Theorem than many of the formal proofs of this fundamental result.

Finally, the computer allows the student to solve problems that do not lend themselves to closed-form formulas such as waiting times in queues. Indeed, the introduction of the computer changes the way in which we look at many problems in probability. For example, being able to calculate exact binomial probabilities for experiments up to 1000 trials changes the way we view the normal and Poisson approximations.

ACKNOWLEDGMENTS FOR FIRST EDITION

Anyone writing a probability text today owes a great debt to William Feller, who taught us all how to make probability come alive as a subject matter. If you find an example, an application, or an exercise that you really like, it probably had its origin in Feller's classic text, *An Introduction to Probability Theory and Applications*.

This book had its start with a course given jointly at Dartmouth College with Professor John Kemeny. I am indebted to Professor Kemeny for convincing me that it is both useful and fun to use the computer in the study of probability. He has continuously and generously shared his ideas on probability and computing with me. No less impressive has been the help of John Finn in making the computing an integral part of the text and in writing the programs so that they not only can be easily used, but they also can be understood and modified by the student to explore further problems. Some of the programs in the text were developed through collaborative efforts with John Kemeny and Thomas Kurtz on a Sloan Foundation project and with John Finn on a Keck Foundation project. I am grateful to both foundations for their support.

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The debt to William Feller has not diminished in the years between the two editions of this book. His book on probability is likely to remain the classic book in this field for many years.

The process of revising the first edition of this book began with some high-level discussions involving the two present co-authors together with Reese Prosser and John Finn. It was during these discussions that, among other things, the first co-author was made aware of the concept of “negative royalties” by Professor Prosser.

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Peter Doyle motivated us to make this book part of a larger project on the Web, to which others can contribute. He also spent many hours actually carrying out the operation of putting the book on the Web.

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