CONTINUOUS AND MEASURABLE EIGENFUNCTIONS OF LINEARLY RECURRENT DYNAMICAL CANTOR SYSTEMS

MARIA ISABEL CORTEZ, FABIEN DURAND, BERNARD HOST, AND ALEJANDRO MAASS

ABSTRACT. The class of linearly recurrent Cantor systems contains the substitution subshifts and some odometers. For substitution subshifts measure–theoretical and continuous eigenvalues are the same. It is natural to ask whether this rigidity property remains true for the class of linearly recurrent Cantor systems. We give partial answers to this question.

1. INTRODUCTION

Let (X, T) be a topological dynamical system and μ a T–invariant probability measure. When a measure-theoretical eigenvalue $\lambda \in \mathbb{C}$ of the system, that is $f \circ T = \lambda f$ for some $f \in L^2(\mu) \backslash \{0\}$, is associated to a continuous eigenfunction $f : X \to \mathbb{C}$? In this paper we are interested in conditions on minimal dynamical Cantor systems that answer this question. Our motivation comes from [Ho] where it is proved that all eigenvalues of minimal substitution subshifts are associated to a continuous eigenfunction. Such a question also appears in [NR] where the authors show that generically interval exchange transformations are not topologically weakly mixing (i.e., they do not have non trivial continuous eigenfunctions) and where they "fully expect" the same holds for (measure-theoretical) weak mixing (i.e., they do not have non trivial eigenfunctions). It is in general not true that all eigenvalues of a minimal dynamical system have a continuous eigenfunction as can be seen for some Toeplitz systems [Iw, DL] and for some interval exchange transformations [FHZ].

In this paper we focus on linearly recurrent dynamical Cantor systems (also called linearly recurrent systems). They naturally extend the notion of substitution subshifts in the sense they share the same return time structure. Linearly recurrent subshifts were studied in [DHS, Du1, Du2, Le].

The paper is organized as follows. In Section 2 we define linearly recurrent systems by means of nested sequence of Kakutani-Rokhlin partitions and obtain some general properties. In particular we prove that these systems are uniquely ergodic but are not strongly mixing.

In the following section, when the dynamical system (X, T) is linearly recurrent and μ is a T–invariant probability measure we give a necessary condition for a complex number to be an eigenvalue. We also exhibit a sufficient condition for a complex number to be a continuous eigenvalue, which involves the underlying matrix structure of the nested sequence of Kakutani-Rokhlin partitions defining (X, T). This is used in the last section to prove for natural probability spaces associated to families

---

1991 *Mathematics Subject Classification.* Primary: 54H20; Secondary: 37B20 .

Key words and phrases. minimal Cantor systems, linearly recurrent dynamical systems, eigenvalues.

of linearly recurrent systems, and under a condition of "hyperbolicity", that almost every system of such family has only continuous eigenvalues. We give in Section 4 several examples to illustrate the results of the paper.

2. Definitions and background

2.1. **Dynamical systems.** By a *topological dynamical system* we mean a couple (X, T) where X is a compact metric space and $T : X \to X$ is a homeomorphism. We say that it is a *Cantor system* if X is a Cantor space; that is, X has a countable basis of its topology which consists of closed and open sets (clopen sets) and does not have isolated points. The topological dynamical system (X, T) is *minimal* whenever X and the empty set are the only T-invariant closed subsets of X. We only deal here with minimal Cantor systems. A complex number λ is a *continuous eigenvalue* of (X, T) if there exists a continuous function $f : X \to \mathbb{C}$, $f \neq 0$, such that $f \circ T = \lambda f$; f is called a *continuous eigenfunction* associated to λ. If (X, T) is minimal, then every continuous eigenvalue is of modulus 1 and every continuous eigenfunction has a constant modulus.

When (X, T) is a topological dynamical system and μ is a T-invariant probability measure, i.e., $T\mu = \mu$, defined on the Borel σ-algebra $\mathcal{B}_X$ of X, we call the triple (X, T, μ) a *dynamical system*. We do not recall the definitions of ergodicity, weak mixing and strong mixing (see [Wa] for example). A complex number λ is an *eigenvalue* of the dynamical system (X, T, μ) if there exists $f \in L^2(\mu)$, $f \neq 0$, such that $f \circ T = \lambda f$, μ-a.e.; f is called an *eigenfunction* (associated to λ). If the system is ergodic, then every eigenvalue is of modulus 1, and every eigenfunction has a constant modulus. By abuse of language we will also say that an eigenvalue is continuous when the associated eigenfunction is continuous.

In this paper we mainly consider topological dynamical systems (X, T) which are uniquely ergodic, that is systems that admit a unique invariant probability measure; this measure is then ergodic.

2.2. **Partitions.** Sequences of partitions of a minimal Cantor system were used in [HPS] to build a representation of the system as an adic transformation on an ordered Bratteli diagram. We recall some definitions and fix some notations we shall use along this paper.

Let (X, T) be a minimal Cantor system. A *clopen Kakutani-Rokhlin partition* (CKR partition) is a partition $\mathcal{P}$ of X of the kind:

$$\mathcal{P} = \{T^{-j}B_k ; 1 \leq k \leq C,\ 0 \leq j < h_k\} \tag{2.1}$$

where C is a positive integer, $B_1, \ldots, B_C$ are clopen subsets of X and $h_1, \ldots, h_k$ are positive integers. For $1 \leq k \leq C$, the k-th *tower* of $\mathcal{P}$ is the family $\{T^{-j}B_k ; 0 \leq j < h_k\}$, and the *base* of $\mathcal{P}$ is the set $B = \bigcup_{1 \leq k \leq C} B_k$. Let

$$\big(\mathcal{P}(n) = \{T^{-j}B_k(n) : 1 \leq k \leq C(n),\ 0 \leq j < h_k(n)\}\ ;\ n \in \mathbb{N}\big) \tag{2.2}$$

be a sequence of CKR partitions. For every n we write $B(n)$ for the base of $\mathcal{P}(n)$, and we assume that $\mathcal{P}(0)$ is the trivial partition, that is $B(0) = X$, $C(0) = 1$ and $h_1(0) = 1$.

We say that this sequence $(\mathcal{P}(n); n \in \mathbb{N})$ is *nested* if for every $n \geq 0$ it satisfies:

(KR1) $B(n+1) \subset B(n)$ and

(KR2) $\mathcal{P}(n+1) \succeq \mathcal{P}(n)$; i.e., for all $A \in \mathcal{P}(n+1)$ there exists $A^{'} \in \mathcal{P}(n)$ such that $A \subset A^{'}$.

We consider mostly nested sequences of CKR partitions which satisfy also the properties:

(KR3) $\cap_{n=0}^{\infty} B(n)$ consists of a unique point;

(KR4) the sequence of partitions spans the topology of X;

In [HPS] it is proven that for each minimal Cantor system (X,T) there exists a nested sequence of CKR partitions fulfilling (KR1)-(KR4) (i.e., (KR1), (KR2), (KR3) and (KR4)) and the following conditions:

(KR5) for all $n \geq 1$, $1 \leq k \leq C(n-1)$, $1 \leq l \leq C(n)$, there exists $0 \leq j < h_l(n)$ such that $T^{-j}B_l(n) \subset B_k(n-1)$;

(KR6) for all $n \in \mathbb{N}$, $B(n+1) \subset B_1(n)$.

To such a sequence of partitions we associate a sequence of matrices $(M(n); n \geq 1)$, where the matrix $M(n) = (m_{l,k}(n); 1 \leq l \leq C(n), 1 \leq k \leq C(n-1))$ is given by

$$m_{l,k}(n) = \#\{0 \leq j < h_l(n); T^{-j}B_l(n) \subset B_k(n-1)\}.$$

We notice that Property (KR5) is equivalent to the condition that for every $n \geq 1$ the matrix $M(n)$ has positive entries. As the sequence of partitions is nested, we get

$$h_l(n) = \sum_{k=1}^{C(n-1)} m_{l,k}(n) h_k(n-1), \quad n \geq 1, \ 1 \leq l \leq C(n).$$

We rewrite this formula in a matrix form. For every $n \geq 0$, let $H(n) = (h_l(n); 1 \leq l \leq C(n))^t$, that is the column vector with entries $h_1(n), h_2(n), \ldots, h_{C(n)}(n)$. Then we have $H(n) = M(n)H(n-1)$ for $n > 0$. For $n > m \geq 0$ we define

$$P(n,m) = M(n)M(n-1)\ldots M(m+1) \text{ and } P(n) = P(n,1) \ .$$

We have:

$$P_{l,k}(n,m) = \#\big\{j; 0 \leq j \leq h_l(n), \ T^{-j}B_l(n) \subset B_k(m)\big\}$$

and

$$P(n,m)H(m) = H(n) = P(n)H(1) \ .$$

2.3. **Linearly recurrent systems.**

Definition 1. *A minimal Cantor system (X,T) is linearly recurrent (with constant L) if there exists a nested sequence of CKR partitions $(\mathcal{P}(n) = \{T^{-j}B_k(n); 1 \leq k \leq C(n), 0 \leq j < h_k(n)\}; n \in \mathbb{N})$ satisfying (KR1)-(KR6) and*

(LR) *there exists L such that for all $(l,k) \in \{1,\ldots,C(n)\} \times \{1,\ldots,C(n-1)\}$ and all $n \geq 1$*

$$h_l(n) \leq L \ h_k(n-1) \ .$$

The notion of linearly recurrent dynamical Cantor system (also called linearly recurrent system) is the extension of the concept of linearly recurrent subshift introduced in [DHS]. Of course it can be proved that linearly recurrent subshifts are linearly recurrent Cantor systems (see [Du1, Du2]). Examples of such systems are substitution subshifts [DHS] and Sturmian subshifts whose associated rotation number has a continued fraction with bounded coefficients [Du1, Du2].

Lemma 2. *Let (X,T) be a linearly recurrent system and $(\mathcal{P}(n); n \in \mathbb{N})$ a sequence of CKR partitions satisfying Properties (KR1)-(KR6), and Property (LR) with constant L. Then:*

1. *for every $n \in \mathbb{N}$ we have $C(n) \le L$;*
2. *for every $n \in \mathbb{N}$, $1 \le k \le C(n)$ and $1 \le k' \le C(n)$ we have $h_k(n) \le L\ h_{k'}(n)$.*

Proof. Property (1) follows directly from the hypotheses (KR5) and (LR). Indeed $C(n)h_i(n) \le h_1(n+1) \le Lh_i(n)$ where $h_i(n) = \min\{h_k(n); 1 \le k \le C(n)\}$.

In a similar way we prove Property (2). From (KR5) it comes that all $h_i(n+1)$ are greater than $\sum_{j=1}^{C(n)} h_j(n)$. Consequently, from (LR) we obtain for all $1 \le k \le C(n)$ and $1 \le k' \le C(n)$

$$h_k(n) \le \sum_{j=1}^{C(n)} h_j(n) \le h_i(n+1) \le Lh_{k'}(n).$$

This ends the proof. □

From this lemma we deduce that the set $\{M(n); n \ge 1\}$ is finite. The following proposition, whose proof is left to the reader, tells us this is in fact a necessary and sufficient condition to be linearly recurrent.

Proposition 3. *Let (X,T) be a minimal Cantor system. The system (X,T) is linearly recurrent if and only if there exist a nested sequence of CKR partitions $(\mathcal{P}(n); n \in \mathbb{N})$, satisfying (KR1)-(KR6), and a constant K such that: for all $n \ge 1$ and all $(l,k) \in \{1,\ldots,C(n)\} \times \{1,\ldots,C(n-1)\}$,*

$$1 \le m_{l,k}(n) \le K,$$

where $(M(n) = (m_{l,k}(n); 1 \le l \le C(n), 1 \le k \le C(n-1)); n \ge 1)$ be the associated sequence of matrices.

2.4. **Unique ergodicity and absence of strong mixing of linearly recurrent systems.** In this subsection (X,T) is a linearly recurrent system with a nested sequence of CKR partitions $(\mathcal{P}(n) = \{T^{-j}B_k(n); 1 \le k \le C(n), 0 \le j < h_k(n)\}; n \in \mathbb{N})$ satisfying (KR1)-(KR6) and (LR) with constant L. Let $(M(n) = (m_{l,k}(n); 1 \le l \le C(n), 1 \le k \le C(n-1)); n \ge 1)$ be the associated sequence of matrices.

We notice that for each T–invariant probability measure μ and for every $n \ge 1$ and $1 \le k \le C(n-1)$ we have

$$\mu(B_k(n-1)) = \sum_{l=1}^{C(n)} m_{l,k}(n)\, \mu(B_l(n)) \tag{2.3}$$

and

$$\sum_{k=1}^{C(n)} h_k(n)\mu(B_k(n)) = 1 \ . \tag{2.4}$$

To prove that linearly recurrent systems are uniquely ergodic we need the following lemma that is used through all this paper.

Lemma 4. *Let μ be an invariant measure of (X,T). Then, for all $n \in \mathbb{N}$ and $1 \le k \le C(n)$ we have*

$$h_k(n)\mu(B_k(n)) \ge \frac{1}{L}.$$

Proof. Fix k with $1 \le k \le C(n)$. By Equation (2.3), since all the entries of $M(n+1)$ are positive, we get

$$\mu(B_k(n)) \ge \sum_{\ell=1}^{C(n+1)} \mu(B_l(n+1)) \ .$$

By (LR), for every l we have $h_k(n) \ge h_l(n+1)/L$ thus

$$h_k(n)\mu(B_k(n)) \ge \sum_{\ell=1}^{C(n+1)} \frac{h_l(n+1)}{L}\mu(B_l(n+1)) = \frac{1}{L} \ .$$

□

Proposition 5. *Every linearly recurrent system is uniquely ergodic.*

Proof. Let (X,T) be a linearly recurrent system. Given a T–invariant probability measure μ, we define the numbers

$$\mu_{n,k} = \mu(B_k(n)),\ n \ge 0,\ 1 \le k \le C(n) \ .$$

These nonnegative numbers satisfy the relations

$$\mu_{0,1} = 1 \text{ and } \mu_{n-1,k} = \sum_{l=1}^{C_n} \mu_{n,l} m_{l,k}(n) \text{ for } n \ge 1,\ 1 \le k \le C(n-1) \ . \tag{2.5}$$

In a matrix form: with $V(n) = (\mu_{n,1}, \dots, \mu_{n,C(n)})$, we have $V(n-1) = V(n)M(n)$. Conversely, let the nonnegative numbers $(\mu_{n,k}; n \ge 0,\ 1 \le k \le C(n))$ satisfy these conditions. As the partitions $\mathcal{P}(n)$ are clopen and span the topology of X, it is immediate to check that there exists a unique invariant probability measure μ on X with $\mu_{n,k} = \mu(B_k(n))$ for every $n \in \mathbb{N}$ and $k \in \{1, \dots, C(n)\}$.

From Lemma 4, there exists a constant $\delta > 0$ such that

$$\mu_{n,i} \ge \delta \mu_{n-1,k}$$

for every $n \ge 1$ and $(i,k) \in \{1, \dots, C(n)\} \times \{1, \dots, C(n-1)\}$, and every invariant measure μ. Without loss of generality we can assume $\delta < 1/2$.

Let μ, μ' be two invariant measures, and $\mu_{n,k}, \mu'_{n,k}$ be defined as above. We define

$$S_n = \max_k \frac{\mu'_{n,k}}{\mu_{n,k}} = \frac{\mu'_{n,i}}{\mu_{n,i}}\ ,\ s_n = \min_k \frac{\mu'_{n,k}}{\mu_{n,k}} = \frac{\mu'_{n,j}}{\mu_{n,j}}\ ,\ \text{and } r_n = \frac{S_n}{s_n}$$

for some i,j. For every $k \in \{1, \dots, C(n-1)\}$ we have:

$$\begin{aligned}
\mu'_{n-1,k} &= \sum_{l \ne j} \mu'_{n,l} m_{l,k}(n) + \mu'_{n,j} m_{j,k}(n) \\
&\le S_n \sum_{l \ne j} \mu_{n,l} m_{l,k}(n) + s_n \mu_{n,j} m_{j,k}(n) \\
&= S_n \mu_{n-1,k} - (S_n - s_n)\mu_{n,j} m_{j,k}(n) \le \mu_{n-1,k} S_n - (S_n - s_n)\mu_{n,j} \\
&\le \mu_{n-1,k} s_n \big(r_n(1-\delta) + \delta \big) \ .
\end{aligned}$$

And in similar way, for every $k \in \{1, \dots, C(n-1)\}$ we have

$$\mu'_{n-1,k} \geq \mu_{n-1,k} s_n \big(\delta r_n + (1-\delta)\big) \ .$$

We deduce that

$$r_{n-1} \leq \phi(r_n) \text{ where } \phi(x) = \frac{(1-\delta)x + \delta}{\delta x + (1-\delta)} \ .$$

The function ϕ is increasing on $[0, +\infty)$ and tends to $(1-\delta)/\delta$ at $+\infty$. Writing $\phi^m = \phi \circ \cdots \circ \phi$ (m times), for every $n, m \in \mathbb{N}$ we have $1 \leq r_n \leq \phi^m(r_{n+m}) \leq \phi^{m-1}((1-\delta)/\delta)$. Taking the limit with $m \to +\infty$, we get $r_n = 1$. □

From now on we call μ the unique invariant measure on (X, T). Let $m \geq 1$ and $0 \leq k \leq C(m)$. By unique ergodicity,

$$\frac{1}{N} \#\{0 \leq j < N;\ T^{-j}x \in B_k(m)\} \to \mu(B_k(m))$$

uniformly as $N \to \infty$. But for $n > m$, $1 \leq l \leq C(n)$, for every $x \in B_k(n)$ we have

$$\#\{0 \leq j < h_l(n);\ T^{-j}x \in B_k(m)\} = P_{l,k}(n, m) \ .$$

We deduce:

$$\max_{1 \leq l \leq C(n)} \Big| \frac{P_{l,k}(n,m)}{h_l(n)} - \mu\big(B_k(m)\big) \Big| \to 0 \text{ as } n \to +\infty \ . \tag{2.6}$$

Proposition 6. *Linearly recurrent systems are not strongly mixing.*

Proof. Let m be an integer such that $\mu(B_1(m)) < 1/L^2$ and for $n > m$ let $D(n) = B_1(m) \cap T^{h_1(n)} B_1(m)$. We prove that $\underline{\lim}_n\, \mu(D(n)) > \mu(B_1(m))^2$ which will imply that (X, T, μ) is not strongly mixing.

For $n > m$ we write

$$E(n) = \{0 \leq j < h_1(n-1);\ T^{-j} B_1(n-1) \subset B_1(m)\} \ .$$

By hypothesis (KR6) we have $B(n) \subset B_1(n-1)$ and for $j \in E(n)$ we get

$$T^{-j-h_1(n)} B_1(n) \subset T^{-j} B(n) \subset T^{-j} B_1(n-1) \subset B_1(m)$$

and $T^{-j} B_1(n) \subset D(n)$. It follows that $\mu(D(n)) \geq \#E(n) \cdot \mu(B_1(n))$. But $\#E(n) = P_{1,1}(n-1, m)$ thus $\#E(n)/h_1(n-1)$ converges to $\mu(B_1(m))$ as $n \to +\infty$ by Equation (2.6). Therefore

$$\begin{aligned}
\underline{\lim}_n\, \mu(D(n)) &\geq \underline{\lim}_n\, h_1(n-1)\mu(B_1(n))\mu(B_1(m)) \\
&\geq \underline{\lim}_n\, \frac{1}{L} h_1(n)\mu(B_1(n))\mu(B_1(m)) && \text{(by (LR))} \\
&\geq \frac{1}{L^2}\mu(B_1(m)) && \text{(by Lemma 4)}
\end{aligned}$$

and the proof is complete. □

It is well-known that there exist substitution subshifts, and *a fortiori* linearly recurrent systems, which are weakly-mixing (see [Qu]).

3. Some conditions to be an eigenvalue

In this section we suppose (X, T, μ) is linearly recurrent, that is to say $(\mathcal{P}(n); n \geq 0)$ satisfies (KR1)-(KR6) and (LR) (with constant L). Let $(M(n); n \geq 1)$ be its associated sequence of matrices.

We give a sufficient condition to be a continuous eigenvalue and a necessary condition to be an eigenvalue. We define for $n \geq 1$, $1 \leq k \leq C(n-1)$, $1 \leq l \leq C(n)$,

$$J(n,k,l) = \big\{0 \leq j < h_l(n);\ T^{-j}B_l(n) \subset B_k(n-1)\big\},\ J(n) = \bigcup_{\substack{1 \leq k \leq C(n-1) \\ 1 \leq l \leq C(n)}} J(n,k,l) \ .$$

so that $\#J(n,k,l) = m_{l,k}(n)$.

Proposition 7. *Let $\lambda \in \mathbb{C}$ satisfy*

$$\sum_{n=1}^{\infty} \max_{1 \leq k \leq C(n)} |\lambda^{h_k(n)} - 1| < \infty \ .$$

Then λ is a continuous eigenvalue of (X, T, μ).

Proof. For every $n \in \mathbb{N}$, let f_n be the function on X defined by

$$f_n(x) = \lambda^{-j} \text{ for } x \in T^{-j}B_k(n),$$

$1 \leq k \leq C(n)$ and $0 \leq j < h_k(n)$.

We compare f_n and f_{n-1}. By construction, for every x, $f_n(x)/f_{n-1}(x)$ belongs to the set $\{\lambda^{-j}; j \in J(n)\}$. But each integer in $J(n)$ is a sum of terms of the form $h_k(n-1)$, and this sum contains at most L terms. We get

$$||f_n - f_{n-1}||_\infty \leq L \max_{1 \leq k \leq C(n-1)} |\lambda^{h_k(n-1)} - 1| \ .$$

By hypothesis, the series $\sum_{n \geq 1} ||f_n - f_{n-1}||_\infty$ converges. Thus the sequence $(f_n; n \in \mathbb{N})$ converges uniformly to a continuous function f, which is clearly an eigenfunction for λ. $\square$

Proposition 8. *If $\lambda \in \mathbb{C}$ is an eigenvalue of (X, T, μ) then*

$$\sum_{n=1}^{\infty} \max_{1 \leq k \leq C(n)} \left|\lambda^{h_k(n)} - 1\right|^2 < \infty \ .$$

Proof. We use the sets $J(n,k,l)$ defined above.

Assume that $\lambda = \exp(2i\pi\alpha)$, $\alpha \in \mathbb{R}$, is an eigenvalue, and that f is a corresponding eigenfunction of modulus 1. For every $n \in \mathbb{N}$, let f_n be the conditional expectation of f with respect to the σ-algebra spanned by $\mathcal{P}_n$. For $1 \leq k \leq C(n)$, f_n is constant on $B_k(n)$, and we write $c(n,k)$ this constant.

The sequence $(f_n; n \in \mathbb{N})$ is a martingale ([Do]), and converges to f in $L^2(\mu)$. Moreover the functions $f_n - f_{n-1}$, $n \geq 1$, are mutually orthogonal in $L^2(\mu)$, hence we have

$$\sum_{n=1}^{\infty} ||f_n - f_{n-1}||_2^2 < \infty \tag{3.1}$$

(see [Do] for the details).

We fix $n \geq 1$, $1 \leq l \leq C(n)$ and $1 \leq k \leq C(n-1)$, and we choose some $j \in J(n,k,l)$. Looking at the structure of the towers, we see that $j + h_k(n-1) \leq h_l(n)$. For $0 \leq p < h_k(n-1)$ we have $j + p < h_l(n)$, and $T^{-(j+p)}B_l(n) \subset T^{-p}B_k(n-1)$.

For $x \in T^{-(j+p)}B_l(n)$, we have $f_n(x) = \exp(-2i\pi(j+p)\alpha)c(n,l)$ and $f_{n-1}(x) = \exp(-2i\pi p\alpha)c(n-1,k)$. We get

$$||f_n - f_{n-1}||_2^2 \geq h_k(n-1)\mu(B_l(n))|\exp(-2i\pi j\alpha)c(n,l) - c(n-1,k)|^2 \ .$$

By Lemma 4 and (LR), $h_k(n-1)\mu(B_l(n)) \geq L^{-2}$, and from Equation (3.1) we get

$$\sum_{n=1}^{\infty} \max_{1\leq l\leq C(n)} \max_{1\leq k\leq C(n-1)} \max_{j\in J(n,k,l)} |\exp(-2i\pi j\alpha)c(n,l) - c(n-1,k)|^2 < \infty \ . \tag{3.2}$$

We use this bound first with $k=1$ and an arbitrary l. By (KR6), $0 \in J(n,1,l)$ and from Equation (3.2) we get

$$\sum_{n=1}^{\infty} \max_{1\leq l\leq C(n)} |c(n,l) - c(n-1,1)|^2 < \infty \ .$$

Using this three times, we get

$$\sum_{n=1}^{\infty} \max_{1\leq l\leq C(n)} \max_{1\leq k\leq C(n-1)} |c(n,l) - c(n-1,k)|^2 < \infty \ . \tag{3.3}$$

For each $n \in \mathbb{N}$ and $1 \leq k \leq C(n)$, the function $|f_n|$ is constant and equal to $|c(n,k)|$ on the k-th tower of $\mathcal{P}(n)$. By Lemma 4, the measure of this tower is not less than $1/L$. Since $\|f_n\|_2 \to \|f\|_2 = 1$, we get that $\inf_k |c(n,k)|$ converges to 1 when $n \to +\infty$. Hence from Equations (3.2) and (3.3), we get

$$\sum_{n=1}^{\infty} \max_{1\leq l\leq C(n)} \max_{1\leq k\leq C(n-1)} \max_{j\in J(n,l,k)} |\exp(-2i\pi j\alpha) - 1|^2 < \infty \ .$$

We use this bound with two consecutive elements of the same set $J(n,l,k)$ and get the announced result. □

The following sufficient condition for weak mixing follows from Proposition 8. A similar condition appears in [FHZ].

Corollary 9. *For every $n \in \mathbb{N}$, let*

$$K_n = \inf\big\{|h_i(n) - h_j(n)| : 1 \leq i,j \leq C(n),\ h_i(n) \neq h_j(n)\big\}$$

and let $K = \underline{\lim}_n K_n$. If K is finite, then (X,T,μ) has at most K eigenvalues. In particular, if $K=1$ then this system is weakly mixing.

Now we restate Proposition 7 and Proposition 8 in terms of matrices.

Notation. For every real number x we write $\lvert\!\lvert\!\lvert x \rvert\!\rvert\!\rvert$ for the distance of x to the nearest integer. For a vector $V = (v_1, \ldots, v_m) \in \mathbb{R}^m$, we write

$$\|V\| = \max_{1\leq j\leq m} |v_j| \text{ and } \lvert\!\lvert\!\lvert V \rvert\!\rvert\!\rvert = \max_{1\leq j\leq m} \lvert\!\lvert\!\lvert v_j \rvert\!\rvert\!\rvert \ .$$

We use similar notations for real matrices. With these notations, the two preceding Theorems can be written as follows.

Theorem 10. *Let $\alpha \in \mathbb{R}$ and $\lambda = \exp(2i\pi\alpha)$.*

(1) *If λ is an eigenvalue of (X,T,μ) then $\displaystyle\sum_{n\geq 1} \lvert\!\lvert\!\lvert \alpha P(n)H(1) \rvert\!\rvert\!\rvert^2 < \infty$.*

(2) *If $\displaystyle\sum_{n\geq 1} \lvert\!\lvert\!\lvert \alpha P(n)H(1) \rvert\!\rvert\!\rvert < \infty$ then λ is a continuous eigenvalue of (X,T,μ).*

Proposition 11. *Let $\alpha \in \mathbb{R}$ and $\lambda = \exp(2i\pi\alpha)$.*

If λ is an eigenvalue of (X, T, μ) then it satisfies at least one of the two following properties:

1. *α is rational, with a denominator dividing* $\gcd(h_i(m) : 1 \le i \le C(m))$ *for some $m \in \mathbb{N}$. In this case λ is a continuous eigenvalue.*
2. *There exist $m \in \mathbb{N}$ and integers w_j, $1 \le j \le C(m)$, such that $\alpha = \sum_{j=1}^{C(m)} w_j \mu(B_j(m))$.*

Moreover, if α is rational, with a denominator dividing $\gcd(h_i(m) : 1 \le i \le C(m))$ *for some $m \in \mathbb{N}$, then λ is an eigenvalue of (X, T, μ).*

The proof of Proposition 11 needs the following lemma.

Lemma 12. *Let u be a real vector such that $|\!|\!| P(n)u |\!|\!| \to 0$ as $n \to +\infty$. Then there exist $m \in \mathbb{N}$, an integer vector w and a real vector v with*

$$P(m)u = w + v \text{ and } \|P(n,m)v\| \to 0 \text{ as } n \to +\infty \ .$$

Proof. By hypothesis, for every $n \in \mathbb{N}$ we can write $P(n)u = v_n + w_n$, where w_n is an integer vector and v_n a real vector with $\|v_n\| \to 0$ as $n \to +\infty$. Since all the matrices $M(m)$ belong to a finite family, $\|M(m)v_m - v_{m+1}\|$ converges to 0 as m goes to infinity. But for every $m \in \mathbb{N}$ we have $P(m+1)u = M(m+1)P(m)u$, thus

$$M(m)v_m - v_{m+1} = -M(m)w_m + w_{m+1}$$

and $M(m)v_m - v_{m+1}$ is an integer vector. Therefore the sequence $(M(m)v_m - v_{m+1})$ is eventually zero. There exists $m \in \mathbb{N}$ such that $v_n = P(n,m)v_m$ for every $n > m$. The vectors $v = v_m$ and $w = w_m$ satisfy the announced properties. □

Proof of Proposition 11. Let $u = \alpha H(1)$. Since λ is an eigenvalue, $|\!|\!| P(n)u |\!|\!| \to 0$ as $n \to \infty$ by Theorem 10. Let m, v and w be as in Lemma 12. We recall that $P(m)u = \alpha H(m)$. We distinguish two cases.

First we assume that $v = 0$. Then $\alpha H(m)$ is equal to the integer vector w, and α is rational with a denominator dividing $\gcd(h_i(m) : 1 \le i \le C(m))$. For $n \ge m$ the vector $\alpha H(m)$ has integer entries, thus $|\!|\!| \alpha H(m) |\!|\!| = 0$ and λ is a continuous eigenvalue by Theorem 10.

Now suppose $v \neq 0$. For $n > m$ we have

$$\begin{aligned}\sum_{k=1}^{C(m)} \mu(B_k(m))v_k &= \sum_{k=1}^{C(m)} \sum_{l=1}^{C(n)} P_{l,k}(n,m)\, \mu(B_l(n))v_k && \text{by (2.3)} \\ &= \sum_{l=1}^{C(n)} (P(n,m)v)_l\, \mu(B_l(n)) \le \|P(n,m)v\|\end{aligned}$$

and the last term converges to 0 as $n \to +\infty$, thus

$$\sum_{k=1}^{C(m)} \mu(B_k(m))v_k = 0 \ .$$

As $w = \alpha H(m) - v$, that is, $w_j = \alpha h_j(m) - v_j$ for $1 \le j \le C(m)$, we get

$$\sum_{j=1}^{C(m)} w_j \mu(B_j(m)) = \alpha \sum_{j=1}^{C(m)} h_j(m)\mu(B_j(m)) = \alpha \ .$$

The rest of the proof is left to the reader. □

4. Examples

We study some examples where we can explicitly say that the eigenfunctions are continuous or there do not exist non trivial eigenvalues. We keep the notations and hypotheses of the preceding section.

4.1. **Example 1: The sequence $(M(n); n \geq 2)$ is ultimately constant.** Let $(M(n); n \geq 1)$ be the sequence of matrices associated to the linearly recurrent system (X, T, μ). We say that (X, T, μ) has *a stationary sequence of matrices* if there exist a square matrix M and an integer $n_0 \in \mathbb{N}$ such that $M(n) = M$ for all $n \geq n_0$. Without loss we can assume that $n_0 = 2$. We have $P_n = M^{n-1}$ for $n \geq 2$.

Substitution subshifts and odometers with constant base belong to the family of linearly recurrent systems with a stationary sequence of matrices (see [DHS]). The following lemma was used in [Ho] to prove that eigenvalues of substitution subshifts are continuous.

Lemma 13. *Let M be a matrix with integer entries. If u is a real vector such that $\|M^n u\| \to 0$ when $n \to \infty$, then the convergence is exponential, i.e., there exist $0 \leq r < 1$ and a constant K such that $\|M^n u\| \leq K r^n$ for all $n \in \mathbb{N}$.*

From this Lemma and Theorem 10 we get:

Proposition 14. *Let (X, T, μ) be a linearly recurrent Cantor system with a stationary sequence of matrices. Then every eigenfunction of this system is continuous.*

Moreover all the linearly recurrent Cantor systems with the same stationary sequence of matrices have the same eigenvalues.

4.2. **Example 2: A family of weakly mixing systems.** We build a family of linearly recurrent systems which are the Cantor analogues of interval exchange transformations considered in [FHZ] (Theorem 2.2). Let N be a positive integer. Let $(M(n); n \geq 2)$ be a sequence of matrices in the family

$$\left\{ \begin{bmatrix} l & k-1 & 1 \\ l-1 & k & 1 \\ l-1 & k-1 & 1 \end{bmatrix}, \begin{bmatrix} l-1 & k & 1 \\ l & k-1 & 1 \\ l & k & 1 \end{bmatrix} \ : \ 1 \leq l, k \leq N \right\}$$

and let (X, T) be a linearly recurrent system with this sequence of matrices. For any n and any $v = (v_1, v_2, v_3)^t$, the vector $u = M(n)v$ satisfies $|u_1 - u_2| = |v_1 - v_2|$.

Consequently if we suppose $|h_1(1) - h_2(1)| = 1$ then it follows by induction that for all $n \geq 1$ we have $|h_1(n) - h_2(n)| = 1$. By Corollary 9 the system (X, T, μ) does not have non trivial eigenvalues, i.e. it is weakly mixing.

4.3. **Example 3: The sequence $(M(n); n \geq 1)$ has infinitely many rank 1 matrices.**

Proposition 15. *Let (X, T, μ) be a linearly recurrent Cantor system and let the associated sequence of matrices be $(M(n); n \geq 1)$. Suppose that $M(n)$ has rank one for infinitely many values of n. Then $\lambda = \exp(2i\pi\alpha)$ is an eigenvalue of (X, T, μ) if and only if α is rational with a denominator equal to $\gcd(h_i(m) : 1 \leq i \leq C(m))$ for some $m \in \mathbb{N}$. Moreover every eigenfunction is continuous.*

Proof. Let $\lambda = \exp(2i\pi\alpha)$ be an eigenvalue of (X, T, μ). As in the proof of Proposition 11, we write $u = \alpha H(1)$, and take m, v and w as given by Lemma 12. We choose $n > m$ such that $M(n)$ is of rank 1. We have

$$\ker(P(n,m)) \subset \big\{x \in \mathbb{R}^{C(m)} : \|P(l,m)x\| \to_{l\to\infty} 0\big\}$$
$$\subset \Big\{x \in \mathbb{R}^{C(m)} : \sum_{j=1}^{C(m)} x_j \mu(B_j(m)) = 0\Big\}$$

where the last inclusion follows from the proof of Proposition 11. The third of these three spaces is of codimension 1 in $\mathbb{R}^{C(m)}$. The matrix $P(n,m)$ is not zero and has a rank ≤ 1, thus the first of these three linear spaces is of codimension 1 also, and these spaces are actually equal. Since v belongs to the second space, it belongs to the first one, and $P(n,m)v = 0$. Thus $\alpha H(n) = P(n,m)w$ and it has integer entries. We conclude as in the proof of Proposition 11. □

4.4. **Example 4: 2×2 matrices with determinant equal to ± 1.** Here we assume that the matrices $(M(n); n \geq 2)$ of the linearly recurrent system (X, T, μ) are 2×2 matrices with entries in $\{1, \ldots, L\}$ and determinant ± 1. We assume also that $h_1(1) = h_2(1) = 1$. We set

$$P_n = \begin{bmatrix} x_n & y_n \\ z_n & w_n \end{bmatrix} \text{ and } \Delta_n = \det(P(n)) = \pm 1 \ .$$

We notice that

$$x_n + y_n = h_1(n) \text{ and } z_n + w_n = h_2(n) \ .$$

Since for every $n \geq 2$ all the entries of $M(n)$ are positive, we get easily that $h_1(n) \geq 2^{n-1}$ and $h_2(n) \geq 2^{n-1}$.

By Equation (2.6), as $n \to \infty$, we have

$$\frac{x_n}{h_1(n)} \to \mu(B_1(1)) \ , \ \frac{y_n}{h_1(n)} \to \mu(B_2(1)) \ , \ \frac{z_n}{h_2(n)} \to \mu(B_1(1)) \ , \ \frac{w_n}{h_2(n)} \to \mu(B_2(1)) \ .$$

Lemma 16. *For a vector $v \in \mathbb{R}^2$ we have $\lim_{n\to\infty} \|P(n)v\| = 0$ if and only if v is collinear with $\big(\mu(B_2(1)), -\mu(B_1(1))\big)$. In this case the convergence is exponential.*

Proof. We write $\mu_1 = \mu(B_1(1))$ and $\mu_2 = \mu(B_2(1))$; clearly $\mu_1 + \mu_2 = 1$.

Let $v \in \mathbb{R}^2$. As in the proof of Proposition 11, if $\|P(n)v\| \to 0$ then $\mu_1 v_1 + \mu_2 v_2 = 0$, and v is collinear with $(\mu_2, -\mu_1)$. It remains to show that $\|P(n)v\| \to 0$ exponentially when v is collinear with $(\mu_2, -\mu_1)$. Obviously we can restrict ourselves to the case where these vectors are equal.

We check that $x_{n+1}h_1(n) - x_n h_1(n+1) = -m_{1,2}(n+1)\Delta_n$, and deduce that

$$\Big|\frac{x_{n+1}}{h_1(n+1)} - \frac{x_n}{h_1(n)}\Big| = \Big|\frac{m_{1,2}(n+1)\Delta_n}{h_1(n+1)h_1(n)}\Big| \leq \frac{L}{h_1(n+1)h_1(n)} \ .$$

Since $x_n/h_1(n) \to \mu_1$ as $n \to \infty$, we get

$$\Big|\frac{x_n}{h_1(n)} - \mu_1\Big| \leq \sum_{i=n}^{\infty} \frac{L}{h_1(i+1)h_1(i)} \leq \frac{C}{2^n h_1(n)}$$

for some constant C. From $x_n + y_n = h_1(n)$ and $\mu_1 + \mu_2 = 1$, we obtain

$$|x_n\mu_2 - y_n\mu_1| = |x_n - h_1(n)\mu_1| \leq C 2^{-n} \ .$$

In a similar way, we have $|z_n\mu_2 - w_n\mu_1| \le C2^{-n}$. We conclude $\|P(n)v\| \le C2^{-n}$. □

Proposition 17. *Let (X, T, μ) be a linearly recurrent Cantor system and let the associated sequence of matrices $(M(n); n \ge 1)$ be 2×2 matrices with entries in $\{1, \dots, L\}$ and determinant ± 1. Suppose also that $h_1(1) = h_2(1) = 1$. Let $\alpha \in \mathbb{R}$ and $\lambda = \exp(2i\pi\alpha)$. Then λ is an eigenvalue of (X, T, μ) if and only if α belongs to the set $\{p_1\mu(B_1(1)) + p_2\mu(B_2(1)) : p_1, p_2 \in \mathbb{Z}\}$. Moreover every eigenfunction of (X, T, μ) is continuous.*

Proof. Let $\lambda = \exp(2i\pi\alpha)$ be an eigenvalue of (X, T, μ). By Theorem 10 we have $|||\alpha P(n)H(1)||| \to 0$ as $n \to \infty$. We take $u = \alpha H(1)$, and let v, w and m as given by Lemma 12. We write $v' = P(m)^{-1}v$ and $w' = P(m)^{-1}w$. We have $\alpha H(1) = v' + w'$, $\|P(n)v'\| \to 0$ as $n \to \infty$, and the vector w' is an integer vector because the matrix $P(m)$ has integer entries and $|det(P(m))| = 1$. By Lemma 16, there exists $k \in \mathbb{R}$ with $v'_1 = k\mu(B_2(1))$ and $v'_2 = -k\mu(B_1(1))$. Since $u_1 = u_2 = \alpha$ and $\mu(B_1(1)) + \mu(B_2(1)) = 1$, we have $k = v'_1 - v'_2 = -w'_1 + w'_2 \in \mathbb{Z}$, and α has the announced form.

Conversely, if $\alpha = p_1\mu(B_1(1)) + p_2\mu(B_2(1))$ for some integers p_1 and p_2, the vector $\alpha H(1)$ can be written as the sum of an integer vector w and a vector v collinear with $\big(\mu(B_2(1)), -\mu(B_1(1))\big)$. By Lemma 16, $\|P(n)v\| \to 0$ exponentially as $n \to \infty$, thus $|||\alpha P(n)H(1)||| \to 0$ exponentially as $n \to \infty$, and α is a continuous eigenvalue by Theorem 10. □

4.5. **Example 5: Two commuting matrices.** Let (X, T, μ) be a linearly recurrent system with $H(1) = (1, 1)^t$. We assume that each matrix $M(n)$, $n \ge 2$, is one of the two following ones:

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

We notice that the matrices A, B have the same eigenvectors and commute. We write α_1, α_2 for the eigenvalues of A, with $\alpha_1 > \alpha_2 > 1$, and β_1, β_2 for the eigenvalues of B, with $\beta_1 > 1 > \beta_2 > 0$. We set $\delta = -\log(\beta_2)/\log(\alpha_2/\beta_2)$. For every n we write a_n (respectively b_n) for the number of occurrences of A (respectively B) in the sequence $M(2), \dots, M(n)$. For every n the eigenvalues of $P(n)$ are $\alpha_1^{a_n}\beta_1^{b_n}$ and $\alpha_2^{a_n}\beta_2^{b_n}$, and we have

$$H(n) = \alpha_1^{a_n}\beta_1^{b_n}u_1 + \alpha_2^{a_n}\beta_2^{b_n}u_2$$

where u_1, u_2 are two non-zero vectors.

It is not difficult to show by induction that $\gcd(h_m(1), h_m(2)) = 1$ for every m. Assume first that $\limsup a_n/n > \delta$. There does not exist any $m \ge 2$ and $0 \ne v \in \mathbb{R}^2$ with $\|P(n, m)v\| \to 0$. By Lemma 12, there does not exist any $u \in \mathbb{R}^2 \setminus \mathbb{Z}^2$ such that $|||P(n)u||| \to 0$. By Theorem 10 the system is weakly mixing.

Let us assume now that $\limsup a_n/n < \delta$. If m is an integer and v a vector such that $\|P(n, m)v\| \to 0$ as $n \to +\infty$, then v is collinear with u_2 and the convergence is exponential. It then follows from Lemma 12 and Theorem 10 that every eigenfunction is continuous. Moreover, there exist real numbers α, s with $\alpha \notin \mathbb{Z}$ such that the vector $(\alpha, \alpha) - sv_2$ belongs to $\mathbb{Z}^2$. Then $\exp(2\pi i\alpha)$ is an eigenvalue, and the system is not weakly mixing. We summarize:

- If $\limsup a_n/n > \delta$ then the system is weakly mixing.

- If $\limsup a_n/n < \delta$ then the system is not weakly mixing, and all of its eigenfunctions are continuous.

5. A random linearly recurrent system

The preceding examples lead to the somewhat vague intuition that for "most" of the linearly recurrent systems the eigenfunctions are continuous. We test here this guess by building random linearly recurrent systems in a natural and relatively general way.

Let $\mathcal{M}$ be a finite set of matrices, not assumed to be of the same size, and let $\mathcal{M}^{\mathbb{N}}$ be endowed with the shift S and with the product topology. We write Ω for the subset of $\mathcal{M}^{\mathbb{N}}$ consisting in sequences (M_n) such that the product $M(n+1)M(n)$ is defined for every n and we assume that Ω is not empty. Ω is a closed S-invariant subset of $\mathcal{M}^{\mathbb{N}}$, and (Ω, S) is a subshift of finite type. We write $M = (M(n))$ for an element of Ω.

For every sequence M in Ω let (X_M, T) be a linearly recurrent system associated to this sequence. By choosing a probability measure ν on Ω we get a random linearly recurrent system. We henceforth assume that ν is invariant and ergodic under S. We show now that under some natural hypothesis the eigenfunctions of X_M are continuous for ν-almost every M.

Let k be the maximum size of the elements of $\mathcal{M}$. By completing each of these elements by zero entries we can consider them as $k \times k$ matrices. We choose a norm on $\mathbb{R}^k$ and a norm on $\mathcal{L}(\mathbb{R}^k)$. Let A be the map from Ω to $\mathcal{L}(\mathcal{R}^k)$ which maps each sequence M to $M(2)$. The function $M \mapsto \log^+(\|A(M)\|)$ is bounded and thus belongs to $L^1(\nu)$.

By Oseledets Theorem (see [Wa]) there exists a measurable subset B of Ω, invariant under S and of full measure, such that for every $M \in B$ the limit

$$\lim_{n\to+\infty} \frac{1}{n} \log\|A(S^{n-1}M) \circ \cdots \circ A(SM) \circ A(M)(v)\| \tag{5.1}$$

exists in $\mathbb{R} \cup \{\pm\infty\}$.

We say that (Ω, S, ν) is *hyperbolic* if the set B can be chosen so that for every $M \in B$ and every $v \in \mathbb{R}^k$ this limit is non-zero. Henceforth we assume that this condition holds and show that for any $M \in B$ the eigenfunctions of X_M are continuous.

Let m be an integer and $v \in \mathbb{R}^{C(m)}$ a vector such that $\|P(n,m)v\| \to 0$ as $n \to +\infty$. As B is invariant under S, the sequence $(M(n) : n > m)$ belongs to B and the limit (5.1) exists. By hypothesis this limit can not be positive, and it is non zero by hyperbolicity, then it is negative. It follows that $\|P(n,m)v\| \to 0$ exponentially.

By Theorem 10 and Lemma 12, every eigenfunction of X_M is continuous for $M \in B$ and thus ν-almost everywhere.

6. Questions

Is it true that all eigenvalues of linearly recurrent systems are continuous ? If it is not true, is the result of Section 5 true without the assumption of "hyperbolicity" ? If the answer is again negative could we find some necessary and sufficient condition to have only continuous eigenvalues ? In fact, it seems that the existence of non continuous eigenfunctions is not only a property of the sequence of matrices, but it depends on other elements of the dynamics.

Acknowledgments. The authors acknowledge financial support from FONDAP program in Mathematical Modeling, FONDECYT 1010447, ECOS-Conicyt C99E10 and Programa Iniciativa Cientifica Milenio P01-005. This paper was mainly written when the second author was "détaché au CNRS" at the Centro de Modelamiento Matemático UMR 2071.

References

[Co] M. I. Cortez, *Funciones propias continuas de sistemas linealmente recurrentes*, Memoria Ingeniería Civil Matemática, Universidad de Chile (2001).

[Do] Doob, J. L., Stochastic Processes, Wiley publications in statistics (1953).

[DL] T. Downarowicz, Y. Lacroix, *A non-regular Toeplitz flow with preset pure point spectrum*, Studia Math. 120 (1996), 235-246.

[Du1] F. Durand, *Linearly recurrent subshifts have a finite number of non-periodic subshift factors*, Ergodic Theory and Dynamical Systems 20 (2000), 1061-1078.

[Du2] F. Durand, *Corrigendum and addendum to: Linearly recurrent subshifts have a finite number of non-periodic subshift factors*, to appear in Ergodic Theory and Dynamical Systems.

[DHS] F. Durand, B. Host, C. Skau, *Substitutive dynamical systems, Bratteli diagrams and dimension groups*, Ergodic Theory and Dynamical Systems 19 (1999), 953–993.

[FHZ] S. Ferenczi, C. Holton, L. Q. Zamboni, *Structure of three interval exchange transformations III: ergodic and spectral properties*, preprint (2001).

[HPS] R. H. Herman, I. Putnam, C. F. Skau, *Ordered Bratteli diagrams, dimension groups and topological dynamics*, Internat. J. of Math. 3 (1992), 827–864.

[Ho] B. Host, *Valeurs propres des systèmes dynamiques définis par des substitutions de longueur variable*, Ergodic Theory and Dynamical Systems 6 (1986), 529–540.

[Iw] A. Iwanik, *Toeplitz flows with pure point spectrum*, Studia Math. 118 (1996), 27–35.

[Le] D. Lenz, Uniform ergodic theorems on subshifts over a finite alphabet, Ergodic Theory and Dynamical Systems 22 (2002), 245–255.

[NR] A. Nogueira, D. Rudolph, Topologically weak-mixing of interval exchange maps, Ergodic Theory and Dynamical Systems 17 (1997), 1183-1209.

[Qu] M. Queffélec, *Substitution Dynamical Systems-Spectral Analysis*, Lecture Notes in Mathematics, 1294, Springer-Verlag, Berlin, 1987.

[Wa] P. Walters, An Introduction to Ergodic Theory, Graduate Texts in Mathematics, 79. Springer-Verlag, New York-Berlin, 1982.

Departamento de Ingeniería Matemática, Universidad de Chile, Casilla 170/3 correo 3, Santiago, Chile.
E-mail address: `mcortez@dim.uchile.cl`

Laboratoire Amiénois de Mathématiques Fondamentales et Appliquées, CNRS-UMR 6140, Université de Picardie Jules Verne, 33 rue Saint Leu, 80000 Amiens, France.
E-mail address: `fdurand@u-picardie.fr`

Équipe d'Analyse et Mathématiques Appliquées, CNRS-UMR 8050, Université de Marne-la-Vallée, 93166 Noisy-le-Grand, France.
E-mail address: `host@math.univ-mlv.fr`

Departamento de Ingeniería Matemática, Universidad de Chile and Centro de Modelamiento Matemático, UMR 2071 UCHILE-CNRS, Casilla 170/3 correo 3, Santiago, Chile.
E-mail address: `amaass@dim.uchile.cl`