SELF-SIMILAR TILING SYSTEMS, TOPOLOGICAL FACTORS AND STRETCHING FACTORS

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Abstract. In this paper we prove that if two self-similar tiling systems, with respective stretching factors λ_1 and λ_2, have a common factor which is a non periodic tiling system, then λ_1 and λ_2 are multiplicatively dependent.

1. Introduction

Given a non periodic self-similar tiling $\mathcal{T}$ generated by some similarity S_1 with stretching factor λ_1, it is rather natural to ask if we could generate $\mathcal{T}$ using another similarity with a different stretching factor λ_2. This is of course possible taking a power of the similarity S_1, where λ_2 is in this case a power of λ_1. Holton, Radin and Sadun show in [HRS] that the stretching factor of any other similarity which generates $\mathcal{T}$ is equal to a rational power of λ_1. More precisely, they prove that the stretching factors of conjugate tiling systems which are the orbit closure under Euclidean motions of some self similar tilings are multiplicatively dependent. In this paper we look at tiling systems which are the orbit closure under translations of some self similar tilings, in order to give a necessary condition to have non periodic common factors. The result we present in this paper is the following:

Theorem 1. *Let $S_1(\mathcal{T}_1) = \mathcal{T}_1$ and $S_2(\mathcal{T}_2) = \mathcal{T}_2$ be two self-similar tilings satisfying the Finite Pattern Condition, where S_1 and S_2 are primitive substitutions. Let λ_1 and λ_2 be the Perron eigenvalues of the substitution matrices associated to S_1 and S_2 respectively. If there exist a non periodic tiling $\mathcal{T}$ and factors maps $\pi_i : \Omega_{\mathcal{T}_i} \to \Omega_{\mathcal{T}}$, for $i \in \{1, 2\}$, then λ_1 and λ_2 are multiplicatively dependent.*

The problem we are interested in has been considered a long time ago by A. Cobham in [Co1] and [Co2] for fixed points of substitutions of constant length. He showed that if $p, q > 1$ are two multiplicatively independent integers then a sequence x on a finite alphabet is both p-substitutive and q-substitutive if and only if x is ultimately periodic, where p-substitutive means that x is the image by a letter to letter morphism of a fixed point of a substitution of constant length p. This theorem was the starting point of a lot of work in many different directions such as : numeration systems for $\mathbb{N}$, substitutive sequences and subshifts, automata theory and logic (for more details see [Be, BH1, BH2, BHMV, Du1, Du2, Du3, Ei, Fab, Fag, Ha1, Ha2, MV]). Later, in [Se] A. Semenov proved a "multidimensional" Cobham type theorem, that is to say a Cobham theorem for recognizable subsets of $\mathbb{N}^d$. This result can be stated in terms of self similar tilings, and in the case these tilings are repetitive, our result is a generalization of Semenov Theorem.

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This paper is organized as follows: in Section 2 we give some basic definitions relevant for the study of tiling systems and substitution tiling systems. In Section 3 we study the frequencies of the patches in self-similar tilings and in their factors. First we prove that the frequencies of the patches in a self-similar tiling $\mathcal{T}$ are included in a finite union of geometric progressions of rate λ, where λ is the stretching factor of $\mathcal{T}$ (In [HZ] the authors remarked this fact for minimal substitution subshifts). Next, we prove that the frequencies of the patches in a tiling $\mathcal{T}$, which is a factor of two self-similar tiling systems with stretching factors λ_1 and λ_2 respectively, are included in the intersection of two finite unions of geometric progressions, one of rate λ_1 and the other of rate λ_2. The proof of this result would be easier if the factor maps were given by a kind of "sliding block code" (as it can be the case for subshifts), because in this case the preimage of a patch would be a finite collection of patches. Nevertheless, this is no longer the case for the tiling systems we consider here (examples of factor maps, and even conjugacies, that are not given by a "sliding block codes" are given in [Pe] and [RS]), but we overcome this problem selecting carefully some patches in the preimages we considered. Finally, in Section 4 we deduce the main Theorem.

2. Definitions and background

In this section we give the classical definitions concerning tilings. For more details we refer to [So1]. A *tiling* of $\mathbb{R}^d$ is a countable collection $\mathcal{T} = \{t_i : i \geq 0\}$ of closed subsets of $\mathbb{R}^d$ (which are known as *tiles*) whose union is the whole space and their interiors are pairwise disjoint. We assume that the tiles are homeomorphic to closed balls and that they belong, up to translations, to a finite collection of closed subsets of $\mathbb{R}^d$ whose elements are called *prototiles*. We say that two tiles are *equivalent* if they are equal up to translations. It is often useful to consider every prototile as a closed set endowed with a label. In this case, two tiles are equivalent if, in addition, their labels coincide.

The translation of the tiling $\mathcal{T}$ by a vector $v \in \mathbb{R}^d$ is the tiling $\mathcal{T} + v$ obtained after translating every tile of $\mathcal{T}$ by $-v$. The tiling $\mathcal{T}$ is said to be *aperiodic* (or *non periodic*) if $\mathcal{T} + v = \mathcal{T}$ implies $v = 0$.

The *support* of a tile t_i, denoted by $\mathrm{supp}(t_i)$, is the closed set that defines t_i. For every subset A of $\mathbb{R}^d$ we define, as usual, $\mathcal{T} \cap A$ to be the set $\{t_i \cap A : i \geq 0\}$. A *patch* P is a finite collection of tiles. The support of a patch P, denoted by $\mathrm{supp}(P)$, is the union of the supports of the tiles in P. The *diameter* of a patch P is the diameter of its support, we call it $\mathrm{diam}(P)$. We define $P + v$ as we defined $\mathcal{T} + v$. The tiling $\mathcal{T}$ satisfies the *finite pattern condition* FPC (or equivalently, we say that it is *locally finite*) if for any $r > 0$, there are up to translation, only finitely many patches with diameter smaller than r. This condition is automatically satisfied in the case of a tiling whose tiles are polyhedra that meet face-to-face. A tiling $\mathcal{T}$ is *repetitive* if for any patch P in $\mathcal{T}$ there exists $r > 0$, such that for every open ball $B_r(v)$ the collection $\mathcal{T} \cap B_r(v)$ contains a patch P' equivalent to P (when it is clear from the context we will say that P "appears" in $B_r(v)$). The non periodic repetitive tilings that satisfy FPC are called *perfect* tilings.

2.1. **Tiling systems.**

Let $\mathcal{A}$ be a finite collection of prototiles. We denote by $T(\mathcal{A})$ (*full tiling space*) the space of all the tilings of $\mathbb{R}^d$ whose tiles are equivalent to some element in $\mathcal{A}$. We always suppose that $T(\mathcal{A})$ is non empty. The group $\mathbb{R}^d$ acts on

$T(\mathcal{A})$ by translations:

$$(v, \mathcal{T}) \to \mathcal{T} + v \text{ for } v \in \mathbb{R}^d \text{ and } \mathcal{T} \in T(\mathcal{A}).$$

Furthermore, this action is continuous with the topology induced by the following distance: take $\mathcal{T}$, $\mathcal{T}'$ in $T(\mathcal{A})$, and define A the set of $\varepsilon \in (0,1)$ such that there exist v and v' in $B_\varepsilon(0)$ with

$$(\mathcal{T} + v) \cap B_{1/\varepsilon}(0) = (\mathcal{T}' + v') \cap B_{1/\varepsilon}(0),$$

we set

$$d(\mathcal{T}, \mathcal{T}') = \begin{cases} \inf A & \text{if } A \neq \emptyset \\ 1 & \text{if } A = \emptyset. \end{cases}$$

Roughly speaking, two tilings are close if they have the same pattern in a large neighborhood of the origin, up to a small translation. A *tiling system* is a pair $(\Omega, \mathbb{R}^d)$ such that Ω is a translation invariant closed subset of some full tiling space. The orbit closure of the tiling $\mathcal{T}$ in $T(\mathcal{A})$ is the set $\Omega_{\mathcal{T}} = \overline{\{\mathcal{T} + v : v \in \mathbb{R}^d\}}$. When $\mathcal{T}$ satisfies the FPC, $\Omega_{\mathcal{T}}$ is compact (see [Ru]). If $\mathcal{T}$ is repetitive then all the orbits are dense in $\Omega_{\mathcal{T}}$. In this case the tiling system $(\Omega_{\mathcal{T}}, \mathbb{R}^d)$ is said to be *minimal.*

A *factor map* between two tiling systems $(\Omega_1, \mathbb{R}^d)$ and $(\Omega_2, \mathbb{R}^d)$ is a continuous map $\pi : \Omega_1 \to \Omega_2$ such that $\pi(\mathcal{T} + v) = \pi(T) + v$, for every $\mathcal{T} \in \Omega_1$ and $v \in \mathbb{R}^d$.
In symbolic dynamics it is well-known that topological factor maps between subshifts are always given by sliding-block-codes. There are examples which show that this result can not be extended to tiling systems ([Pe], [RS]). The following Lemma shows that factor maps between tiling systems are not far to be sliding-block-codes. A similar result can be found in [HRS].

Lemma 2. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two tilings. Suppose $\mathcal{T}_1$ verifies the FPC and $\pi : \Omega_{\mathcal{T}_1} \to \Omega_{\mathcal{T}_2}$ is a factor map. Then, there exists a constant $s_0 > 0$ such that to every $\varepsilon > 0$ it is possible to associate $R_\varepsilon > 0$ satisfying the following: Let $R \geq R_\varepsilon$. If $\mathcal{T}$ and $\mathcal{T}'$ in $\Omega_{\mathcal{T}_1}$ verify*

$$\mathcal{T} \cap B_{R+s_0}(0) = \mathcal{T}' \cap B_{R+s_0}(0),$$

then

$$(\pi(\mathcal{T}) + v) \cap B_R(0) = \pi(\mathcal{T}') \cap B_R(0)$$

for some $v \in B_\varepsilon(0)$.

Proof. The tiling $\mathcal{T}_2$ also satisfies the FPC because $\Omega_{\mathcal{T}_2}$ is compact. Since the tilings in $\Omega_{\mathcal{T}_2}$ have a finite number of tiles, up to translations, there exists $\delta_0' > 0$ such that if $y_1 \neq y_2 \in \mathbb{R}^d$ satisfy $(\mathcal{T} + y_1) \cap B_R(0) = (\mathcal{T} + y_2) \cap B_R(0)$ for some $\mathcal{T} \in \Omega_{\mathcal{T}_2}$ and some $R > \max\{\text{diam}(p) : p \text{ prototile in } \mathcal{T}\}$, then $\|y_1 - y_2\| \geq \delta_0'$ (for the details see [So1]).

Let $0 < \delta_0 < \frac{\delta_0'}{2}$. Since π is uniformly continuous, there exists $s_0 > 1$ such that if $\mathcal{T}$ and $\mathcal{T}'$ in $\Omega_{\mathcal{T}_1}$ verify $\mathcal{T} \cap B_{s_0}(0) = \mathcal{T}' \cap B_{s_0}(0)$ then

$$(\pi(\mathcal{T}) + v) \cap B_{\frac{1}{\delta_0}}(0) = \pi(\mathcal{T}') \cap B_{\frac{1}{\delta_0}}(0),$$

for some $v \in B_{\delta_0}(0)$.

Let $0 < \varepsilon < \delta_0$. By uniform continuity of π there exists $0 < \delta < \frac{1}{s_0}$ such that if $\mathcal{T}$ and $\mathcal{T}'$ in $\Omega_{\mathcal{T}_1}$ verify $\mathcal{T} \cap B_{\frac{1}{\delta}}(0) = \mathcal{T}' \cap B_{\frac{1}{\delta}}(0)$ then

$$(\pi(\mathcal{T}) + v) \cap B_{\frac{1}{\varepsilon}}(0) = \pi(\mathcal{T}') \cap B_{\frac{1}{\varepsilon}}(0), \tag{2.1}$$

for some $v \in B_\varepsilon(0)$.
Now fix $R \geq R_\varepsilon = \frac{1}{\delta} - s_0$ and $\mathcal{T}$ and $\mathcal{T}'$ two tilings in $\Omega_{\mathcal{T}_1}$ verifying

$$\mathcal{T} \cap B_{R+s_0}(0) = \mathcal{T}' \cap B_{R+s_0}(0). \tag{2.2}$$

Then, on one hand, the tilings $\mathcal{T}$ and, $\mathcal{T}'$ satisfy (2.1), and on the other hand, we obtain that $(\mathcal{T} + a) \cap B_{s_0}(0) = (\mathcal{T}' + a) \cap B_{s_0}(0)$ for every a in $B_R(0)$. From the choice of s_0, this implies that

$$(\pi(\mathcal{T}) + a + t_a) \cap B_{\frac{1}{\delta_0}}(0) = (\pi(\mathcal{T}') + a) \cap B_{\frac{1}{\delta_0}}(0), \tag{2.3}$$

for some $t_a \in B_{\delta_0}(0)$.
Since $\delta_0 > \varepsilon$, from (2.1) we get

$$(\pi(\mathcal{T}) + v) \cap B_{\frac{1}{\delta_0}}(0) = \pi(\mathcal{T}') \cap B_{\frac{1}{\delta_0}}(0). \tag{2.4}$$

We will show that $t_a = v$ for every a in $B_R(0)$. This property together with (2.3) and (2.4) imply that

$$(\pi(\mathcal{T}) + v) \cap B_R(0) = \pi(\mathcal{T}') \cap B_R(0).$$

For $a = 0$, from (2.3) and (2.4) we have that $t_0 = v$ or $\|v - t_0\| \geq \delta_0'$. Since $\|t_0 - v\| \leq \delta_0 + \varepsilon < 2\delta_0 < \delta_0'$, we conclude $t_0 = v$.
For $a \in B_R(0)$, consider $s > 0$ such that for every $a' \in B_s(a)$ the patch

$$P = ((\pi(\mathcal{T}') + a) \cap B_{\frac{1}{\delta_0}}(0)) \cap ((\pi(\mathcal{T}') + a + (a' - a)) \cap B_{\frac{1}{\delta_0}}(0),$$

contains a tile.
From (2.3) we get $\pi(\mathcal{T}) + a + t_a + (a - a') \cap \mathrm{supp}(P) = P$. Replacing a by a' in (2.3), we obtain $\pi(\mathcal{T}) + a + t_a' + (a' - a) \cap \mathrm{supp}(P) = P$. This implies the norm of $t_a - t_a'$ is equal to 0 or greater than δ_0'. Since $\|t_a - t_a'\| \leq 2\delta_0 < \delta_0'$, we get $t_a = t_a'$. Thus we conclude that the function that associates t_a to a is constant, which implies that $t_a = t_0 = v$ for every a in $B_R(0)$. □

2.2. **Linearly recurrent tilings.** A tiling $\mathcal{T}$ is *linearly recurrent* (or strongly repetitive, or linearly repetitive) if there exists a constant $L > 0$ such that for every patch P in $\mathcal{T}$, any ball of radius $L\mathrm{diam}(P)$ contains a translate of P. Every tiling in the orbit closure of a linearly recurrent tiling is linearly recurrent with the same constant. When $\mathcal{T}$ is linearly recurrent, we call $(\Omega_{\mathcal{T}}, \mathbb{R}^d)$ a *linearly recurrent* tiling system.

Lemma 3. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two tilings verifying the FPC. If $\pi : \Omega_{\mathcal{T}_1} \to \Omega_{\mathcal{T}_2}$ is a factor map and $\mathcal{T}_1$ is linearly repetitive, then $(\Omega_{\mathcal{T}_2}, \mathbb{R}^d)$ is linearly recurrent.*

Proof. Let $\mathcal{T} \in \Omega_{\mathcal{T}_1}$. Consider $\varepsilon > 0$ and $R > 0$ the positive number of Lemma 2 associated to ε. Since $\mathcal{T}$ is linearly repetitive with some constant L, for any $y \in \mathbb{R}^d$ there exists $v \in B_{L(R+s_0)}(y)$ such that $B_{R+s_0}(v) \subseteq B_{L(R+s_0)}(y)$ and $(\mathcal{T} + v) \cap B_{R+s_0}(0) = \mathcal{T} \cap B_{R+s_0}(0)$. From Lemma 2, there exists $t \in B_\varepsilon(0)$ such that $(\pi(\mathcal{T}) + v + t) \cap B_R(0) = \pi(\mathcal{T}) \cap B_R(0)$. This implies that any ball of radius $L(R + s_0) + 2\varepsilon$ in $\pi(\mathcal{T})$ contains a copy of $\pi(\mathcal{T}) \cap B_R(0)$. Since $Ls_0 + 2\varepsilon$ is smaller than some constant, it follows that $\pi(\mathcal{T})$ is linearly recurrent. □

2.3. **Substitution tiling systems.** Let M be a linear map on $\mathbb{R}^d$. It is called *expansive* if there exists $\lambda > 1$ such that

$$\|Mv\| \geq \lambda\|v\|, \text{ for all } v \in \mathbb{R}^d.$$

The map M is a *similarity* if $\|Mv\| = \lambda\|v\|$ for all $v \in \mathbb{R}^d$.
Let α be an eigenvalue of the expansive (resp. similar) linear map M, and let $v \neq 0$ be an eigenvector associated to α. We have $\|Mv\| = |\alpha|\|v\|$, which implies that $|\alpha| \geq \lambda$ (resp. $|\alpha| = \lambda$) and then, $|\det(M)| \geq \lambda^d$ (resp. $|\det(M)| = \lambda^d$). Thus, if Θ is a Borel set in $\mathbb{R}^d$, we obtain

$$\mathrm{vol}(M\Theta) = |\det(M)|\mathrm{vol}(\Theta) \geq \lambda^d \mathrm{vol}(\Theta) \text{ if } M \text{ is expansive.}$$

$$\mathrm{vol}(M\Theta) = |\det(M)|\mathrm{vol}(\Theta) = \lambda^d \mathrm{vol}(\Theta) \text{ if } M \text{ is a similarity.}$$

Let $\mathcal{A}$ be a finite collection of prototiles and let M be a expansive linear map on $\mathbb{R}^d$. A *substitution* is a function S on the set of prototiles $\mathcal{A}$ that associates to each p in P a patch $S(p)$ such that

- the support of $S(p)$ is $M\mathrm{supp}(p)$.
- for every $q \in \mathcal{A}$ there exist $n_{p,q} \geq 0$ and $v_{p,q,k} \in \mathbb{R}^d$ for each $1 \leq k \leq n_{p,q}$, such that

$$S(p) = \{q + v_{p,q,k} : 1 \leq k \leq n_{p,q},\, q \in \mathcal{A}\}.$$

The *substitution matrix* of S is the matrix $A \in \mathcal{M}_{\mathcal{A}\times\mathcal{A}}(\mathbb{Z}^+)$ which contains, in the coordinate (p, q), the number of different tiles in $S(p)$ which are equivalent to q. That is, $A_{p,q} = n_{p,q}$ for each $p, q \in \mathcal{A}$.
The substitution S can be defined on $T(\mathcal{A})$ in the following way: if t is a tile in $\mathcal{T} \in T(\mathcal{A})$, such that t is equivalent to the prototile $p \in \mathcal{A}$, we define

$$S(t) = S(p) + Mv,$$

where $v \in \mathbb{R}^d$ is such that $\mathrm{supp}(t) = \mathrm{supp}(p) + v$. Then, we define

$$S(\mathcal{T}) = \bigcup_{t\in\mathcal{T}} S(t) \in T(\mathcal{A}).$$

The substitution is *primitive* if A is primitive, that is, there exists $k > 0$ such that $A^k > 0$. In this case, the Perron eigenvalue of A is $|\det(M)|$ ([So1]).
In this paper, we always suppose that S is primitive.
The *substitution tiling system* associated to S is the tiling system $(X_S, \mathbb{R}^d)$, where X_S is the space of all the tilings $\mathcal{T}$ in $T(\mathcal{A})$ such that for every patch P of $\mathcal{T}$ there exist a prototile $p \in \mathcal{A}$ and $k > 0$ satisfying $P \subseteq S^k(p)$. The action of $\mathbb{R}^d$ on X_S is the translation. Because S is primitive, there always exist a tiling $\mathcal{T}_0 \in T(\mathcal{A})$ and $k_0 > 0$ such that $S^{k_0}(\mathcal{T}_0) = \mathcal{T}_0$. It is classical (in the primitive case) that $\Omega_{\mathcal{T}_0} = X_S = X_{S^k}$ for every $k > 0$. So, without loss of generality we can suppose that $S(\mathcal{T}_0) = \mathcal{T}_0$. In addition, we will always suppose that the fixed point of S satisfies the FPC. In this case X_S is a compact metric space and $(X_S, \mathbb{R}^d)$ is minimal.
A tiling $\mathcal{T}$ in $T(\mathcal{A})$ which satisfies the FPC is *self-affine* if it is the fixed point of a substitution. The tiling $\mathcal{T}$ is said to be *self-similar* if it is the fixed point of a substitution S which is defined by a similarity M with constant λ (For more details see [So1]). We say λ is the *stretching factor* of S or $\mathcal{T}$.
Let $\mathcal{T}_0$ be a self-similar tiling which is the fixed point of a primitive substitution S satisfying the FPC. The following two results are included in [So2].

Lemma 4. *$\mathcal{T}_0$ is linearly recurrent.*

Lemma 5. *There exists $N > 0$ such that if P is a patch in $\mathcal{T}_0$ whose support contains a ball of radius R, then whenever $P + v$ is a patch of $\mathcal{T}_0$ with $v > 0$, $\|v\| > \frac{R}{N}$.*

These two lemmata mean that the minimal distance between two equivalent patches in a self-similar tiling is neither too large nor too small compared to their sizes.

3. Frequencies

Consider a tiling $\mathcal{T}$ of $\mathbb{R}^d$. For a set $F \subseteq \mathbb{R}^d$, we write

$$\mathcal{T}[[F]] = \{t \in \mathcal{T} : t \cap F \neq \emptyset\}.$$

A $\mathcal{T}$-*corona* is a patch $\mathcal{T}[[\mathrm{supp}(t)]]$, where t is a tile in $\mathcal{T}$. Remark that for some $\epsilon \in \mathbb{R}^d$ we could have $\mathcal{T}[[F + \epsilon]] = \mathcal{T}[[F]]$. To avoid this situation we define, for $v \in \mathbb{R}^d$, $\mathcal{T}[F, v] = \mathcal{T}[[F]] - v$. When F is a ball $B_R(v)$ we write $\mathcal{T}[B_R(v)]$ instead of $\mathcal{T}[B_R(v), v]$.
In the sequel we suppose that $\mathcal{T}_0$ is a self-similar tiling which is the fixed point of a primitive substitution S, with stretching factor λ, satisfying the FPC.

3.1. **Van Hove sequences.**

In order to define the notion of frequency of a patch we need the concept of Van Hove sequences.

Let P be a patch in $\mathcal{T}_0$ and let $\Theta \subset \mathbb{R}^d$. Denote by $L_P(\Theta)$ the number of patches included in $\mathcal{T}_0 \cap \Theta$ which are equivalent to P ([So1]).
A sequence $(\Theta_n)_{n\geq 0}$ of subsets of $\mathbb{R}^d$ is a *Van Hove* sequence if for any $r > 0$,

$$\lim_{n\to\infty} \frac{\mathrm{vol}((\partial\Theta_n)^{+r})}{\mathrm{vol}(\Theta_n)} = 0,$$

where

$$\Theta^{+r} = \{x \in \mathbb{R}^d : \mathrm{dist}(x, \Theta) \leq r\},$$

and $\partial\Theta$ is the border of Θ.
In [So1], it was shown for any patch P in $\mathcal{T}_0$ there is a number $\mathrm{freq}(P) > 0$ such that for any Van Hove sequence $(\Theta_n)_{n\geq 0}$,

$$\lim_{n\to\infty} \frac{L_P(\Theta_n)}{\mathrm{vol}(\Theta_n)} = \mathrm{freq}(P).$$

Suppose that P and Q are two patches in $\mathcal{T}_0$. In order to simplify the notation, we will write $L_P(Q)$, $\mathrm{vol}(P)$ and $(\partial P)^{+r}$ instead of $L_P(\mathrm{supp}(Q))$, $\mathrm{vol}(\mathrm{supp}(P))$ and $(\partial \mathrm{supp}(P))^{+r}$ respectively.

It is easy to show that $(M^n\Theta)_{n\geq 0}$ is a Van Hove sequence when $M : \mathbb{R}^d \to \mathbb{R}^d$ is an expansive linear map and Θ is a compact subset of $\mathbb{R}^d$ with non empty interior and such that $\mathrm{vol}(\partial\Theta) = 0$. Consequently, to compute $\mathrm{freq}(P)$ we will use the following limit

$$\mathrm{freq}(P) = \lim_{k\to\infty} \frac{L_P(S^k(p))}{\mathrm{vol}(S^k(p))},$$

for any prototile p in $\mathcal{A}$.

3.2. **Patch frequencies of a self-similar tiling.** The next proposition extends a result of C. Holton and L. Zamboni [HZ] obtained for minimal substitution subshifts. But before we will need the following technical lemma:

Lemma 6. *Suppose that $\mathcal{T}$ satisfies the FPC. Then there exists a constant $\eta > 0$ such that for every $y \in \mathbb{R}^d$ the ball $B_\eta(y)$ is contained in the support of a corona in $\mathcal{T}$.*

Proof. Let t be a tile in $\mathcal{T}$. The number

$$\eta_t = \text{dist}(\partial t, \partial \mathcal{T}[[\text{supp}(t)]])$$

is positive for every tile t. The FPC implies there is a finite number of coronas up translations. Hence we get

$$\eta = \min\{\eta_t : t \in \mathcal{T}\} > 0.$$

Notice that the set

$$\{x \in \mathbb{R}^d : \text{dist}(x,t) \leq \eta\}$$

is contained in the support of $\mathcal{T}[[\text{supp}(t)]]$ for every tile t in $\mathcal{T}$. Thus if y is a point in $\mathbb{R}^d$ belonging to the tile $t \in \mathcal{T}$ then the ball $B_\eta(y)$ is contained in the support of $\mathcal{T}[[\text{supp}(t)]]$. □

Proposition 7. *There exists a finite set $F \subset \mathbb{R}$ such that for every patch P in $\mathcal{T}_0$ satisfying $P = \mathcal{T}_0[B_R(y)]$, for some $R > 0$ and $y \in \mathbb{R}^d$,*

$$\text{freq}(P) = \frac{f}{\lambda^{dk}},$$

where $f \in F$ and $k > 0$ is such that

$$\lambda^{k-1}\eta \leq \text{diam}(P) < \lambda^k \eta,$$

with η is the constant of Lemma 6.

Proof. Let $\mathcal{A}$ be the prototile set associated to $\mathcal{T}_0$. We define

$$\bar{l} = \max\{\text{diam}(p) : p \in \mathcal{A}\}.$$

Let P be a patch in $\mathcal{T}_0$ such that $P = \mathcal{T}_0[[B_R(y)]]$, for some $R > 0$ and $y \in \mathbb{R}^d$. This implies that

$$\text{diam}(P) \leq 2(R + \bar{l}). \tag{3.1}$$

Let $k \geq 0$ be such that

$$\lambda^{k-1}\eta \leq \text{diam}(P) < \lambda^k \eta. \tag{3.2}$$

By Lemma 6, there exists a corona B which support contains the ball $B_\eta(M^{-k}y)$. Because the support of $S^k(B)$ contains the ball $B_{\lambda^k\eta}(y)$, by (3.2) we deduce that $S^k(B)$ contains the patch P. From Lemma 5, we have

$$L_P(S^k(B)) \leq \frac{\text{vol}(S^k(B))}{\text{vol}(B_{\frac{R}{N}}(0))} = \frac{\lambda^{kd}}{\frac{R^d}{N^d}} \frac{\text{vol}(B)}{\text{vol}(B_1(0))}. \tag{3.3}$$

From (3.1) and (3.2) we obtain

$$\frac{1}{2(R+\bar{l})} \leq \frac{1}{\text{diam}(P)} \leq \frac{1}{\lambda^{k-1}\eta},$$

which implies there exists C not depending on k such that

$$\frac{\lambda^{kd}}{R^d} \leq \left(\frac{2\lambda}{\eta - \frac{2\bar{l}}{\lambda^{k-1}}} \right)^d \leq C. \tag{3.4}$$

From (3.3) and (3.4) we conclude there exists a constant K, independent on P, k and B, such that

$$L_P(S^k(B)) \leq K.$$

Let P' be any patch in $\mathcal{T}_0$ and let D be the set of all the $\mathcal{T}_0$-coronas, up to translation. We have

$$L_P(S^k(P')) = \sum_{B \in D} L_B(P') N(P', P, B)$$

where $N(P', P, B)$ is some integer in $\{0, \cdots, L_P(S^k(B))\} \subseteq \{0, \cdots, K\}$. Thus, for $p \in \mathcal{A}$ and $n > k$,

$$\begin{aligned}
\frac{L_P(S^n(p))}{\text{vol}(S^n(p))} &= \frac{L_P(S^k(S^{n-k}(p)))}{\text{vol}(S^n(p))} \\
&= \sum_{B \in D} \frac{L_B(S^{n-k}(p)) N(S^{k-n}(p), P, B)}{\text{vol}(S^n(p))} \\
&= \sum_{B \in D} \frac{L_B(S^{n-k}(p))}{\text{vol}(S^{n-k}(p))} \frac{\text{vol}(S^{n-k}(p))}{\text{vol}(S^n(p))} N(S^{k-n}(p), P, B) \\
&= \frac{1}{\lambda^{kd}} \sum_{B \in D} \frac{L_B(S^{n-k}(p))}{\text{vol}(S^{n-k}(p))} N(S^{k-n}(p), P, B)
\end{aligned}$$

Because $N(S^{k-n}(p), P, B)$ is in $\{1, \cdots, K\}$ for every $n > k$, we can take a convergent subsequence to obtain

$$\begin{aligned}
\text{freq}(P) &= \frac{1}{\lambda^{kd}} \lim_{n \to \infty} \sum_{B \in D} \frac{L_B(S^{n-k}(p))}{\text{vol}(S^{n-k}(p))} N(S^{k-n}(p), P, B) \\
&= \frac{1}{\lambda^{kd}} \sum_{B \in D} \text{freq}(B) N(P, B),
\end{aligned}$$

where $N(P, B)$ is some integer in $\{0, \cdots, K\}$ for every $B \in D$. Because D is finite, to conclude it suffices to take

$$F = \left\{ \sum_{B \in D} \text{freq}(B) N_B : N_B \in \{0, \cdots, K\} \right\}.$$

□

Remark 8. From [So1] we know $(\Omega_{\mathcal{T}_0}, \mathbb{R}^d)$ is uniquely ergodic. Hence, the frequency of a patch P does not depend on the tiling. That is, freq(P) is the same for every $\mathcal{T}$ in $\Omega_{\mathcal{T}_0}$.

3.3. **Patch frequency in the factor.** The next result extends Proposition 7 to tiling factors of self-similar tiling systems. The main problem we have to overcome is that the factor map is not necessarily given by a sliding block code. Hence the first part of the next proof consists in selecting carefully the preimages of a given patch P by means of a finite induction procedure. Then, we show that the frequency of the patch P is the sum of the frequencies of the selected patches.

Proposition 9. *Let $\mathcal{T}$ be a non periodic tiling. If there exists a factor map $\pi : \Omega_{\mathcal{T}_0} \to \Omega_{\mathcal{T}}$ then there exists a finite set $F \subseteq \mathbb{R}$ such that for every patch P in $\mathcal{T}$ satisfying $P = \mathcal{T}[B_R(y)]$, for some $R > 0$ and $y \in \mathbb{R}^d$,*

$$\mathrm{freq}(P) = \frac{f}{\lambda^{dk}},$$

where $f \in F$ and $k > 0$ is such that

$$\eta\lambda^{k-3} \leq \mathrm{diam}(P) < \eta\lambda^{k-1},$$

if R is big enough.

Proof. Let $\mathcal{T}_2 \in \Omega_{\mathcal{T}}$ and let $\mathcal{T}_1 \in \Omega_{\mathcal{T}_0}$ be such that $\pi(\mathcal{T}_1) = \mathcal{T}_2$. Let $s_0 > 0$ be the constant of Lemma 2.
The linear recurrence of $\mathcal{T}_1$ implies that the tiling $\mathcal{T}_2$ is also linearly recurrent. Let L be the constant of linear recurrence of $\mathcal{T}_1$ and let M and N be the constants of Lemma 5 associated to $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively. We set

$$K = \max\{(8LN)^d, (8LM)^d\}$$

and

$$\eta_i = \max\{\mathrm{diam}(t) : t \text{ is a tile in } \mathcal{T}_i\}, \text{ for } i \in \{1, 2\}.$$

Let $\varepsilon > 0$. Let $R_\varepsilon > 0$ be the positive number associated to ε as in Lemma 2. Notice that R_ε can be chosen big enough in order that

$$R_\varepsilon \geq \max \begin{cases} s_0 + \eta_1 + \eta_2 + \varepsilon \\ 4N(2K+1)\varepsilon \\ 2M\varepsilon - s_0 \\ 2(\eta_1 + \varepsilon) - (s_0 + \eta_2) \\ \eta\lambda^{\lceil \log_\lambda \frac{2\eta_1}{\eta(\lambda-1)} \rceil} \\ \eta\lambda^{\lceil \log_\lambda \frac{2(s_0+\eta_1+\eta_2+2\varepsilon)}{\eta(\lambda-1)} \rceil + 2} \\ \eta/2 \end{cases} \tag{3.5}$$

Let $R \geq R_\varepsilon$ and let $P = \mathcal{T}_2[B_R(y)]$, $y \in \mathbb{R}^d$.
Suppose that $v_1, \cdots, v_l$ are all the points in $B_{2L(R+s_0+\varepsilon+\eta_1+\eta_2)}(0)$ such that

$$\mathcal{T}_2[B_R(v_i)] = P.$$

If $v_i \neq v_j$ we have $\|v_i - v_j\| > \frac{R}{N}$. This implies that in a ball of radius $\frac{R}{2N}$ there is at most one point v such that $\mathcal{T}_2[B_R(v)] = P$. Using (3.5) It follows that in $B_{2L(R+s_0+\varepsilon+\eta_1+\eta_2)}(0)$ there are at most

$$\frac{\mathrm{vol}(B_{2L(R+s_0+\varepsilon+\eta_1+\eta_2)}(0))}{\mathrm{vol}(B_{\frac{R}{2N}}(0))} \leq (8LN)^d \leq K$$

points v such that $\mathcal{T}_2[B_R(v)] = P$. This implies that for any patch P we have $l \leq K$.
For every $1 \leq i \leq l$ we set

$$P_i = \mathcal{T}_1[B_{R+s_0+\eta_2}(v_i)].$$

Now, for every $1 \leq i \leq l$ we will define, by induction on i, k_i different patches as follows (see figure 1).

For $i = 1$, we take all the patches P' in $\mathcal{T}_1$ satisfying the following two conditions:

$$P' = \mathcal{T}_1[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(v)] \text{ for some } v \in \mathbb{R}^d \tag{3.6}$$

$$P_1 = \mathcal{T}_1[B_{R+s_0+\eta_2}(v)]. \tag{3.7}$$

Because $\mathcal{T}_1$ satisfies the FPC, there exists a finite number k_1 of different patches satisfying the previous condition. We call these patches $P_{1,1}, \cdots, P_{1,k_1}$. Moreover, k_1 is bounded by K. Indeed, if v and v' are two different points in $\mathbb{R}^d$ such that

$$\begin{aligned} P_{1,j} &= \mathcal{T}_1[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(v)] \\ P_{1,i} &= \mathcal{T}_1[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(v')], \end{aligned}$$

for some $1 \leq i, j \leq k_1$, then

$$P_1 = \mathcal{T}_1[B_{R+s_0+\eta_2}(v)] = \mathcal{T}_1[B_{R+s_0+\eta_2}(v')].$$

From Lemma 5, this implies that

$$\|v - v'\| > \frac{R + s_0 + \eta_2}{M}.$$

It follows that in a ball of radius $\frac{R+s_0+\eta_2}{2M}$ there is at most one point w which is the center of some $P_{1,j}$. Since $\mathcal{T}_1$ is linearly recurrent with constant L and for every $1 \leq j \leq k_1$

$$\text{diam}(P_{1,j}) \leq 2(R + s_0 + \eta_1 + \eta_2 + 2\varepsilon) + 2\eta_1,$$

all the patches $P_{1,j}$ appear in the ball $B_{2L(R+s_0+2\eta_1+\eta_2+2\varepsilon)}(0)$ in $\mathcal{T}_1$. Using (3.5) this implies

$$k_1 \leq \frac{\text{vol}(B_{2L(R+s_0+2\eta_1+\eta_2+2\varepsilon)}(0))}{\text{vol}(B_{\frac{R+s_0+\eta_2}{2M}}(0))} \leq (8LM)^d \leq K.$$

For $1 < i \leq l$, we take all the patches P' in $\mathcal{T}_1$ satisfying the following three conditions:

$$P' = \mathcal{T}_1[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(v)] \text{ for some } v \in \mathbb{R}^d, \tag{3.8}$$

$$P_i = \mathcal{T}_1[B_{R+s_0+\eta_2}(v)], \tag{3.9}$$

$$\text{if } \mathcal{T}_1[B_{R+s_0+\eta_2}(v + t)] = P_j \text{ for some } t \in B_{2\varepsilon}(0) \text{ then } j \geq i. \tag{3.10}$$

As for the case $i = 1$, we remark there is a finite number k_i of different patches satisfying the previous conditions, and that k_i is smaller than K. We call these patches $P_{i,1}, \cdots, P_{i,k_i}$.

Remark 10. The linear recurrence of $\mathcal{T}_1$ and (3.5) imply that if $v \in \mathbb{R}^d$ satisfies

$$\mathcal{T}_1[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(v)] = P_{i,j},$$

for some $1 \leq i \leq l$ and $1 \leq j \leq k_i$, then $\mathcal{T}_1[B_{R+s_0+\eta_2}(v + t)] \neq P_i$ for every $t \in B_{2\varepsilon}(0) \setminus \{0\}$.

Remark 11. From Remark 10 and from (3.10), if $v \in \mathbb{R}^d$ satisfies

$$\mathcal{T}_1[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(v)] = P_{i,j},$$

for some $1 \leq i \leq l$ and $1 \leq j \leq k_i$, then $\mathcal{T}_1[B_{R+s_0+\eta_2}(v+t)] \neq P_s$ for every $1 \leq s \leq i$ and $t \in B_{2\varepsilon}(0) \setminus \{0\}$.

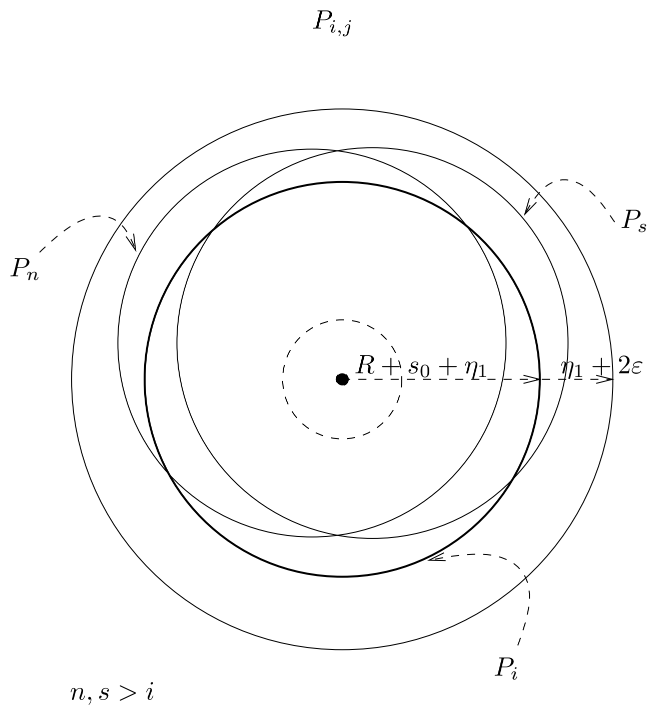

Figure 1

Remark 12. From the construction of the patches $P_{i,j}$, if $v \in \mathbb{R}^d$ satisfies

$$\mathcal{T}_1[B_{R+s_0+\eta_2}(v)] = P_i,$$

for some $1 \le i \le l$ and, $j > i$ whenever $\mathcal{T}_1[B_{R+s_0+\eta_2}(v + t)] = P_j$ for some $t \in B_{2\varepsilon}(0) \setminus \{0\}$, then

$$\mathcal{T}_1[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(v)] = P_{i,k},$$

for some $1 \le k \le k_i$.

In the sequel we will show that $\text{freq}(P) = \sum_{i=1}^{l} \sum_{j=1}^{k_i} \text{freq}(P_{i,j})$.

Lemma 13. *Let $v \in \mathbb{R}^d$ be such that*

$$\mathcal{T}_1[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(v)] = P_{i,j},$$

for some $1 \le i \le l$ and $1 \le j \le k_i$. Then there exists a point $w(v) \in B_\epsilon(v)$ verifying $\mathcal{T}_2[B_R(w(v))] = P$ Moreover, if $v' \neq v$ then $w(v') \neq w(v)$, and,

$$\sum_{i=1}^{l} \sum_{j=1}^{k_i} \text{freq}(P_{i,j}) \le \text{freq}(P). \tag{3.11}$$

Proof. Consider $v \in \mathbb{R}^d$ such that

$$\mathcal{T}_1[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(v)] = P_{i,j},$$

for some $1 \le i \le l$ and $1 \le j \le k_i$. Since $\mathcal{T}_1[B_{R+s_0+\eta_2}(v)] = P_i$, we have

$$(\mathcal{T}_1 + v) \cap B_{R+s_0+\eta_2}(0) = (\mathcal{T}_1 + v_i) \cap B_{R+s_0+\eta_2}(0).$$

Thus from Lemma 2 we obtain that there exists $t \in B_\varepsilon(0)$ verifying

$$(\mathcal{T}_2 + v + t) \cap B_{R+\eta_2}(0) = (\mathcal{T}_2 + v_i) \cap B_{R+\eta_2}(0),$$

which implies that $\mathcal{T}_2[B_R(v+t)] = P$. Now, if $v' \in \mathbb{R}^d$ is another point such that

$$\mathcal{T}_1[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(v')] = P_{i',j'},$$

for some $1 \leq i' \leq l$ and $1 \leq j' \leq k'_i$, in a similar way we get that there exists $t' \in B_\varepsilon(0)$ satisfying $\mathcal{T}_2[B_R(v'+t')] = P$. Suppose that $v + t = v' + t'$. This implies that $\|v - v'\| < 2\varepsilon$, i.e $v - v' \in B_{2\varepsilon}(0)$. But since

$$\begin{aligned} P_{i,j} &= \mathcal{T}_1[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(v)], \\ P_i &= \mathcal{T}_1[B_{R+s_0+\eta_2}(v)], \\ P_{i'} &= \mathcal{T}_1[B_{R+s_0+\eta_2}(v + (v'-v))], \end{aligned}$$

the condition (3.10) implies that $i' \geq i$. In the same way we obtain that $i' \leq i$, which implies $i = i'$. Since $2\varepsilon < \frac{R+s_0}{M}$, we get that $v' - v = 0$. Hence we deduce that it is possible to associate to each v in $\mathbb{R}^d$ which satisfies

$$\mathcal{T}_1[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(v)] = P_{i,j},$$

for some $1 \leq i \leq l$ and $1 \leq j \leq k_i$, a point $w(v) \in \mathbb{R}^d$ verifying

$$\mathcal{T}_2[B_R(w(v))] = P,$$

and such that $w(v) \neq w(v')$ if $v \neq v'$. Thus we deduce that

$$\sum_{i=1}^{l} \sum_{j=1}^{k_i} \text{freq}(P_{i,j}) \leq \text{freq}(P).$$

□

Lemma 14. *Let $v \in \mathbb{R}^d$ be such that $\mathcal{T}_2[B_R(v)] = P$. Then there exists a point $p(v) \in B_{(2l+1)\epsilon}(v)$ verifying*

$$\mathcal{T}_1[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(p(v))] = P_{i,j},$$

for some $1 \leq i \leq l$ and $1 \leq j \leq k_i$. Moreover, if $v' \neq v$ then $p(v') \neq p(v)$, and,

$$\sum_{i=1}^{l} \sum_{j=1}^{k_i} \text{freq}(P_{i,j}) \geq \text{freq}(P). \tag{3.12}$$

Proof. Let $v \in \mathbb{R}^d$ be such that

$$\mathcal{T}_2[B_R(v)] = P,$$

and consider

$$P' = \mathcal{T}_1[B_{R+s_0+\eta_2+\varepsilon}(v)].$$

Since L is the constant of linear recurrence of $\mathcal{T}_1$ and

$$\text{diam}(P') \leq 2(R + s_0 + \eta_2 + \varepsilon) + 2\eta_1,$$

there exists a translated of P' which support is included in the ball

$$B_{2L(R+s_0+\eta_1+\eta_2+\varepsilon)}(0).$$

In other words, there exists $v' \in B_{2L(R+s_0+\eta_1+\eta_2+\varepsilon)}(0)$ such that the support of the patch $\mathcal{T}_1[[B_{R+s_0+\eta_2+\varepsilon}(v')]]$ is contained in the ball $B_{2L(R+s_0+\eta_1+\eta_2+\varepsilon)}(0)$ and satisfies

$$\begin{aligned} P' &= \mathcal{T}_1[B_{R+s_0+\eta_2+\varepsilon}(v')] \\ &= \mathcal{T}_1[B_{R+s_0+\eta_2+\varepsilon}(v)]. \end{aligned}$$

This implies that

$$(\mathcal{T}_1 + v) \cap B_{R+s_0+\eta_2}(0) = (\mathcal{T}_1 + v') \cap B_{R+s_0+\eta_2}(0).$$

So, from Lemma 2 there exists $t \in B_\varepsilon(0)$ verifying

$$(\mathcal{T}_2 + v' + t) \cap B_{R+\eta_2}(0) = (\mathcal{T}_2 + v) \cap B_{R+\eta_2}(0).$$

It follows that $\mathcal{T}_2[B_R(v'+t)] = P$ and, since $v'+t$ is in $B_{L(R+s_0+\eta_1+\eta_2+\varepsilon)}(0)$, we deduce that $v'+t = v_i$, for some $1 \le i \le l$. Because $\mathcal{T}_1[B_{R+s_0+\eta_2}(v'+t)] = P_i$ is included in $\mathcal{T}_1[B_{R+s_0+\eta_2+\varepsilon}(v')] = P'$, we obtain that

$$\mathcal{T}_1[B_{R+s_0+\eta_2}(v+t)] = P_i.$$

Now, we will show that in the ball $B_{(2l+1)\varepsilon}(v)$ there is a point $p(v)$ such that

$$\mathcal{T}_1[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(p(v))] = P_{m,j},$$

for some $1 \le m \le l$ and $1 \le j \le k_m$. For that, consider the following algorithm (see figure 3):

Step 0: We put $v_0 = v + t$ and $i_0 = i$.

Step 1: We have $\mathcal{T}_1[B_{R+s_0+\eta_2}(v_0)] = P_{i_0}$.
If $\mathcal{T}_1[B_{R+s_0+\eta_2}(v_0+s)] = P_j$ for some $s \in B_{2\varepsilon}(0)$ implies $j \ge i_0$, then from the definition of the patches $P_{i,k}$ we obtain that

$$\mathcal{T}_1[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(v_0)] = P_{i_0,m},$$

for some m in $\{1, \cdots, k_{i_0}\}$.

Step 2: If there exists $s \in B_{2\varepsilon}(0)$ such that $\mathcal{T}_1[B_{R+s_0+\eta_2}(v_0+s)] = P_j$ with $j < i_0$, then we put

$$i_0 = \min\{j : \exists s \in B_{2\varepsilon}(0) \text{ such that } \mathcal{T}_1[B_{R+s_0+\eta_2}(v_0+s)] = P_j\}.$$

If $s \in B_{2\varepsilon}(0)$ is such that $\mathcal{T}_1[B_{R+s_0+\eta_2}(v_0+s)] = P_{i_0}$ then we put $v_0 = v_0 + s$. With these new values of v_0 and i_0 we go to the step 1.

This algorithm finishes in at most l steps. The result is a point $p(v) = v_0$ which distance to v is at most $(2l+1)\varepsilon$ and such that

$$\mathcal{T}_1[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(v_0)] = P_{i_0,m},$$

for some m in $\{1, \cdots, k_{i_0}\}$.
If $w \in \mathbb{R}^d$ is another point satisfying $\mathcal{T}_2[B_R(w)] = P$, we have

$$\begin{aligned} \frac{R}{N} &\le \|v - w\| \\ &\le \|p(v) - v\| + \|p(v) - p(w)\| + \|p(w) - w\| \\ &\le 2(2l+1)\varepsilon + \|p(v) - p(w)\|. \end{aligned}$$

Thus we get

$$0 < \frac{R}{2N} < \frac{R}{N} - 2(2l+1)\varepsilon \le \|p(v) - p(w)\|.$$

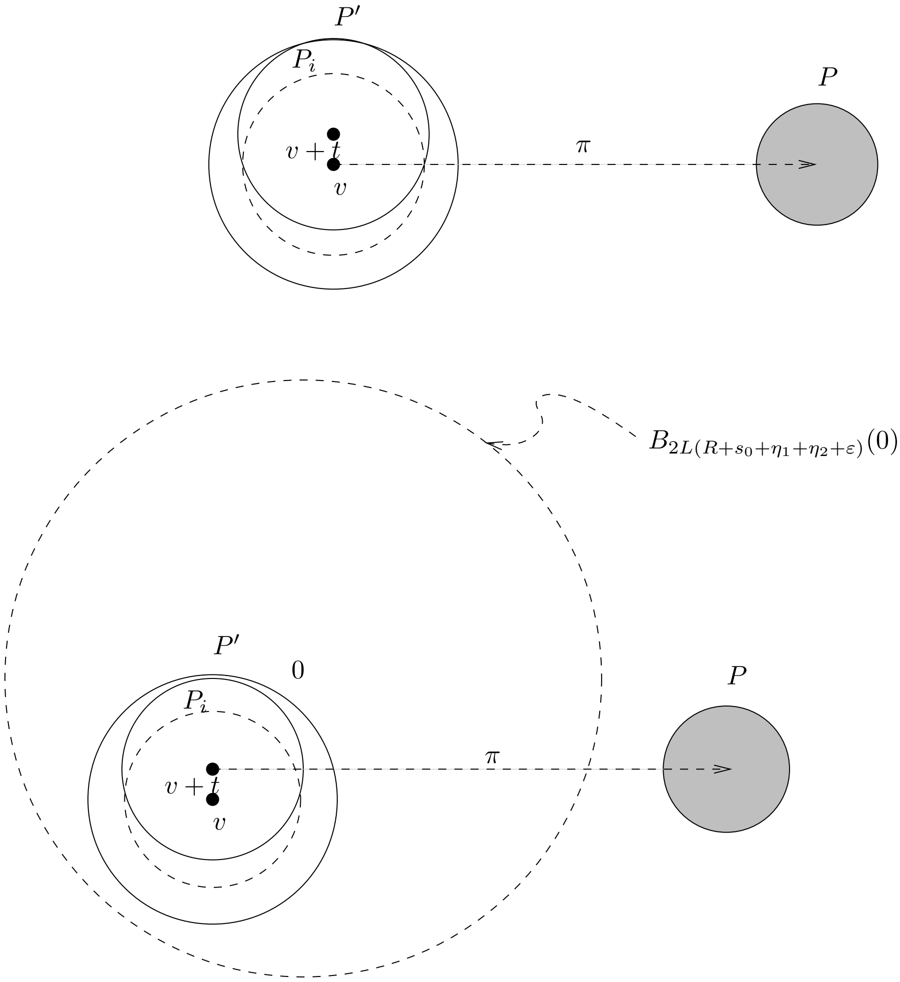

Figure 2

This implies it is possible to associate to each v in $\mathbb{R}^d$ which satisfies $\mathcal{T}_2[B_R(v)] = P$ a point $p(v) \in \mathbb{R}^d$ verifying

$$\mathcal{T}_1[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(p(v))] = P_{i,j},$$

for some $1 \leq i \leq l$ and $1 \leq j \leq k_i$, and such that $p(v) \neq p(w)$ if $v \neq w$. Hence we deduce that

$$\text{freq}(P) \leq \sum_{i=1}^{l} \sum_{j=1}^{k_i} \text{freq}(P_{i,j}).$$

□

From (3.11) and (3.12) we get

$$\text{freq}(P) = \sum_{i=1}^{l} \sum_{j=1}^{k_i} \text{freq}(P_{i,j}). \tag{3.13}$$

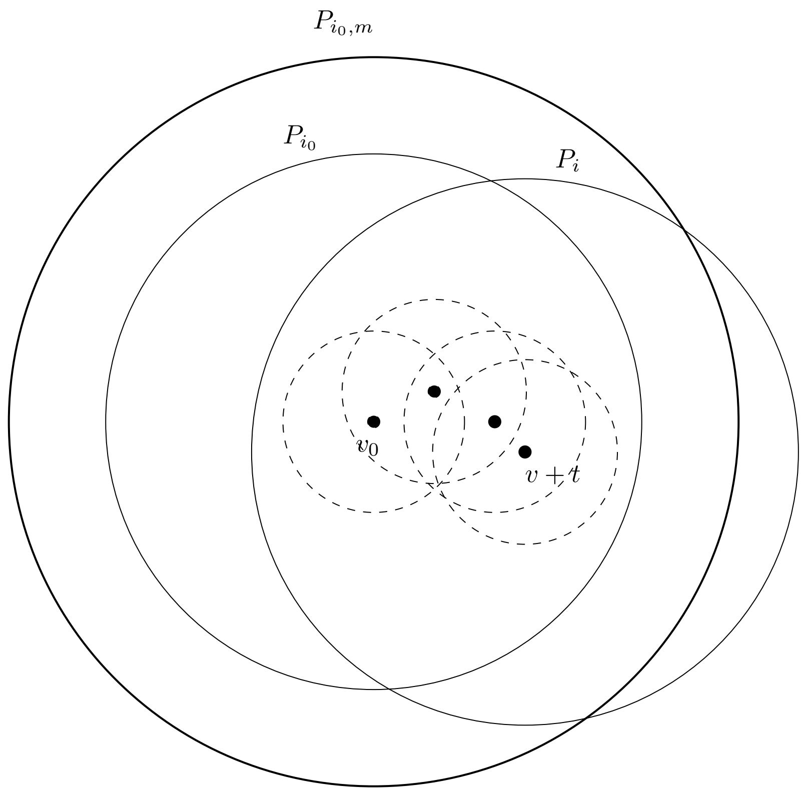

FIGURE 3

As $R > \eta/2$, there exists $k > 0$ such that

$$\eta\lambda^{k-2} \leq 2(R + s_0 + \eta_1 + \eta_2 + 2\varepsilon) < \eta\lambda^{k-1}. \tag{3.14}$$

Since

$$2(R + s_0 + \eta_1 + \eta_2 + 2\varepsilon) \leq \operatorname{diam}(P_{i,j}) \leq 2(R + s_0 + \eta_1 + \eta_2 + 2\varepsilon) + 2\eta_1$$

and $R \geq \eta\lambda^{\lceil \log_\lambda \frac{2\eta_1}{\eta(\lambda-1)} \rceil}$, we have

$$\eta\lambda^{k-2} \leq \operatorname{diam}(P_{i,j}) < \eta\lambda^{k}.$$

Hence, by Proposition 7, we get

$$\operatorname{freq}(P_{i,j}) \in \left\{ \frac{f}{\lambda^{dk}}, \frac{f}{\lambda^{d(k-1)}} : f \in F \right\},$$

where F is the finite set of Proposition 7. Thus we obtain

$$\operatorname{freq}(P) = \frac{f}{\lambda^{dk}},$$

where f is an element in

$$F' = \left\{ \sum_{i=1}^{K} f_i \ : f_i \in F \cup \lambda^d F, \, \forall\, 1 \leq i \leq K \right\},$$

which is a finite subset of $\mathbb{R}^d$.

Notice that

$$2R \le \operatorname{diam}(P) \le 2(R + \eta_2).$$

Thus from (3.14) we have

$$\eta\lambda^{k-2} - 2(s_0 + \eta_1 + \eta_2 + 2\varepsilon) \le \operatorname{diam}(P) < \eta\lambda^{k-1},$$

and by the choice of R in (3.5), we obtain

$$\eta\lambda^{k-3} \le \operatorname{diam}(P) < \eta\lambda^{k-1}.$$

□

4. Proof of Theorem 1

From Proposition 9, there exist two finite sets F_1 and F_2 such that for $R > 0$ and $P = \mathcal{T}[B_R(0)]$ there exist k_1 and k_2 such that

$$\operatorname{freq}(P) = \frac{f_1}{\lambda_1^{k_1}} = \frac{f_2}{\lambda_2^{k_2}},$$

for some $f_1 \in F_1$ and $f_2 \in F_2$.
Because F_1 and F_2 are finite, we can find $a \in F_1$, $b \in F_2$, $n_2 > n_1$, $m_2 > m_1$ and patches P_1 and P_2 in $\mathcal{T}$ such that

$$\operatorname{freq}(P_1) = \frac{a}{\lambda_1^{n_1}} = \frac{b}{\lambda_2^{m_1}},$$

$$\operatorname{freq}(P_2) = \frac{a}{\lambda_1^{n_2}} = \frac{b}{\lambda_2^{m_2}}.$$

This implies that

$$\lambda_1^{n_2-n_1} = \lambda_2^{m_2-m_1},$$

which means that λ_1 and λ_2 are multiplicatively dependent.

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