$\mathbb{Z}^d$-TOEPLITZ ARRAYS

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Abstract. In this paper we give a definition of Toeplitz sequences and odometers for $\mathbb{Z}^d$ actions for $d \geq 1$ which generalizes that in dimension one. For these new concepts we study properties of the induced Toeplitz dynamical systems and odometers classical for $d = 1$. In particular, we characterize the $\mathbb{Z}^d$-regularly recurrent systems as the minimal almost 1-1 extensions of odometers and the $\mathbb{Z}^d$-Toeplitz systems as the family of subshifts which are regularly recurrent.

1. Introduction

Toeplitz sequences have been introduced in dynamical systems by Jacobs and Keane in [11]. Since then, they have been extensively studied in different contexts and they have been used to provide a series of examples with interesting dynamical properties (see for example [16], [10], [7], [6]).

Toeplitz flows are characterized as minimal almost 1-1 symbolic extensions of odometers systems by Markley and Paul. In [8] Downarowicz and Lacroix publish a proof of this theorem. In addition, as it was shown by Gjerde and Johansen in [10], Toeplitz systems also correspond, up to conjugacy, to the family of expansive Bratteli-Vershik systems associated to Bratteli diagrams with the equal path number property.

The aim of this paper is to extend the definition of both odometers and Toeplitz flows to $\mathbb{Z}^d$-actions and to settle down a characterization result, in this general context, in the sense of Markley and Paul. A first approach to this problem was made by Downarowicz in [5], where he introduces the $\mathbb{Z}^2$-Toeplitz arrays.

Since any element of a $\mathbb{Z}^d$-subshift may be seen as a tiling of $\mathbb{R}^d$, $\mathbb{Z}^d$-Toeplitz arrays are a class of interesting examples of perfect tilings.

In Section 2, we give some basic definitions relevant for the study of $\mathbb{Z}^d$-actions and in Section 3 we introduce the generalized notion of an odometer. In Section 4, we introduce $\mathbb{Z}^d$-regularly recurrent systems and we characterize them as the minimal almost 1-1 extensions of odometers. In Section 5, we identify the set of eigenvalues of odometers and the set of continuous eigenvalues of regularly recurrent systems and we characterize those which are measure-theoretically conjugate to their maximal equicontinuous factors. In Section 6, we define $\mathbb{Z}^d$-Toeplitz arrays and we show that they are the family of regularly recurrent $\mathbb{Z}^d$-subshifts. We prove that every Topelitz array has, as in the case $d = 1$, a periodic structure that allows to identify the maximal equicontinuous factor of the associated Toeplitz system. We generalize the notion of a regular Toeplitz sequence to higher dimensions. In Section 7, we introduce the concept

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of a semicocycle. The last section contains an example of a $\mathbb{Z}^2$-Toeplitz system with a determined finite number of ergodic measures and another example of a uniquely ergodic $\mathbb{Z}^2$-Toeplitz system with positive entropy.

2. Basic Definitions and Background

Let $d \geq 1$ be an integer. In this article, by a *topological dynamical system* we mean a pair $(X, \mathbb{Z}^d)$, where $\mathbb{Z}^d$ acts, by homeomorphism, on a compact metric space X. Given $v \in \mathbb{Z}^d$ and $x \in X$ we will identify v with the associated homeomorphism and we denote by $v(x)$ the action of v on x. The dynamical system $(X, \mathbb{Z}^d)$ is *free* if $v(x) = x$ for some $x \in X$ implies $v = 0$. For a subgroup $Z \subseteq \mathbb{Z}^d$ isomorphic to $\mathbb{Z}^d$, the *Z-orbit* of $x \in X$ is $O_Z(x) = \{v(x) : v \in Z\}$ and the *Z-system associated* to x is $(\Omega_Z(x), Z)$, where $\Omega_Z(x)$ is the closure of $O_Z(x)$ and the action of Z on $\Omega_Z(x)$ is the restriction to Z and $\Omega_Z(x)$ of the action of $\mathbb{Z}^d$ on X. When $Z = \mathbb{Z}^d$ we write orbit and associated system instead of $\mathbb{Z}^d$-orbit and $\mathbb{Z}^d$-associated system, respectively. The set of *return times* of $x \in X$ to $A \subseteq X$ is $T_A(x) = \{v \in \mathbb{Z}^d : v(x) \in A\}$. The topological dynamical system $(X, \mathbb{Z}^d)$ is *minimal* if the orbit of any $x \in X$ is dense in X, and it is said to be *equicontinuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in X$ satisfy $d(x, y) < \delta$ then $d(v(x), v(y)) < \varepsilon$ for all $v \in \mathbb{Z}^d$. We say that $(X, \mathbb{Z}^d)$ is an *extension* of $(Y, \mathbb{Z}^d)$, or that $(Y, \mathbb{Z}^d)$ is a factor of $(X, \mathbb{Z}^d)$, if there exists a continuous surjection $\pi : X \to Y$ such that π preserves the action. We call π a *factor map*. When the factor map is bijective, we say that $(X, \mathbb{Z}^d)$ and $(Y, \mathbb{Z}^d)$ are *conjugate*. The factor map π is an *almost 1-1 factor map* and $(X, \mathbb{Z}^d)$ is an *almost 1-1 extension* of $(Y, \mathbb{Z}^d)$ by π if the set of points having one pre-image is residual (contains a dense G_δ set) in Y. In the minimal case it is equivalent to the existence of a point with one pre-image.
The set $\mathcal{M}(X)$ of *invariant probability measures* of X is the set of probability measures μ defined on $\mathcal{B}(X)$, the Borel σ-algebra of X, such that $\mu(v(B)) = \mu(B)$ for all $v \in \mathbb{Z}^d$ and $B \in \mathcal{B}(X)$. We say that $(X, \mu, \mathbb{Z}^d)$, the topological dynamical system $(X, \mathbb{Z}^d)$ equipped with $\mu \in \mathcal{M}(X)$, is a *measure-theoretic* dynamical system. A *measure-theoretic factor map* ϕ from $(X, \mu, \mathbb{Z}^d)$ to $(Y, \nu, \mathbb{Z}^d)$, is a measurable function preserving the action and such that $\mu(\phi^{-1}(B)) = \nu(B)$ for all $B \in \mathcal{B}(Y)$. If ϕ is bijective we say that $(X, \mu, \mathbb{Z}^d)$ and $(Y, \nu, \mathbb{Z}^d)$ are *measure-theoretically conjugate*.
Consider a finite alphabet Σ endowed with the discrete topology and $\Sigma^{\mathbb{Z}^d}$ with the product topology. The elements $x = \{x(z)\}_{z \in \mathbb{Z}^d}$ of $\Sigma^{\mathbb{Z}^d}$ are called *$\mathbb{Z}^d$-arrays*. The *shift* action on $\Sigma^{\mathbb{Z}^d}$ is defined by $v(x) = \{x(z+v)\}_{z \in \mathbb{Z}^d}$, for all $v \in \mathbb{Z}^d$ and $x = \{x(z)\}_{z \in \mathbb{Z}^d} \in \Sigma^{\mathbb{Z}^d}$. In this context, we will write $x + v$ instead of $v(x)$. If Z is a subset of $\mathbb{Z}^d$, $x(Z)$ denotes $\{x(z) : z \in Z\} \in \Sigma^Z$. When $X \subseteq \Sigma^{\mathbb{Z}^d}$ is closed and invariant by the shift action, we say that $(X, \mathbb{Z}^d)$ is a *subshift*.

3. d-dimensional odometers

Let $\{Z_i\}_{i \geq 0} \subseteq \mathbb{Z}^d$ be a decreasing sequence of subgroups isomorphic to $\mathbb{Z}^d$ (or of rank d) and let $\pi_i : \mathbb{Z}^d / Z_{i+1} \to \mathbb{Z}^d / Z_i$ the function induced by the inclusion $Z_{i+1} \subseteq Z_i$, $i \geq 0$. Consider the inverse limit

$$G = \lim_{\leftarrow i} (\mathbb{Z}^d / Z_i, \pi_i)$$

More precisely, G is defined as the subset of the product $\Pi_{i\geq 0}\mathbb{Z}^d/Z_i$ consisting of the elements $\mathbf{g} = (g_i)_{i\geq 0}$ such that $\pi_i(g_{i+1}) = g_i$ for all $i \geq 0$. The set G is a group equipped with the addition defined by

$$\mathbf{g} + \mathbf{h} = (g_i +_i h_i)_{i\geq 0},$$

where $+_i$ is the operation induced on $\mathbb{Z}^d/Z_i$ by the addition in $\mathbb{Z}^d$.
Every $\mathbb{Z}^d/Z_i$ is endowed with the discrete topology and $\Pi_{i\geq 0}\mathbb{Z}^d/Z_i$ with the product topology. Thus G is a compact topological group whose topology is spanned by the cylinder sets

$$[i;a] = \{\mathbf{g} \in G : g_i = a\}, \text{ with } a \in \mathbb{Z}^d/Z_i \text{ and } i \geq 0.$$

If H is a subgroup of G then it acts by homeomorphisms on G by $\mathbf{h}(\mathbf{g}) = \mathbf{h}+\mathbf{g}$, $\mathbf{h} \in H$, $\mathbf{g} \in G$. Since for all $\mathbf{h} \in H$ and for all cylinders $[i;a]$ we have $\mathbf{h}([i;a]) \subseteq [i; a +_i h_i]$, the topological dynamical system (G,H) is equicontinuous. Moreover, if H is dense in G then (G,H) is a minimal equicontinuous system.
Consider the homomorphism $\tau : \mathbb{Z}^d \to \Pi_{i\geq 0}\mathbb{Z}^d/Z_i$ defined for $v \in \mathbb{Z}^d$ by

$$\tau(v) = \{\tau_i(v)\}_{i\geq 0},$$

where $\tau_i : \mathbb{Z}^d \to \mathbb{Z}^d/Z_i$ is the canonical projection. The image of $\mathbb{Z}^d$ by τ is dense in G, which implies that the $\mathbb{Z}^d$-action $v(\mathbf{g}) = \tau(v) + \mathbf{g}$, $v \in \mathbb{Z}^d$, $\mathbf{g} \in G$, is well defined and $(G,\mathbb{Z}^d)$ is a minimal equicontinuous system. We call $(G,\mathbb{Z}^d)$ an *odometer system* or simply an *odometer*. It is straightforward that an odometer $(G,\mathbb{Z}^d)$ is a free dynamical system if and only if $\tau : \mathbb{Z}^d \to G$ is one to one, which is equivalent to $\bigcap_{i\geq 0} Z_i = \{0\}$.
Notice that for all $\mathbf{g}$ in a cylinder set $[i;a]$ of an odometer $G = \lim_{\leftarrow i}(\mathbb{Z}^d/Z_i, \pi_i)$, the set of return times of $\mathbf{g}$ to $[i;a]$ is Z_i, $i \geq 0$. Through this paper we will use these properties and we will identify G with $(G,\mathbb{Z}^d)$.

Lemma 1. *Let $G_j = \lim_{\leftarrow i}(\mathbb{Z}^d/Z_i^{(j)}, \pi_i)$ be two odometers ($j = 1,2$). There is a factor map $\pi : (G_1,\mathbb{Z}^d) \to (G_2,\mathbb{Z}^d)$ if and only if for every $Z_i^{(2)}$ there exists some $Z_k^{(1)}$ such that $Z_k^{(1)} \subseteq Z_i^{(2)}$.*

Proof. If $\pi : G_1 \to G_2$ is a factor map then by continuity, given $i \geq 0$ and $a \in \mathbb{Z}^d/Z_i^{(2)}$, there exist $k \geq 0$ and $b \in \mathbb{Z}^d/Z_k^{(1)}$ such that $[k;b] \subseteq \pi^{-1}[i;a]$. Let $v \in Z_k^{(1)}$, we have that $v(\mathbf{g}) \in [k;b]$ for all $\mathbf{g} \in [k;b]$, which implies that $\pi(v(\mathbf{g})) = v(\pi(\mathbf{g})) \in [i;a]$. Since $\pi(\mathbf{g}) \in [i;a]$ it holds that $T_{[i;a]}(\pi(\mathbf{g})) = Z_i^{(2)}$, which proves that $v \in Z_i^{(2)}$.
Suppose that for every $i \geq 0$ there exists $Z_{n_i}^{(1)} \subseteq Z_i^{(2)}$. Since the sequences $\{Z_i^{(j)}\}_{i\geq 0}$ ($j = 1,2$) are decreasing, we can take $n_i \leq n_{i+1}$ for all $i \geq 0$. The function $\pi : G_1 \to G_2$ defined by $\pi((g_i)_{i\geq 0}) = (j_{n_i}(g_{n_i}))_{i\geq 0}$ where $j_{n_i} : \mathbb{Z}^d/Z_{n_i}^{(1)} \to \mathbb{Z}^d/Z_i^{(2)}$ is the function induced by the inclusion $Z_{n_i}^{(1)} \subseteq Z_i^{(2)}$, is a factor map. □

A *scale* is a sequence $\{A_i\}_{i\geq 0} \subseteq GL(d,\mathbb{Z})$ such that for every $i \geq 0$ there exists $Q_i \in GL(d,\mathbb{Z})$ satisfying $A_{i+1} = A_iQ_i$.
Let $G = \lim_{\leftarrow i}(\mathbb{Z}^d/Z_i, \pi_i)$ be an odometer. Any sequence $\{A_i\}_{i\geq 0}$ of integer matrices such that for all $i \geq 0$ the columns of A_i represent a base of Z_i is a scale. We say that $\{A_i\}_{i\geq 0}$ is a scale associated to G if the odometer $\lim_{\leftarrow i}(\mathbb{Z}^d/A_i\mathbb{Z}^d, \pi_i)$ is conjugate to G.

It is direct that the scale $\{A_i\}_{i\geq 0}$ is associated to the odometer $G = \lim_{\leftarrow i}(\mathbb{Z}^d/A_i\mathbb{Z}^d, \pi_i)$, but an odometer can be associated to several scales.
We can formulate Lemma 1 in terms of scales:

Lemma 2. Let $G_j = \lim_{\leftarrow i}(\mathbb{Z}^d/Z_i^{(j)}, \pi_i)$ be two odometers $(j = 1, 2)$. There is a factor map $\pi : G_1 \to G_2$ if and only if given $\{A_i^{(j)}\}_{i\geq 0}$ a scale associated to G_j $(j = 1, 2)$ for all $A_i^{(2)}$ there exists $A_k^{(1)}$ such that $A_k^{(1)} = A_i^{(2)}Q$ for some $Q \in GL(d, \mathbb{Z})$.

We could think that all d-dimensional odometers correspond to a product (up to conjugation) of d one-dimensional odometers. It can be proved that a product of d one-dimensional odometers coincides with an odometer having an associated scale consisting of diagonal matrices. However, it is not true that all d-dimensional odometers $(d \geq 2)$ admit a scale formed by diagonal matrices. Examples can be constructed in any dimension $d \geq 2$:

Example 3. Examples of d-dimensional odometers which are not conjugate to a product of d one-dimensional odometers.

If $d = 2$, consider the sequence $\{A_i\}_{i\geq 0} \in GL(2, \mathbb{Z})$ given by

$$A_i = \begin{bmatrix} 3^{i+1} & 7\cdot 11^i \\ 7\cdot 3^i & 11^{i+1} \end{bmatrix}.$$

If $d > 2$ consider $\{A_i\}_{i\geq 0} \in GL(d, \mathbb{Z})$ defined by

$$A_i(k,k) = \begin{cases} 3^{i+1} & \text{if } k = 1 \bmod \mathbb{Z}_3 \\ 11^{i+1} & \text{if } k = 2 \bmod \mathbb{Z}_3 \\ 7^{i+1} & \text{if } k = 0 \bmod \mathbb{Z}_3 \end{cases}$$

$$A_i(k,k+1) = \begin{cases} 7\cdot 11^i & \text{if } k = 1 \bmod \mathbb{Z}_3 \\ 3\cdot 7^i & \text{if } k = 2 \bmod \mathbb{Z}_3 \\ 11\cdot 3^i & \text{if } k = 0 \bmod \mathbb{Z}_3 \end{cases}$$

$$A_i(k,j) = 0 \; if \; j \in \{1,..,d\} \setminus \{k, k+1\}, \; k = 1,..,d.$$

In both cases $\{A_i\}_{i\geq 0}$ is a scale and $\bigcap_{i\geq 0} A_i\mathbb{Z}^d = \{0\}$. This means that $G = \lim_{\leftarrow i}(\mathbb{Z}^d/A_i\mathbb{Z}^d, \pi_i)$ contains a copy of $\mathbb{Z}^d$ and therefore $G \neq \{0\}$. Suppose there exists a factor map $\pi : G \to G'$ with G' an odometer having an associated scale formed by diagonal matrices $\{D_i\}_{i\geq 0}$ with $D_i(k,k) = d_i^{(k)}$ for $k \in \{1,..,d\}$. By Lemma 2, we have that $d_i^{(k)}$ divides every element in the k-th row of some A_j. Since $m.c.d\{A_j(k,l) : l = 1,..,d\} = 1$, we have that $D_i = id$, $d_i^{(k)} = 1$ and then $G' = \{0\}$. This proves that G is not conjugate to a product of d one-dimensional odometers.

4. Characterization of minimal almost 1-1 extensions of odometers

Let $(X, \mathbb{Z}^d)$ and $(Y, \mathbb{Z}^d)$ be two topological dynamical systems. $(Y, \mathbb{Z}^d)$ is said to be the *maximal equicontinuos factor* of $(X, \mathbb{Z}^d)$ if it is an equicontinuos factor of $(X, \mathbb{Z}^d)$ such that for any other equicontinuous factor $(Y', \mathbb{Z}^d)$ of $(X, \mathbb{Z}^d)$ there exists a factor map $\pi : Y \to Y'$ that satisfies $\pi \circ f = f'$, with $f : X \to Y$ and $f' : X \to Y'$ factor maps.

It is well known that every topological dynamical system has a maximal equicontinuous factor and if $(X, \mathbb{Z}^d)$ is a minimal almost 1-1 extension of a minimal equicontinuous system $(Y, \mathbb{Z}^d)$, then $(Y, \mathbb{Z}^d)$ is the maximal equicontinuous factor of $(X, \mathbb{Z}^d)$ (for more details see [1]).

4.1. **Regularly recurrent systems.** A subset S of $\mathbb{Z}^d$ is said to be *syndetic* if there exists a finite subset K of $\mathbb{Z}^d$ such that $\mathbb{Z}^d = S + K = \{s + k : s \in S,\, k \in K\}$.
Let $(X, \mathbb{Z}^d)$ be a topological dynamical system and let $x \in X$. The point x is *uniformly recurrent* if for every open neighborhood V of x the set $T_V(x)$ is syndetic. It is well known that $(\Omega_{\mathbb{Z}^d}(x), \mathbb{Z}^d)$ is minimal if and only if x is uniformly recurrent.
A point $x \in X$ is *regularly recurrent* if for every open neighborhood V of x there is a subgroup Z of $\mathbb{Z}^d$ isomorphic to $\mathbb{Z}^d$ such that $Z \subseteq T_V(x)$. We say that a system is regularly recurrent if it is the orbit closure of a regularly recurrent point. Since every subgroup Z of $\mathbb{Z}^d$ isomorphic to $\mathbb{Z}^d$ is syndetic, regularly recurrent systems are minimal.
In this section we will show that regularly recurrent systems are exactly the minimal almost 1-1 extensions of the odometers.

Lemma 4. *Let $(X, \mathbb{Z}^d)$ be a minimal topological dynamical system and let $x \in X$. If $Z \subseteq \mathbb{Z}^d$ is a group isomorphic to $\mathbb{Z}^d$ then $(\Omega_Z(x), Z)$ is minimal.*

Proof. Let $V \subseteq X$ be a neighborhood of x. Pick a minimal set M in $X \times \mathbb{Z}^d/Z$ (with the natural product action). This set projects onto a minimal subset of X, hence onto X. Thus for every $x \in X$ there exists a point $(x, a) \in M$ and this point is uniformly recurrent. Adding $-a$ on the second axis is a conjugacy, hence $(x, 0)$ is also uniformly recurrent. This implies that $\{z : z(x) \in V,\, z \in Z\}$ is syndetic. □

Lemma 5. *Let $(X, \mathbb{Z}^d)$ be a topological dynamical system and let $x \in X$ be a regularly recurrent point. For all closed neighborhood V of x there exists a subgroup Z of $\mathbb{Z}^d$ isomorphic to $\mathbb{Z}^d$ such that $Z \subseteq T_V(x)$ and $\{w(\Omega_Z(x))\}_{w \in \mathbb{Z}^d/Z}$ is a clopen partition of X.*

Proof. Let $Z \subseteq \mathbb{Z}^d$ be a subgroup isomorphic to $\mathbb{Z}^d$. If u and w are two elements of $\mathbb{Z}^d$ in the same class of $\mathbb{Z}^d/Z$ then $u(\Omega_Z(x)) = w(\Omega_Z(x))$. So, it makes sense to speak about $w(\Omega_Z(x))$ for $w \in \mathbb{Z}^d/Z$. By minimality of $(X, \mathbb{Z}^d)$ we have that $X = \bigcup_{w \in \mathbb{Z}^d/Z} w(\Omega_Z(x))$. From Lemma 4, for every $w \in \mathbb{Z}^d/Z$ the system $(w(\Omega_Z(x)), Z)$ is minimal. Thus if $u, w \in \mathbb{Z}^d/Z$ satisfy $w(\Omega_Z(x)) \cap u(\Omega_Z(x)) \neq \emptyset$ then $u(\Omega_Z(x)) = w(\Omega_Z(x))$. This implies that $\{w(\Omega_Z(x))\}_{w \in \mathbb{Z}^d/Z}$ is a clopen covering of X.
Let $V \subseteq X$ be a closed neighborhood of x and let $Z \subseteq \mathbb{Z}^d$ be a subgroup isomorphic to $\mathbb{Z}^d$ such that $Z \subseteq T_V(x)$. Consider the subgroup Z' of $\mathbb{Z}^d$ spanned by the set $\{w \in \mathbb{Z}^d : \Omega_Z(x) = w(\Omega_Z(x))\}$. Since $Z \subseteq Z'$, we have that Z' is isomorphic to $\mathbb{Z}^d$, and, because $\Omega_{Z'}(x) = \Omega_Z(x)$, the group Z' is contained in $T_V(x)$. Finally, for $w \in \mathbb{Z}^d$ due to $\Omega_{Z'}(x) = w(\Omega_{Z'}(x))$ if and only if $\Omega_Z(x) = w(\Omega_Z(x))$, it holds that $\{w(\Omega_{Z'}(x))\}_{w \in \mathbb{Z}^d/Z}$ is a clopen partition of X. □

Corollary 6. *Let $(X, \mathbb{Z}^d)$ be a topological dynamical system and let $x \in X$. The point x is regularly recurrent if and only if there exists $\{C_i\}_{i \geq 0}$, a fundamental system of* *clopen neighborhoods of x, such that there is a subgroup $Z_i \subseteq \mathbb{Z}^d$ isomorphic to $\mathbb{Z}^d$ such that for all $y \in C_i$ the set of return times of y to C_i is Z_i, for every $i \geq 0$.*

Proof. If $x \in X$ has a fundamental system of neighborhoods as is written above, it is a regularly recurrent point.
If x is a regularly recurrent point we take V_1 an open neighborhood of x and we apply Lemma 5 to $\overline{V_1}$. We obtain a group $Z_1 \subseteq T_{\overline{V_1}}(x)$, isomorphic to $\mathbb{Z}^d$, such that $\{w(\Omega_{Z_1}(x))\}_{w\in\mathbb{Z}^d/Z_1}$ is a clopen partition of X. We set $C_1 = \Omega_{Z_1}(x)$ which is a clopen set with $T_{C_1}(y) = Z_1$ for all $y \in C_1$.
So, given C_n and Z_n, we take an open neighborhood $V_{n+1} \subset C_n$ of x and we apply Lemma 5 to $\overline{V_{n+1}}$. As in the case $n = 1$, we obtain C_{n+1} and Z_{n+1}.
If we take $\lim_{i\to\infty} \operatorname{diam}(V_n) = 0$, we obtain that $\{C_i\}_{i\geq 0}$ is a fundamental system of clopen neighborhoods of x. □

Theorem 7. *A minimal topological dynamical system $(X, \mathbb{Z}^d)$ is an almost 1-1 extension of an odometer G by π if and only if $(X, \mathbb{Z}^d)$ is a regularly recurrent system. Moreover, the set of regularly recurrent points of X is exactly the pre-image of the set of points in G which have only one pre-image by π.*

Proof. Let $(X, \mathbb{Z}^d)$ be a minimal 1-1 extension of an odometer $G = \lim_{\leftarrow i}(\mathbb{Z}^d/Z_i, \pi_i)$. Let $\pi : X \to G$ be the almost 1-1 factor map and let $x \in X$ be such that $\{x\} = \pi^{-1}\{\pi(x)\}$. Since π is continuous, if $\pi(x) = (a_i)_{i\geq 0} \in G$ then $\{\pi^{-1}([i; a_i])\}_{i\geq 0}$ is a decreasing sequence of clopen neighborhoods of x that satisfies $\bigcap_{i\geq 0} \pi^{-1}([i; a_i]) = \{x\}$. We know that for every $\mathbf{g} \in [i; a_i]$ it holds $T_{[i;a_i]}(\mathbf{g}) = Z_i$, therefore for all $\mathbf{g} \in \pi^{-1}([i; a_i])$, $T_{\pi^{-1}([i;a_i])}(\mathbf{g}) = Z_i$. So, by Corollary 6 we conclude that x is a regularly recurrent point of X.
Let X be a regularly recurrent system and let $x \in X$ be a regularly recurrent point. By Corollary 6 there exists a decreasing sequence $\{C_i\}_{i\geq 0}$ of clopen neighborhoods of x such that $\bigcap_{i\geq 0} C_i = \{x\}$, and there is a subgroup Z_i isomorphic to $\mathbb{Z}^d$ such that $T_{C_i}(y) = Z_i$ for all $y \in C_i$, $i \geq 0$. Since $C_{i+1} \subseteq C_i$, we have that $Z_{i+1} \subseteq Z_i$, $i \geq 0$. So, we can define the odometer $G = \lim_{\leftarrow i}(\mathbb{Z}^d/Z_i, \pi_i)$. We define $\pi : X \to G$ by $\pi = (f_i)_{i\geq 0}$ where f_i is the continuous map $f_i : X \to \mathbb{Z}^d/Z_i$ given by $f_i(y) = z$ if and only if $y \in z(C_i)$ for $y \in X$, $z \in Z_i$ and $i \geq 0$. The function π is a factor map, and, since $\bigcap_{i\geq 0} C_i = \{x\}$, we have that $f^{-1}\{\mathbf{0}\} = \{x\}$. So, π is an almost 1-1 extension.
If $\pi' : X \to G'$ is another almost 1-1 factor map and G' an odometer, G and G' are the maximal equicontinuous factor of $(X, \mathbb{Z}^d)$ (therefore, they are conjugate). Thus there exists a factor map $\pi'' : G' \to G$ such that $\pi'' \circ \pi' = \pi$, which implies that $\{x\} = \pi'{-1}\{\pi(x)\}$. We conclude that the set of regularly recurrent points is exactly the pre-image of the points in G which have only one pre-image. □

5. Eigenvalues of odometers, measure-theoretic factor maps.

5.1. **Eigenvalues.** Let $(X, \mu, \mathbb{Z}^d)$ be a measure-theoretic dynamical system. A vector $\alpha \in \mathbb{R}^d$ is an *eigenvalue* of X if there exists $f \in L^2_\mu(X) \setminus \{0\}$ such that
$f(v(x)) = \exp(2i\pi\alpha^T v)f(x)$ for all $x \in X$ and $v \in \mathbb{Z}^d$. We call f an *eigenfunction* associated to α. We say that an eigenvalue is a *continuous eigenvalue* if it has an

associated continuous eigenfunction.
Since an odometer G is a compact group, the normalized Haar measure λ of G is the only invariant probability measure of G. Thus when we speak about G as a measure-theoretic dynamical system, we mean G equipped with the measure λ and on G we consider the action of $\mathbb{Z}^d$ viewed as a subset of G.

Proposition 8. *Let $G = \lim_{\leftarrow n}(\mathbb{Z}^d/Z_n, \pi_n)$ be an odometer. The set of eigenvalues of G is given by $E_G = \bigcup_{n\geq 0}\{\alpha \in \mathbb{R}^d : \alpha^T z \in \mathbb{Z},\ \forall z \in Z_n\} \subseteq \mathbb{Q}^d$. Moreover, every eigenvalue of G is a continuous eigenvalue.*

Proof. It is clear that $E_G \subseteq \mathbb{Q}^d$ because $\alpha^T z \in \mathbb{Z}$ for all $z \in Z_n$ if and only if $\alpha = v^T A_n^{-1}$ for some $v \in \mathbb{Z}^d$ and $A_n \in GL(d, \mathbb{Z})$ such that $Z_n = A_n\mathbb{Z}^d$.
For $n \geq 0$ we call $C_n = [n; 0]$. Since $v, w \in \mathbb{Z}^d$ satisfy $C_n + v = C_n + w$ if and only if w and v belong to the same class in $\mathbb{Z}^d/Z_n$, it makes sense to write $C_n + v$ for $v \in \mathbb{Z}^d/Z_n$. Notice that the collection $\mathcal{P}_n = \{C_n + v : v \in \mathbb{Z}^d/Z_n\}$ is a clopen partition of G.
Let $\alpha \in E_G$ and let $n \geq 0$ be such that $\alpha^T z \in \mathbb{Z}$ for all $z \in Z_n$. This means that $\exp(2i\pi\alpha^T v) = \exp(2i\pi\alpha^T w)$ for all $v \in \mathbb{Z}^d$ and $w \in v + Z_n$, which implies that
$f = \sum_{v\in\mathbb{Z}^d/Z_n} \exp(2i\pi\alpha^T v)1_{C_n+v}$ is a well defined continuous function that verifies
$f(\mathbf{g} + w) = \exp(2i\pi\alpha^T w)f(\mathbf{g})$ for all $\mathbf{g} \in G$ and $w \in \mathbb{Z}^d$.
Let $\alpha \in \mathbb{R}^d$ be an eigenvalue of G and let $f \in L^2_\lambda(G)\backslash\{0\}$ be an associated eigenfunction. For $v \in \mathbb{Z}^d$ we have that

$$\exp(2i\pi\alpha^T v)\left(\int_{C_n} f d\lambda\right) = \int_{C_n+v} f d\lambda.$$

Since $C_n + v = C_n + v + z$ for all $z \in Z_n$, it holds that

$$\exp(2i\pi\alpha^T z)\left(\int_{C_n} f d\lambda\right) = \int_{C_n} f d\lambda \quad \text{for all } z \in Z_n. \tag{5.1}$$

Observe that

$$\mathbb{E}(f|\mathcal{P}_n) = \sum_{v\in\mathbb{Z}^d/Z_n} \frac{\exp(2i\pi\alpha^T v)}{\lambda(C_n)}\left(\int_{C_n} f d\lambda\right)1_{C_n+v}.$$

Since $\mathcal{B}(\mathcal{P}_n) \uparrow \mathcal{B}(G)$, by the increasing Martingale theorem, we have that $\mathbb{E}(f|\mathcal{P}_n)$ converges to f in $L^2_\lambda(G)$. Because $f \neq 0$, this implies there exists $m \geq 0$ such that $\int_{C_m} f d\lambda \neq 0$ and, by (5.1), we conclude that $\alpha^T z \in \mathbb{Z}$ for all $z \in Z_m$, which means that $\alpha \in E_G$. □

Corollary 9. *Let $(X, \mathbb{Z}^d)$ be a regularly recurrent system and let G be its maximal equicontinuous factor. The set of continuous eigenvalues of X is E_G.*

Proof. It is clear that E_G is contained in the set of continuous eigenvalues of X. Conversely, if α is a continuous eigenvalue of X we can take $f : X \to S^1$ an associated continuous eigenfunction which is a factor map between $(X, \mathbb{Z}^d)$ and the dynamical system $(f(X), \mathbb{Z}^d)$, where the action of $v \in \mathbb{Z}^d$ on $\exp(2i\pi x) \in f(X)$ is given by $v(\exp(2i\pi x)) = \exp(2i\pi(\alpha^T v + x))$, which is an isometry. Thus the system $(f(X), \mathbb{Z}^d)$ is equicontinuous and therefore there exists a factor map $\pi : G \to f(X)$. Since π is an eigenfunction associated to α we conclude that $\alpha \in E_G$. □

5.2. **Measure-theoretic conjugation.**

Proposition 10. *Let $(X_1, \mathbb{Z}^d)$ and $(X_2, \mathbb{Z}^d)$ be two minimal equicontinuous systems. If $\phi : X_1 \to X_2$ is a measure-theoretic factor map then there exists a topological factor map $\pi : X_1 \to X_2$ such that $\pi = \phi$ a.e.*

Proof. A minimal equicontinuous system $(X, \mathbb{Z}^d)$ is conjugate to a system $(G, \mathbb{Z}^d)$, where G is a topological compact group with a continuous homomorphism $\varphi : \mathbb{Z}^d \to G$ satisfying $\overline{\varphi(\mathbb{Z}^d)} = G$, and the action of $\mathbb{Z}^d$ on G is defined by $v(g) = \varphi(v) + g$ for all $v \in \mathbb{Z}^d$ and $g \in G$, ([1] Theorem 3.6, [9] Theorem 1.8). The Haar measure λ is the only invariant probability measure of $(G, \mathbb{Z}^d)$ ([15] Theorem 6.20) and every eigenfunction of this system is continuous because is a constant multiple of a character of G ([15] Theorem 3.5), which implies that there exists an orthonormal basis of $L^2_\lambda(G)$ consisting of continuous eigenfunctions of $(G, \mathbb{Z}^d)$.
Let μ_i be the only invariant probability measure of $(X_i, \mathbb{Z}^d)$, for $i = 1, 2$, and let $\{f_n\}_{n\geq 0}$ be an orthonormal basis of $L^2_{\mu_2}(X_2)$ consisting of continuous eigenfunctions of $(X_2, \mathbb{Z}^d)$. If $\phi : X_1 \to X_2$ is a measure-theoretic factor map then $f_n \circ \phi$ is an eigenfunction of $(X_1, \mathbb{Z}^d)$, for all $n \geq 0$. Thus the ergodicity of the system implies that for every $n \geq 0$ there exists a continuous eigenfunction g_n of $(X_1, \mathbb{Z}^d)$, such that $f_n \circ \phi = g_n$ a.e. Thus it is possible to take a full measure Borel subset A of X_1 such that $f_n \circ \phi = g_n$ on A, for all $n \geq 0$. Let $\{x_i\}_{i\geq 0}$ be a sequence in A which converges to $x \in A$, and let $y \in X_2$ an accumulation point of $\{\phi(x_i)\}_{i\geq 0}$. By continuity of f_n on X_2 and by continuity of $f_n \circ \phi$ on A, we have $f_n(y) = f_n \circ \phi(x)$ for all $n \geq 0$. Thus if y_1 and y_2 are two accumulation points of $\{\phi(x_i)\}_{i\geq 0}$ then $g(y_1) = g(y_2)$ for all $g \in L^2_{\mu_2}(X_2)$, which implies that $y_1 = y_2$. This shows that ϕ is continuous on A. Since $(X_1, \mathbb{Z}^d)$ is strictly ergodic, A is dense on X_1, and since f_n and g_n are continuous on the whole spaces, $\phi|_A$ extends to a continuous map π on X_1, which is a factor map. □

Lemma 11. *Let G be an odometer. If $\pi : G \to G$ is a factor map then π is injective.*

Proof. We set $G = \lim_{\leftarrow n}(\mathbb{Z}^d/Z_n, \pi_n)$. Let $\mathbf{g}, \mathbf{h} \in G$ be two elements such that $\pi(\mathbf{g}) = \pi(\mathbf{h}) = \mathbf{j}$. For all $i \geq 0$ there exists $v_i \in \mathbb{Z}^d/Z_i$ such that $[i; g_i] + v_i = [i; h_i]$. Thus for every $i \geq 0$ there exists $n_i \geq 0$ such that $[n; g_n], [n; g_n] + v_n \subseteq \pi^{-1}([i; j_i])$ for all $n > n_i$. This implies that $v_n \in Z_i$. Thus for $n > i$ it holds that $[n; g_n], [n; h_n] \subseteq [i; g_i]$. Because this is true for all $i \geq 0$ we conclude that $\mathbf{g} = \mathbf{h}$. □

Since odometers are uniquely ergodic, the invariant probability measures of a regularly recurrent system $(X, \mathbb{Z}^d)$ coincide on the sub σ-algebra $\pi^{-1}(\mathcal{B}(G))$, where π is the almost 1-1 factor map between X and its maximal equicontinuous factor G. In particular, due to the set of regularly recurrent points of X is the pre-image by π of a G_δ-set in G, its measure does not depend on the chosen measure $\mu \in \mathcal{M}(X)$.
The proof of the next Theorem follows the same ideas used in the proof for $d = 1$ (see [4], [16]).

Theorem 12. *Let $(X, \mathbb{Z}^d)$ be a regularly recurrent system. The following statements are equivalent:*

(1) *The set of regularly recurrent points of X is a full measure set.*

(2) $(X, \mathbb{Z}^d)$ *is uniquely ergodic and it is measure-theoretically conjugate to its maximal equicontinuous factor.*

Proof. Let $\pi : X \to G$ be the almost 1-1 factor map between X and its maximal equicontinuous factor. Let $R \subseteq X$ be the set of regularly recurrent points.
Suppose that R is a full measure set. Let $\mu \in \mathcal{M}(X)$ and let $B \in \mathcal{B}(G)$. We have $B = (B \cap R) \cup (B \setminus R)$ and $\mu(B) = \mu(B \cap R)$. Since π is injective on R, $B \cap R = \pi^{-1}(\pi(B \cap R)) \in \pi^{-1}(\mathcal{B}(G))$. Thus $\mu(B) = \mu(B \cap R) = \lambda(\pi(B \cap R))$. This implies that $(X, \mathbb{Z}^d)$ is uniquely ergodic. Because π is injective on R, a full measure set, it is a measure-theoretic conjugation between X and G. Assume (2). Let $\phi : (X, \mu) \to (G, \lambda)$ be the measure-theoretic conjugation. Then $\pi \circ \phi^{-1}$ is a self-homomorphism of the odometer. By Proposition 10 and Lemma 11, π is injective when restricted to an invariant set $A \subset X$ with $\mu(A) = 1$. If the set of regularly recurrent points of X is not a full measure set for μ, then by ergodicity, invariance and Theorem 7, the set of points in G with non-singleton fibers in X is of full measure λ. Let B be the pre-image of this set. The intersection $A \cap B$ supports μ. On the other hand, $B \setminus A$ has the same projection on G as B, because A removes only one point from each fiber. So, $B \setminus A$ has projection of full measure λ and it is invariant, hence the measure λ lifts to an invariant measure ν supported by this set. Because μ and ν have disjoint supports, $\nu \neq \mu$ contradicting unique ergodicity □

Remark 13. Let us indicate a mistake in the paper [4]: Condition (6) in [4, Theorem 13.1] claims that for regularity of one-dimensional Toeplitz flows it suffices to find one ergodic measure measure-theoretically conjugate to the odometer. This statement is false; for example the Oxtoby sequence of [4, Example 10.3] is not regular and has two ergodic measures, both isomorphic to the odometer. Clearly, similar examples exist in higher dimensions.

6. $\mathbb{Z}^d$-Toeplitz Arrays

Let Σ be a finite alphabet and $Z \subseteq \mathbb{Z}^d$ a subgroup isomorphic to $\mathbb{Z}^d$. For $x = \{x(v)\}_{v \in \mathbb{Z}^d} \in \Sigma^{\mathbb{Z}^d}$ we define:

$$Per(x, Z, \sigma) = \{w \in \mathbb{Z}^d : x(w + z) = \sigma \text{ for all } z \in Z\},\ \sigma \in \Sigma,$$

$$Per(x, Z) = \bigcup_{\sigma \in \Sigma} Per(x, Z, \sigma).$$

When $Per(x, Z) \neq \emptyset$ we say that Z is a *group of periods of* x. We say that x is a $\mathbb{Z}^d$*-Toeplitz array* (or simply a Toeplitz array) if for all $v \in \mathbb{Z}^d$ there exists $Z \subseteq \mathbb{Z}^d$ subgroup isomorphic to $\mathbb{Z}^d$ such that $v \in Per(x, Z)$.

Proposition 14. *The following statements concerning $x \in \Sigma^{\mathbb{Z}^d}$ are equivalent:*

(1) *x is Toeplitz array.*
(2) *There exists a sequence of positive integer numbers $\{p_n\}_{n \geq 0}$ such that $p_n < p_{n+1}$, p_n divides p_{n+1} and $\{-n, \cdots, n\}^d \subseteq Per(x, p_n\mathbb{Z}^d)$ for all $n \geq 0$.*
(3) *x is regularly recurrent.*

Proof. Before we prove the equivalence between sentences (1),(2) and (3) notice that for every subgroup $Z \subseteq \mathbb{Z}^d$ isomorphic to $\mathbb{Z}^d$, there exists an integer $p > 1$ such that $p\mathbb{Z}^d \subseteq Z$. In fact, since $\mathbb{Z}^d/Z$ is finite, for all $w \in \mathbb{Z}^d/Z$ there exists $k > 1$ such that $kw = 0$. This implies that for all $v \in \mathbb{Z}^d$ there exists $k > 1$ such that $kv \in Z$. In particular, there exist $p_1, .., p_d > 1$ with $p_1e_1, .., p_de_d \in Z$, where $e_1, .., e_d$ are the canonical vectors in $\mathbb{Z}^d$. Thus $p\mathbb{Z}^d \subseteq Z$ with $p = \Pi_{i=1}^d p_i$.
We set $D_n = \{-n, \cdots, n\}^d$ and $C_n = \{y \in \Sigma^{\mathbb{Z}^d} : y(D_n) = x(D_n)\}$ for all $n \geq 0$.
Suppose that x is a Toeplitz array. Let $n \geq 0$ and $v \in D_n$. We take $Z_v \subseteq \mathbb{Z}^d$, subgroup isomorphic to $\mathbb{Z}^d$, such that $v \in Per(x, Z_v)$ and $p_v > 1$ such that $p_v\mathbb{Z}^d \subseteq Z_v$. For $p = \Pi_{v \in D_n} p_v$ we have $p\mathbb{Z}^d \subseteq Z_v$ for all $v \in D_n$. Thus $Z_n = \bigcap_{v \in D_n} Z_v$ is a subgroup isomorphic to $\mathbb{Z}^d$ which satisfies $D_n \subseteq Per(x, Z_n)$. We define the sequence $\{p_n\}_{n \geq 0}$ by $p_0 = q_0$ and for $n > 0$ we set $p_n = q_np_{n-1}$, where $q_n > 1$ is an integer such that $q_n\mathbb{Z}^d \subseteq Z_n$ for all $n \geq 0$. Thus $\{p_n\}_{n \geq 0}$ is a sequence of positive integer numbers such that $p_n < p_{n+1}$, p_n divides p_{n+1} and $D_n \subseteq Per(x, Z_n) \subseteq Per(x, p_n\mathbb{Z}^d)$ for all $n \geq 0$.
Suppose there exists a sequence $\{p_n\}_{n \geq 0}$ as in statement (2). Since $D_n \subseteq Per(x, p_n\mathbb{Z}^d)$ the set of return times of x to C_n contains $p_n\mathbb{Z}^d$ which implies that x is regularly recurrent because $\{C_n\}_{n \geq 0}$ is a fundamental system of clopen neighborhoods of x.
Suppose that x is regularly recurrent. For $n \geq 0$ we take Z_n a subgroup isomorphic to $\mathbb{Z}^d$ such that $Z_n \subseteq T_{C_n}(x)$. It holds that $\mathbb{Z}^d = \bigcup_{n \geq 0} Per(x, Z_n)$ which means that x is a Toeplitz array. □

A subshift $(X, \mathbb{Z}^d)$ is a $\mathbb{Z}^d$-*Toeplitz system* (or simply a Toeplitz system) if there exists a Toeplitz array x such that $X = \Omega_{\mathbb{Z}^d}(x)$. From Theorem 7 and Proposition 14 we conclude that the family of minimal subshifts which are almost 1-1 extensions of odometers coincides with the family of Toeplitz systems.
As it was done for the case $d = 1$ in [16], in order to know the maximal equicontinuous factor of a given Toeplitz system, we will introduce the generalization, for $d \geq 1$, of the concepts of essential period and period structure.

Definition 15. Let $x \in \Sigma^{\mathbb{Z}^d}$. A group $Z \subset \mathbb{Z}^d$ of periods of x is called *group generated by essential periods of* x if $Per(x, Z) \subseteq Per(x, Z')$ implies that $Z' \subseteq Z$.

Lemma 16. *Let $x \in \Sigma^{\mathbb{Z}^d}$. If $Z \subseteq \mathbb{Z}^d$ is a group of periods of x then there exists $K \subseteq \mathbb{Z}^d$ a group generated by essential periods of x such that $Per(x, Z) \subseteq Per(x, K)$.*

Proof. Let $Z \subseteq \mathbb{Z}^d$ be a group of periods of x. We call $\hat{Z}$ the set of the groups $H \subseteq \mathbb{Z}^d$ isomorphic to $\mathbb{Z}^d$ which satisfy $Per(x, Z) \subseteq Per(x, H)$. Let K be the subgroup of $\mathbb{Z}^d$ generated by $\bigcup_{H \in \hat{Z}} H$. Let $H \in \hat{Z}$ and let w be an element in $Per(x, Z, \sigma)$ for some $\sigma \in \Sigma$. Since $w + z \in Per(x, Z, \sigma)$ for all $z \in Z$ we have that $w + z \in Per(x, H, \sigma)$ for all $z \in Z$. This means that $\sigma = x(w + z) = x(w + z + h)$ for all $z \in Z$ and for all $h \in H$ which is equivalent to say that $w + h \in Per(x, Z, \sigma)$ for all $h \in H$. Thus, if m is a finite positive integer and h_i is some element in $H_i \in \hat{Z}$ for $1 \leq i \leq m$ then $w + k \in Per(x, Z, \sigma)$ where $k = \sum_{i=1}^m h_i$. So, $w + K \subseteq Per(x, Z, \sigma)$. This implies that $w \in Per(x, K, \sigma)$. It holds that $K \in \hat{Z}$ and since every H which satisfies $Per(x, K) \subseteq Per(x, H)$ is also in $\hat{Z}$, it follows that K is a group generated by essential periods of x. □

Corollary 17. *Let $x \in \Sigma^{\mathbb{Z}^d}$ be a Toeplitz array. There exists a sequence $\{Z_n\}_{n\geq 0}$ of groups generated by essential periods of x such that $Z_{n+1} \subseteq Z_n$ and $\bigcup_{n\geq 0} Per(x, Z_n) = \mathbb{Z}^d$.*

Proof. From Proposition 14 (2) we conclude there exists a decreasing sequence $\{Z'_n\}_{n\geq 0}$ of groups of periods of x such that $\bigcup_{n\geq 0} Per(x, Z'_n) = \mathbb{Z}^d$. We set Z_0 a group generated by essential periods of x such that $Per(x, Z'_0) \subseteq Per(x, Z_0)$. For $n > 0$ we set $Z''_n = Z'_n \cap Z_{n-1}$ which is a subgroup isomorphic to $\mathbb{Z}^d$, and since $Per(x, Z_{n-1}), Per(x, Z'_n) \subseteq Per(x, Z''_n)$, Z''_n is a group of periods of x. Thus, by Lemma 16, there exists a group Z_n generated by essential periods of x, such that $Per(x, Z''_n) \subseteq Per(x, Z_n)$. Since Z_{n-1} is a group generated by essential periods of x, we have $Z_n \subseteq Z_{n-1}$. Thus $\{Z_n\}_{n\geq 0}$ is a decreasing sequence of groups generated by essential periods of x such that $\bigcup_{n\geq 0} Per(x, Z_n) = \mathbb{Z}^d$. □

Definition 18. A sequence of groups as in Corollary 17 is called a *period structure* of x.

In the sequel, we will show that from a period structure $\{Z_n\}_{n\geq 0}$ of a $\mathbb{Z}^d$-Toeplitz array x it is possible to construct a sequence of nested finite clopen partitions of $\Omega_{\mathbb{Z}^d}(x)$. From this sequence of partitions it will be easy to define an almost 1-1 factor map between the Toeplitz system $(\Omega_{\mathbb{Z}^d}(x), \mathbb{Z}^d)$ and the odometer $G = \lim_{\leftarrow n}(\mathbb{Z}^d/Z_n, \pi_n)$.
Let $x \in \Sigma^{\mathbb{Z}^d}$ be a Toeplitz array, let $y \in \Omega_{\mathbb{Z}^d}(x)$ and let $Z \subseteq \mathbb{Z}^d$ be a subgroup isomorphic to $\mathbb{Z}^d$. Since $(\Omega_Z(y), Z)$ is minimal, if Z is a group of periods of y then $\Omega_Z(y) \subseteq C_Z(y)$, where

$$C_Z(y) = \{x' \in \Omega_{\mathbb{Z}^d}(x) : Per(x', Z, \sigma) = Per(y, Z, \sigma),\ \ \forall\ \sigma \in \Sigma\}.$$

We will use the following convention: For a Z-periodic subset C of $\Omega_{\mathbb{Z}^d}(x)$, i.e., such that $C + w = C + w'$ whenever $w - w' \in Z$ we will write $C + v$ instead of $C + w$, where v is the projection of w to $\mathbb{Z}^d/Z$.

Proposition 19. *Let $x \in \Sigma^{\mathbb{Z}^d}$ be a Toeplitz array and let $y \in \Omega_{\mathbb{Z}^d}(x)$. If $Z \subseteq \mathbb{Z}^d$ is a group generated by essential periods of y then $\Omega_Z(y) = C_Z(y)$ and $\{C_Z(y) + v\}_{v\in\mathbb{Z}^d/Z}$ is a clopen partition of $\Omega_{\mathbb{Z}^d}(x)$.*

Proof. It holds that $\Omega_Z(y) + w \subseteq C_Z(y) + w$ for all $w \in \mathbb{Z}^d/Z$. Since $\{\Omega_Z(y) + w\}_{w\in\mathbb{Z}^d/Z}$ is a covering of $\Omega_{\mathbb{Z}^d}(x)$, so is $\{C_Z(y) + w\}_{w\in\mathbb{Z}^d/Z}$. Furthermore, $(C_Z(y) + w) \cap (C_Z(y) + v) \neq \emptyset$ if and only if $C_Z(y) + w = C_Z(y) + v$, for $w, v \in \mathbb{Z}^d/Z$, which implies that $\{C_Z(y) + w\}_{w\in\mathbb{Z}^d/Z}$ is a clopen covering of $\Omega_{\mathbb{Z}^d}(x)$.
If $C_Z + w = C_Z + v$ for some $v, w \in \mathbb{Z}^d/Z$, then $k(v - w) \in T_{C_Z(y)}(y)$ for all $k \in \mathbb{Z}$. This implies that $Per(y, Z) \subseteq Per(y, Z')$, where $Z' \subseteq \mathbb{Z}^d$ is some subgroup isomorphic to $\mathbb{Z}^d$ generated by $v - w$ and $d - 1$ elements of some base of Z. Since Z is a group generated by essential periods of y then $Z' \subseteq Z$. Thus $v = w$ and $\{C_Z(y) + v\}_{v\in\mathbb{Z}^d/Z}$ is a clopen partition of $\Omega_{\mathbb{Z}^d}(x)$. Because $\Omega_Z(y) + w$ is contained in $C_Z(y) + w$, both sets must be equal because $\{\Omega_Z(y) + v\}_{v\in\mathbb{Z}^d/Z}$ is a covering of $\Omega_{\mathbb{Z}^d}(x)$. □

Proposition 20. *Let $x \in \Sigma^{\mathbb{Z}^d}$ be a Toeplitz array. If $\{Z_n\}_{n\geq 0}$ is a period structure of x then the odometer $G = \lim_{\leftarrow n}(\mathbb{Z}^d/Z_n, \pi_n)$ is the maximal equicontinuous factor of $(\Omega_{\mathbb{Z}^d}(x), \mathbb{Z}^d)$.*

Proof. By Proposition 19, if $\{Z_n\}_{\geq 0}$ is period structure of the Toeplitz array x, then $\{C_{Z_n}(x) + w : w \in \mathbb{Z}^d/Z_n\}_{n\geq 0}$ is a sequence of nested clopen partitions of $\Omega_{\mathbb{Z}^d}(x)$. This implies that the function $f_n : \Omega_{\mathbb{Z}^d}(x) \to \mathbb{Z}^d/Z_n$ given by $f_n(y) = w$ if and only if $y \in C_{Z_n}(x) + w$ is a well defined continuous function, $y \in \Omega_{\mathbb{Z}^d}(x)$, $n \geq 0$. The function $\pi : \Omega_{\mathbb{Z}^d}(x) \to G$ given by $\pi = (f_n)_{n\geq 0}$ is a factor map. Since $\bigcap_{n\geq 0} C_{Z_n(x)} = \{x\}$, we have that $\pi^{-1}\{\mathbf{0}\} = \{x\}$ and then π is an almost 1-1 factor map. □

Proposition 21. *For every odometer G there exists a Toeplitz array $x \in \{0,1\}^{\mathbb{Z}^d}$ such that G is the maximal equicontinuous factor of $(\Omega_{\mathbb{Z}^d}(x), \mathbb{Z}^d)$.*

Proof. Let $G = \lim_{\leftarrow n}(\mathbb{Z}^d/Z_n, \pi_n)$ be an odometer. We distinguish two cases:
Case 1: There exists $m \geq 0$ such that $Z_n = Z_m$ for all $n \geq m$. In this case G is the finite group $\mathbb{Z}^d/Z_m$ and then every minimal almost 1-1 extension will be conjugate to G. For example, $x \in \{0,1\}^{\mathbb{Z}^d}$ defined by $x(v) = 0$ for all $v \in Z_m$ and $x(v) = 1$ if not, provides a Toeplitz sequence x such that G is the maximal equicontinuous factor of the system associated to x.
Case 2: For every $m \geq 0$ there exists $n > m$ such that $Z_n \neq Z_m$. In this case we can take a subsequence $\{Z_n\}_{n\geq 0}$ such that $Z_{n+1} \neq Z_n$ and $|Z_n/Z_{n+1}| \geq 3$ for all $n \geq 0$. By Proposition 1, G is conjugate to the odometer obtained from this sequence. In order to construct the Toeplitz array x we will define a sequence $\{(w_n, v_n)\}_{n\geq 0} \subseteq \mathbb{Z}^d \times \mathbb{Z}^d$ as follows: we set $v_0 = 0$ and we choose some element $w_0 \in \mathbb{Z}^d \setminus Z_0$. For $n > 0$, we take $v \in w_{n-1} + Z_{n-1}$ which satisfies $\|v\| = \min\{\|w\| : w \in w_{n-1} + Z_{n-1}\}$, where $\|v\| = \max_{1\leq i\leq d} |v^{(i)}|$ with $v = (v^{(1)}, \cdots, v^{(d)})$. We set $v_n = v$ and we choose $w_n \in w_{n-1} + Z_{n-1} \setminus (v_n + Z_n)$. The sequence is well defined because $|Z_n/Z_{n+1}| \geq 3$ for all $n \geq 0$. We define,

$$K_0 = \mathbb{Z}^d \setminus (v_0 + Z_0) \cup (w_0 + Z_0)$$

$$K_n = \bigcup_{w\in(w_{n-1}+Z_{n-1})\setminus(v_n+Z_n\cup w_n+Z_n)} w + Z_n, \text{ for } n > 0.$$

The family of sets $\{v_n + Z_n, K_n : n \geq 0\}$ is a partition of $\mathbb{Z}^d$. Thus $x \in \{0,1\}^{\mathbb{Z}^d}$ given by:

$$x(z) = \begin{cases} 0 & \text{if } z \in \bigcup_{n\geq 0} v_n + Z_n \\ 1 & \text{if } z \in \bigcup_{n\geq 0} K_n \end{cases} \tag{6.2}$$

is well defined. Since $\bigcup_{j=0}^{n} v_j + Z_j \subseteq Per(x, Z_n, 0)$ and $\bigcup_{j=0}^{n} K_j \subseteq Per(x, Z_n, 1)$, it holds that $\mathbb{Z}^d = \bigcup_{n\geq 0} Per(x, Z_n)$ and then x is a Toeplitz array. To conclude that G is the maximal equicontinuous factor of the system associated to x, by Proposition 20, it suffices to show that $\{Z_n\}_{n\geq 0}$ is a period structure of x.
Let $n \geq 0$ and $Z \subseteq \mathbb{Z}^d$ a subgroup isomorphic to $\mathbb{Z}^d$ such that $Per(x, Z_n) \subseteq Per(x, Z)$. Given $z \in Z$, this implies that $0 = x(v_n) = x(v_n + z)$. Thus $v_n + z \in \bigcup_{j=0}^{n}(v_j + Z_j) \cup (w_n + Z_n)$. If $v_n + z \in w_n + Z_n$ we obtain that $x(w) = 0$ for all $w \in w_n + Z_n$ which is not possible, and if $v_n + z \in v_j + Z_j$ for some $0 \leq j < n$ we get $x(w) = 0$ for all

$w \in w_j + Z_j$ which also contradicts the construction of x. So, $z \in Z_n$ and we conclude that Z_n is a group of essential periods of x. $\square$

Remark 22. Example 3 and Proposition 21 imply that, for $d \geq 2$, there are Toeplitz systems in $\{0,1\}^{\mathbb{Z}^d}$ such that their maximal equicontinuous factors are not products of d one-dimensional odometers.

6.1. **Aperiodic part of a Toeplitz array.** Let $x \in \Sigma^{\mathbb{Z}^d}$ be a Toeplitz array, let $\pi : \Omega_{\mathbb{Z}^d}(x) \to G$ be the almost 1-1 factor map between $\Omega_{\mathbb{Z}^d}(x)$ and its maximal equicontinuous factor G, and let $\{Z_n\}_{n\geq 0}$ be a period structure of x.
We define

$$D_G = \{\mathbf{g} \in G : \exists\, y_1, y_2 \in \pi^{-1}\{\mathbf{g}\} \text{ such that } \quad y_1(0) \neq y_2(0)\},$$

and for $y \in \Omega_{\mathbb{Z}^d}(x)$ the set

$$Aper(y) = \mathbb{Z}^d \setminus \bigcup_{n\geq 0} Per(y, Z_n).$$

In analogy of the case $d = 1$, in the next proposition we will show that $Aper(y)$ does not depend on the choice of a period structure $\{Z_n\}_{n\geq 0}$ and that it is exactly the aperiodic part of y.

Proposition 23. *If $y \in \Omega_{\mathbb{Z}^d}(x)$ and $\pi(y) = \boldsymbol{g} \in G$ then:*

1. *$w \in Aper(y)$ if and only if $\boldsymbol{g} + w \in D_G$.*
2. *$w \notin Aper(y)$ if and only if there exists a subgroup Z of $\mathbb{Z}^d$ isomorphic to $\mathbb{Z}^d$ such that $w \in Per(y, Z)$.*
3. *y is a Toeplitz array if and only if $Aper(y) = \emptyset$.*
4. *If $y' \in \pi^{-1}\{\boldsymbol{g}\}$ then $y'(w) = y(w)$ for all $w \in \mathbb{Z}^d \setminus Aper(y)$.*

Proof. If $w \in Aper(y)$ then for all $n \geq 0$ there exists $z_n \in Z_n$ such that $(y + w)(0) \neq (y + w + z_n)(0)$. Since $\mathbf{g} + w$ and $\mathbf{g} + w + z_n$ are in $[n; g_n]$, we have that $\lim_{n\to\infty} \pi(y + w + z_n) = \lim_{n\to\infty} \mathbf{g} + w + z_n = \mathbf{g} + w$. Thus for every accumulation point y' of $\{y + w + z_n\}_{n\geq 0}$ it holds that $\pi(y') = \mathbf{g} + w$ and $y'(0) \neq (y + w)(0)$. So, $\mathbf{g} + w \in D_G$. If $\mathbf{g} + w \in D_G$ then there is $y' \in \pi^{-1}(\mathbf{g} + w)$ such that $(y + w)(0) \neq y'(0)$. By minimality and since $y', y + w$ are in $\bigcap_{n\geq 0} \pi^{-1}([n; g_n])$ we have that for all $n \geq 0$ there exists $z_n \in Z_n$ such that $(y + w + z_n)(0) = y'(0) \neq (y + w)(0)$ which implies that $w \in Aper(y)$.
To show (2) it is obvious that if $w \notin Aper(y)$ then $w \in Per(y, Z)$ for some subgroup $Z \subseteq \mathbb{Z}^d$ isomorphic to $\mathbb{Z}^d$. Conversely, suppose that $w \in Per(y, Z)$ for some $Z \subseteq \mathbb{Z}^d$ subgroup isomorphic to $\mathbb{Z}^d$. By Lemma 16, we can suppose that Z is a group generated by essential periods of y. From Proposition 19, $\{C_Z(y) + w\}_{w\in\mathbb{Z}^d/Z}$ is a clopen partition of $\Omega_{\mathbb{Z}^d}(x)$. Let $x' \in C_Z(y)$ be a Toeplitz array and let $\{Z''_n\}_{n\geq 0}$ be a periodic structure of x'. Consider the sequence $\{Z'_n\}_{n\geq 0}$ given by $Z'_0 = Z$ and Z'_n a group of essential periods of x' such that $Per(x, Z'_{n-1} \cap Z''_n) \subseteq Per(x, Z'_n)$ for all $n > 0$. Since $\{C_{Z'_n}(x') + w : w \in \mathbb{Z}^d/Z'_n\}_{n\geq 0}$ is a sequence of nested clopen partitions of $\Omega_{\mathbb{Z}^d}(x)$ such that $\bigcap_{n\geq 0} C_{Z'_n}(x') = \{x'\}$, we can prove, as it was done in the proof of Proposition 20, that $G = \lim_{\leftarrow n}(\mathbb{Z}^d/Z'_n, \pi_n)$. Since $\{Z'_n\}_{n\geq 0}$ and $\{Z_n\}_{n\geq 0}$ define the

same odometer, Lemma 1 implies that there exists $n \geq 0$ such that $Z_n \subseteq Z'_0 = Z$. Thus $Per(y,Z) \subseteq Per(y,Z_n)$, which implies that $w \notin Aper(y)$.
Properties (3) and (4) follow of property (1). □

6.2. **Regular Toeplitz arrays.** Let $x \in \Sigma^{\mathbb{Z}^d}$ be a Toeplitz array and $\pi : \Omega_{\mathbb{Z}^d}(x) \to G$ the almost 1-1 factor map between $\Omega_{\mathbb{Z}^d}(x)$ and its maximal equicontinuous factor $G = \lim_{\leftarrow n}(\mathbb{Z}^d/Z_n, \pi_n)$. For all $n \geq 0$ we define

$$r_n = \frac{|Per(x,Z_n) \cap [\mathbb{Z}^d/Z_n]|}{|\mathbb{Z}^d/Z_n|},$$

where $[\mathbb{Z}^d/Z_n]$ is a subset of $\mathbb{Z}^d$ which contains exactly one representative element of every class in $\mathbb{Z}^d/Z_n$.
Since $Per(x,Z_n) \subseteq Per(x,Z_{n+1})$ and $|\mathbb{Z}^d/Z_{n+1}| = |\mathbb{Z}^d/Z_n| \cdot |Z_n/Z_{n+1}|$ we have that $r_{n+1} \geq r_n$. Thus $\lim_{n\to\infty} r_n = r \in (0,1]$ exists. The Topelitz array x is said to be *regular* if $r = 1$.

Proposition 24. *Let $x \in \Sigma^{\mathbb{Z}^d}$ be a Toeplitz array. The following statements are equivalent:*

1. *x is regular.*
2. *The set of Toeplitz arrays of $\Omega_{\mathbb{Z}^d}(x)$ is a full measure set for every $\mu \in \mathcal{M}(X)$.*
3. *$\lambda(D_G) = 0$, where λ is the Haar measure on G.*
4. *$(\Omega_{\mathbb{Z}^d}(x), \mathbb{Z}^d)$ is uniquely ergodic and it is measure-theoretically conjugate to its maximal equicontinuous factor.*

Proof. The statements (2) and (4) are equivalent by Theorem 12. As it was done in [16], the set of Toeplitz arrays of $\Omega_{\mathbb{Z}^d}(x)$ is given by $\bigcap_{v\in\mathbb{Z}^d} C + v \in \pi^{-1}\{\mathcal{B}(G)\}$, where $C = \{y \in \Omega_{\mathbb{Z}^d}(x) : 0 \notin Aper(y)\}$. Thus, for all $\mu \in \mathcal{M}(\Omega_{\mathbb{Z}^d}(x))$ it holds that $\mu(\{y \in \Omega_{\mathbb{Z}^d}(x) : Aper(y) = \emptyset\}) = r$, which shows that (1) and (2) are equivalent.
We have $G\backslash\{\mathbf{g} \in G : |\pi^{-1}\{\mathbf{g}\}| = 1\} = \bigcup_{v\in\mathbb{Z}^d}(D_G+v)$, which means that $\bigcup_{v\in\mathbb{Z}^d}(D_G+v)$ is the complement of $\{\mathbf{g} : \pi^{-1}\{\mathbf{g}\}$ is a Toeplitz array$\}$. Thus if the set of Toeplitz arrays is a full measure set for some $\mu \in \mathcal{M}(\Omega_{\mathbb{Z}^d}(x))$, then the complement of $\bigcup_{v\in\mathbb{Z}^d}(D_G+v)$ is a full measure set for λ, which implies that $\lambda(D_G) = 0$. Conversely, if $\lambda(D_G) = 0$ then $\lambda(\bigcup_{v\in\mathbb{Z}^d}(D_G+v)) = 0$, which implies $\lambda(\{\mathbf{g} : \pi^{-1}\{\mathbf{g}\}$ is a Toeplitz array$\}) = 1$. Let $\mu \in \mathcal{M}(\Omega_{\mathbb{Z}^d}(x))$. Since $\mu(\pi^{-1}A) = \lambda(A)$ for all $A \in \mathcal{B}(G)$, the set of Toeplitz array is a full measure set for μ. This shows that (2) is equivalent to (3). □

7. Semicocycles

The notion of a semicocycle has been extensively used in the theory of one-dimensional Toeplitz flows (see [4]). In this paper it is not used but we develop it for higher dimensional actions for further utility.
Let $x \in \Sigma^{\mathbb{Z}^d}$ be a Toeplitz array and let $\{p_n\}_{n\geq 0}$ be the sequence of integer numbers of Proposition 14(2). Since for all $n \geq 0$ there exists $q_n > 1$ such that $p_{n+1} = q_n p_n$, the odometer $G = \lim_{\leftarrow n}(\mathbb{Z}^d/p_n\mathbb{Z}^d, \pi_n)$ is a free odometer, that is an odometer which is a free dynamical system. Thus the function $\tau : \mathbb{Z}^d \to G$ defined in Section 3 is an homomorphism between the groups $\mathbb{Z}^d$ and $\tau(\mathbb{Z}^d)$. So, we can identify $\tau(\mathbb{Z}^d)$ with $\mathbb{Z}^d$

and write $\mathbb{Z}^d$ instead of $\tau(\mathbb{Z}^d)$. The odometer G induces on $\mathbb{Z}^d$ the topology generated by the family of sets $\{w + p_n\mathbb{Z}^d : w \in \mathbb{Z}^d, n \geq 0\}$ that we call Θ_G. The function $v \to x(v)$ is continuous with respect Θ_G because $\{-n, .., n\}^d \subseteq Per(x, p_n\mathbb{Z}^d)$ for all $n \geq 0$. The last one means that $x : \mathbb{Z}^d \to \Sigma$ is a semicocycle on G in the following sense:

Definition 25. Let $G = \lim_{\leftarrow n}(\mathbb{Z}^d/Z_n, \pi_n)$ be a free odometer and let K be a compact metric space. A function $f : \mathbb{Z}^d \to K$ is a *semicocycle on G* if it is continuous with respect Θ_G, where Θ_G is the topology on $\mathbb{Z}^d$ inherited from G.

The functions $f : \mathbb{Z}^d \to K$ may be seen as elements of the topological dynamical system $(K^{\mathbb{Z}^d}, \mathbb{Z}^d)$, where $K^{\mathbb{Z}^d}$ is endowed with the product topology and the action of $v \in \mathbb{Z}^d$ on $f = \{f(z)\}_{z\in\mathbb{Z}^d} \in K^{\mathbb{Z}^d}$ is the shift action, it means $v(f) = \{f(v + z)\}_{n\geq 0}$.
We will skip the proofs of Theorems 26 and 27 below, because they follow by the same ideas as used in [4] for dimension one.

Theorem 26. *If $f \in K^{\mathbb{Z}^d}$ is a semicocycle on some odometer G then f is a regularly recurrent point of $(K^{\mathbb{Z}^d}, \mathbb{Z}^d)$.*

Theorem 26 provides another characterization of $\mathbb{Z}^d$-Toeplitz arrays: we have showed that every Toeplitz array $x \in \Sigma^{\mathbb{Z}^d}$ is a semicocycle on some odometer G. By Theorem 26, if $x \in \Sigma^{\mathbb{Z}^d}$ is a semicocycle on some odometer G with values in a finite set Σ then x is regularly recurrent and therefore a Toeplitz array.
Proposition 7 and Theorem 26 imply that $(\Omega_{\mathbb{Z}^d}(f), \mathbb{Z}^d)$ is a minimal almost 1-1 extension of some odometer, where $\Omega_{\mathbb{Z}^d}(f)$ represents the closure orbit of the semicocycle f in $K^{\mathbb{Z}^d}$. Notice that G need not be the maximal equicontinuos factor of $(\Omega_{\mathbb{Z}^d}(f), \mathbb{Z}^d)$, for instance, in the first part of this section it was shown that every Toeplitz array is a semicocycle on an odometer which is a product of d one-dimensional odometers. While for $d > 1$ it is not true that any Toeplitz system has a maximal equicontinuous factor which is a product of d one-dimensional odometers.
Let $f \in K^{\mathbb{Z}^d}$ be a semicocycle on an odometer G. Since we have identified $\tau(\mathbb{Z}^d)$ with $\mathbb{Z}^d$ it makes sense to define $F = \overline{\{(v, f(v)) : v \in \mathbb{Z}^d\}} \subseteq G \times K$ and
$F(\mathbf{g}) = \{k \in K : (\mathbf{g}, k) \in F\}$ for $\mathbf{g} \in G$.
We call C_f the set of $\mathbf{g} \in G$ such that $|F(\mathbf{g})| = 1$ and $D_f = G \setminus C_f$. Since f is continuous we have that $F(v) = \{v\}$ for all $v \in \mathbb{Z}^d$. Thus C_f is the subset where f can be continuously extended by $f(\mathbf{g}) = F(\mathbf{g})$.
The semicocycle f is said to be *invariant under no rotation* if $F(\mathbf{g} + \mathbf{g'}) = F(\mathbf{g'})$ for every $\mathbf{g'} \in G$ implies that $\mathbf{g} = \mathbf{0}$.

Theorem 27. *A topological dynamical system $(X, \mathbb{Z}^d)$ is a minimal almost 1-1 extension of $(G, \mathbb{Z}^d)$ if and only if it is conjugate to $(\Omega_{\mathbb{Z}^d}(f), \mathbb{Z}^d)$, where f is a semicocycle on G, invariant under no rotation.*

We say that a Toeplitz array x is *non periodic* if $x + v = x$ implies that $v = 0$. A semicocycle defined by a non periodic Toeplitz array is not extendable to a continuous function on the whole odometer. For contrast, a constant semicocycle defines a periodic array. Notice that x is non periodic if and only if x is a semicocycle on G, its maximal equicontinuous factor. In fact, if x is a semicocycle on G then it is a free dynamical system and therefore x is non periodic. Conversely, if x is non periodic then G is a free dynamical system. Since x is continuous with respect to Θ_G, x is a semicocycle on G. In Proposition 28 we mean x as a semicocycle on its maximal equicontinuous factor G.

Proposition 28. *If x is non periodic and $\boldsymbol{j} \in G$ then:*

(1) *$\sigma \in F(\boldsymbol{j})$ if and only if there exists $y \in \pi^{-1}\{\boldsymbol{j}\}$ such that $y(0) = \sigma$.*
(2) *$D_x = D_G$.*

Proof. Always we can suppose that $\pi(x) = 0$. If $\sigma \in F(\mathbf{j})$ then $\mathbf{j}$ is the limit of some sequence $\{n_i\}_{i\geq 0} \in \mathbb{Z}^d$ such that $\lim_{i\to\infty} x(n_i) = \sigma$. Thus every accumulation point y of $\{x + n_i\}_{i\geq 0}$ satisfies $y(0) = \sigma$ and $\pi(y) = \pi(x) + \mathbf{j} = \mathbf{j}$. By minimality of Ω_x and by continuity of π, if $y \in \pi^{-1}\{\mathbf{j}\}$ satisfies $y(0) = \sigma$ then $\sigma \in F(\mathbf{j})$. Property (2) follows directly from (1). □

8. Examples

In this section we will give two examples of $\mathbb{Z}^2$-Toeplitz arrays. In the first example we will construct a Toeplitz array $\bar{x}$ such that $\mathcal{M}(\Omega_{\mathbb{Z}^2}(\bar{x}))$ has a determined finite number of ergodic measures and in the second one the Toeplitz array $\bar{x}$ will be constructed such that $(\Omega_{\mathbb{Z}^2}(\bar{x}), \mathbb{Z}^2)$ is uniquely ergodic with positive entropy.

We set some notation that we use in both examples.
Let $\{q_n\}_{n\geq 0}$ be a sequence of integer numbers such that $q_n \geq 3$ for all $n \geq 0$. We set $p_0 = 1$ and $p_n = \Pi_{i=0}^{n-1} q_i$ for $n > 0$.
For $n > 0$ we put

$$r_n = \begin{cases} \frac{q_{n-1}}{2} - 1 & \text{if } q_{n-1} \text{ is even} \\ \frac{q_{n-1}-1}{2} & \text{otherwise} \end{cases}$$

and $l_n = q_{n-1} - r_n - 1$. We define $D_0 = \{0\}^2$ and

$$D_n = \{z \in \mathbb{Z} : -\sum_{i=1}^{n} l_i p_{i-1} \leq z \leq \sum_{i=1}^{n} r_i p_{i-1}\}^2 \subseteq \mathbb{Z}^2.$$

Notice that D_n is the disjoint union of the sets $D_{n-1,v} = D_{n-1} + v$, for $v \in S_n$, where $S_n = \{p_{n-1}z \in \mathbb{Z} : -l_n \leq z \leq r_n\}^2$.
The "boundary" of S_n is $\partial S_n = \{(t_1, t_2) \in S_n : t_1 \text{ or } t_2 \text{ is in } \{r_n p_{n-1}, -l_n p_{n-1}\}\}$.
Since $q_n \geq 3$ for all $n \geq 0$, then $\{r_n\}_{n\geq 0}$ and $\{l_n\}_{n\geq 0}$ are increasing sequences and thus $\mathbb{Z}^2 = \bigcup_{n\geq 0} D_n$.
Let $q > 1$ be an integer and consider the alphabet $\Sigma = \Sigma_0 = \{\sigma_1, .., \sigma_q\}$. For $n > 0$ we take $\Sigma_n = \{B_{n,1}, \cdots, B_{n,k_n}\}$ a set of different blocks in Σ^{D_n} such that for all $1 \leq k \leq k_n$,

(1) $B_{n,k}(D_{n-1,0}) = B_{n-1,1}$,
(2) $B_{n,k}(D_{n-1,v}) \in \Sigma_{n-1}$ for all $v \in S_n$,

where $B_{0,i} = \sigma_i$ for all $1 \leq i \leq q$.
From property (1) and since $\{D_n\}_{n>0}$ covers $\mathbb{Z}^2$, we have that there is only one element $\bar{x}$ in $\bigcap_{n\geq 0}\{x \in \Sigma^{\mathbb{Z}^2} : x(D_n) = B_{n,1}\}$. Property (2) implies that $\bar{x}(D_n + v) \in \Sigma_n$ for

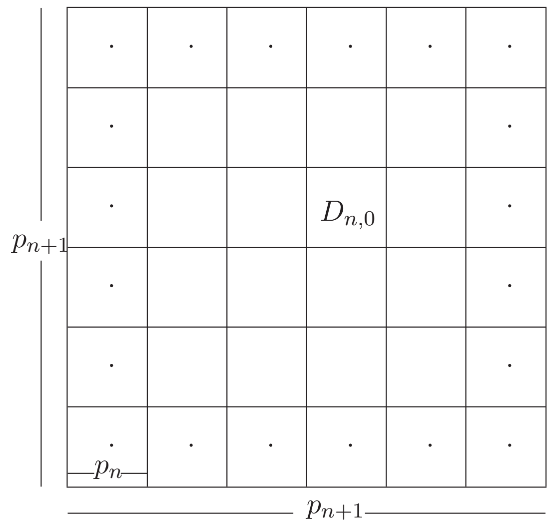

FIGURE 1. The square D_{n+1} for $q_n = 6$. The points "." represent the set ∂S_n.

all $v \in p_n\mathbb{Z}^2$. Thus, property (2) insures that $\bar{x}(D_{n-1} + v) = B_{n-1,1}$ for all $v \in p_n\mathbb{Z}^2$, which means that $D_{n-1} \subseteq Per(\bar{x}, p_n\mathbb{Z}^2)$. So, $\bar{x}$ is a $\mathbb{Z}^2$-Toeplitz array.

For all $n \geq 0$ and $1 \leq k \leq q$ we define $C_{n,k} = \{x \in \Omega_{\mathbb{Z}^2}(\bar{x}) : x(D_n) = B_{n,k}\}$, $C_n = \bigcup_{k=1}^{k_n} C_{n,k}$ and $\mathcal{P}_n = \{C_{n,k} + w : w \in D_n, 1 \leq k \leq k_n\}$. From property (2) we have that $\mathcal{P}_n$ covers the orbit of $\bar{x}$ and since the sets in $\mathcal{P}_n$ are clopen, $\mathcal{P}_n$ is a clopen covering of $\Omega_{\mathbb{Z}^2}(\bar{x})$.

Lemma 29. *If for all $n > 0$ the following statements are satisfied:*

(1) *If there exists $w \in D_n$ such that for some $1 \leq k, k' \leq k_n$, $B_{n,k}(v+w) = B_{n,k'}(v)$ for all $v \in D_n$ such that $v + w \in D_n$, then $w = 0$,*
(2) *$B_{n,k}(v) = B_{n,k'}(v)$ for every $v \in \partial S_n$ and $1 \leq k, k' \leq k_n$,*

then the coverings $\mathcal{P}_n$ are partitions spanning the topology of $\Omega_{\mathbb{Z}^2}(\bar{x})$.

Proof. Let $z \in \mathbb{Z}^2$ and let $w \in D_n$. Suppose that $\bar{x} + p_n z + w \in C_n$. Let $B_{n,k}$ be the block in Σ_n such that $(\bar{x}+p_n z+w)(D_n) = B_{n,k}$. Since $\bar{x}+p_n z$ is also in C_n, there exists $1 \leq k' \leq k_n$ such that $(\bar{x}+p_n z)(D_n) = B_{n,k'}$. This implies that $B_{n,k'}(w+v) = B_{n,k}(v)$ for all $v \in D_n$ satisfying $v + w \in D_n$. From statement (1) we have $w = 0$ and thus we conclude that $T_{C_n}(\bar{x}) = p_n\mathbb{Z}^2$, which implies $T_{C_n}(x) = p_n\mathbb{Z}^2$ for all $x \in C_n$, because $(\Omega_{\mathbb{Z}^2}(\bar{x}), \mathbb{Z}^2)$ is minimal. Thus if $(C_{n,k} + v) \cap (C_{n,k'} + w) \neq \emptyset$ for some $v, w \in D_n$ and $1 \leq k, k' \leq k_n$, then $v - w \in p_n\mathbb{Z}^2$, which implies that $w - v = 0$. If $v = w$ then $C_{n,k} \cap C_{n,k'} \neq \emptyset$, which is possible if and only if $B_{n,k} = B_{n,k'}$, i.e, when $k = k'$. This proves that $\mathcal{P}_n$ is a partition.

Suppose that x_1 and x_2 are two points of $\Omega_{\mathbb{Z}^2}(\bar{x})$ which belong to the same set of $\mathcal{P}_n$. Namely, $x_1, x_2 \in C_{n,j_n} + v_n$ for some $v_n \in D_n$ and $1 \leq j_n \leq k_n$. Let $y_1, y_2 \in C_{n,j_n}$ be such that $x_i = y_i + v_n$ for $i = 1, 2$, and let $u \in \mathbb{Z}^2$ be some vector in D_{n-1}. If $v_n + u \in D_n$ then $y_1(v_n+u) = y_2(v_n+u)$ which implies that $x_1(u) = x_2(u)$. If $v_n + u \notin D_n$, consider $z \in \mathbb{Z}^2 \setminus \{0\}$ and $w \in D_n$ such that $v_n + u = p_n z + w$. Since $y_1 + p_n z$ and $y_2 + p_n z$

are in C_n, $(y_1+p_nz)(D_n) = B_{n,l_1}$ and $(y_2+p_nz)(D_n) = B_{n,l_2}$ for some $1 \le l_1, l_2 \le k_n$. Thus $y_1(v_n+u) = B_{n,l_1}(w)$ and $y_2(v_n+u) = B_{n,l_2}(w)$. It holds $w-u = v_n - p_nz$, which implies that $w-u \notin D_n$ because $z \neq 0$. Since $u \in D_{n-1}$ and $w \in D_n$, this is possible only if $w \in \partial S_n$. Thus, from statement (2), we have $B_{n,l_1}(w) = B_{n,l_2}(w)$ and then $x_1(u) = y_1(v_n+u) = y_2(v_n+u) = x_2(u)$. This proves that $x_1(D_{n-1}) = x_2(D_{n-1})$. So, if x_1 and x_2 are in the same set of $\mathcal{P}_n$, for all $n \ge 0$, then $x_1 = x_2$, which means that $\{\mathcal{P}_n\}_{n\ge 0}$ spans the topology of $\Omega_{\mathbb{Z}^2}(\bar{x})$. □

For $n \ge 0$ we define the set $\triangle_n = \{(x_1,..,x_{k_n})^T \in (\mathbb{R}^+)^{k_n} : \sum_{i=1}^{k_n} x_i = \frac{1}{p_n}\}$ and the incidence matrix $A_n \in \mathcal{M}_{k_n \times k_{n+1}}(\mathbb{Z})$ between $\mathcal{P}_n$ and $\mathcal{P}_{n+1}$ by

$$A_n(k,j) = |\{v \in D_{n+1} : C_{n+1,j} + v \subseteq C_{n,k}\}|,\ 1 \le k \le k_n,\ \ 1 \le j \le k_{n+1}.$$

We denote by $\lim_{\leftarrow n}(\triangle_n, A_n)$ the inverse limit

$$\triangle_0 \xleftarrow{A_0} \triangle_1 \xleftarrow{A_1} \triangle_2 \xleftarrow{A_2} \cdots,$$

that is, $\lim_{\leftarrow n}(\triangle, A_n) = \{(x_n)_{n\ge 0} \in \Pi_{n\ge 0}\triangle_n : A_n x_{n+1} = x_n,\ \forall\, n \ge 0\}$.

Lemma 30. *If the coverings $\mathcal{P}_n$ are partitions spanning the topology of $\Omega_{\mathbb{Z}^2}(\bar{x})$ then we can identify $\mathcal{M}(\Omega_{\mathbb{Z}^2}(\bar{x}))$ with the inverse limit $\lim_{\leftarrow n}(\triangle_n, A_n)$.*

Proof. Suppose that the coverings $\mathcal{P}_n$ are partitions that span the topology of $\Omega_{\mathbb{Z}^2}(\bar{x})$. By property (2) we have that $\mathcal{P}_{n+1}$ is finer that $\mathcal{P}_n$. This implies that $C_{n,k} = \bigcup_{j=1}^{k_{n+1}} \bigcup_{v \in J(n,k,j)} C_{n+1,j} + v$, with $J(n,k,j) = \{v \in D_{n+1} : C_{n+1,j} + v \subseteq C_{n,k}\}$. Thus $\sum_{k=1}^{k_n} A_n(k,j) = q_n$ and for $\mu \in \mathcal{M}(\Omega_{\mathbb{Z}^2}(\bar{x}))$, $\mu(C_{n,k}) = \sum_{j=1}^{k_{n+1}} A_n(k,j)\mu(C_{n+1,j})$ for all $n \ge 0$. The first one implies that $\lim_{\leftarrow n}(\triangle_n, A_n)$ is well defined and the second one that $(\mu_n = (\mu(C_{n,1}),..,\mu(C_{n,k_n})))_{n\ge 0}$ is in this inverse limit. Conversely, given $(u_n = (u_{n,1},..,u_{n,k_n}))_{n\ge 0} \in \lim_{\leftarrow n}(\triangle_n, A_n)$, since the $\mathcal{P}_n$ are clopen and span the topology of $\Omega_{\mathbb{Z}^2}(\bar{x})$, there exists only one $\mu \in \mathcal{M}(\Omega_{\mathbb{Z}^2}(\bar{x}))$ satisfying $\mu(C_{n,k}) = u_{n,k}$ for all $1 \le k \le k_n$ and $n \ge 0$. □

We will construct the different examples by choosing appropriate sequences $\{\Sigma_n\}_{n\ge 0}$ and $\{q_n\}_{n\ge 0}$.

8.1. **An example of a $\mathbb{Z}^2$-Toeplitz system with a determined finite number of ergodic measures.** Let $n > 0$ and let $k \in \{1,..,q\}$. We set $q_n = s_nq + 1$ for some $s_n > 1$ and

$$\partial S_{n,k} = \bigcup_{i \in \{0,..,q_{n-1}-1\} \cap (k+q\mathbb{Z})} \{(t_1,t_2) \in \partial S_n : t_1 \text{ or } t_2 \text{ is equal to } p_{n-1}(-l_n+i)\},$$

We have that ∂S_n is the disjoint union of the sets $\partial S_{n,k}$ and the cardinality of every one of these sets is $4s_{n-1}$.

For $n > 0$ we set $k_n = q$ and we define $B_{n,k}$, for $1 \le k \le q$, as follows:

1. $B_{n,k}(D_{n-1,0}) = B_{n-1,1}$,
2. $B_{n,k}(D_{n-1,v}) = B_{n-1,i}$ if $v = p_{n-1}(-l_n+i-1,1)$, for $i \in \{2,..,q\}$,
3. $B_{n,k}(D_{n-1,v}) = B_{n-1,i}$, for $v \in \partial S_{n,i}$, with $i \in \{1,..,q\}$.
4. $B_{n,k}(D_{n-1,v}) = B_{n-1,k}$, for all $v \in S_n$ such that $B_{n,k}(D_{n-1,v})$ was not defined in the previous steps.

$B_{n,3}$	$B_{n,1}$	$B_{n,2}$	$B_{n,3}$	$B_{n,1}$	$B_{n,2}$	$B_{n,3}$
$B_{n,2}$						$B_{n,2}$
$B_{n,1}$	$B_{n,2}$	$B_{n,3}$				$B_{n,1}$
$B_{n,3}$			$B_{n,1}$			$B_{n,3}$
$B_{n,2}$						$B_{n,2}$
$B_{n,1}$						$B_{n,1}$
$B_{n,3}$	$B_{n,1}$	$B_{n,2}$	$B_{n,3}$	$B_{n,1}$	$B_{n,2}$	$B_{n,3}$

FIGURE 2. For $q = 3$, $s_n = 2$ and $k \in \{1, 2, 3\}$, the picture represents the block $B_{n+1,k}$ if we consider that every empty square corresponds to the block $B_{n,k}$.

Point (3) from the construction insures that statement (2) of Lemma 29 is satisfied. The existence of $w \in D_1 \setminus \{0\}$ such that $B_{1,k}(v + w) = B_{1,k'}(v)$ for some $1 \leq k, k' \leq k_1$ and for every $v \in D_n$ satisfying $v + w \in D_1$, contradicts (3) and (4) from the construction. Using the same argument for $n > 1$, it is possible to show by an induction argument that statement (1) of Lemma 29 is also satisfied. Thus we conclude that $\{\mathcal{P}_n\}_{n\geq 0}$ is a sequence of partitions spanning the topology of $\Omega_{\mathbb{Z}^2}(\bar{x})$, and by Lemma 30, the set $\mathcal{M}(\Omega_{\mathbb{Z}^2}(\bar{x}))$ is given by $\lim_{\leftarrow n}(\triangle_n, A_n)$.
Let $n \geq 0$. In this case, the incidence matrix $A_n \in \mathcal{M}_{q\times q}(\mathbb{N})$ between $\mathcal{P}_n$ and $\mathcal{P}_{n+1}$ is given by

$$A_n(i, j) = \begin{cases} 4s_n + 1 & \text{if } j \neq i \\ q_n^2 - (q - 1)(4s_n + 1) & \text{if } j = i \end{cases}$$

For $i \in \{1, .., q\}$. For $m \geq n$ and $j \in \{1, .., q\}$, we define $u_m^{(j)} = \frac{1}{p_m^2} e_j$, where e_j is the j-th unitary vector in $\mathbb{R}^q$. Simple computations yields

$$A_n \cdots A_m u_{m+1}^{(j)} = \frac{1}{qp_n^2} \left[\begin{pmatrix} 1 \\ \vdots \\ \vdots \\ \vdots \\ 1 \end{pmatrix} + l_{n,m} \begin{pmatrix} -1 \\ \vdots \\ q-1 \\ \vdots \\ -1 \end{pmatrix} \right]$$

where $l_{n,m} = \frac{(q_n^2 - q(4s_n+1))\cdots(q_m^2 - q(4s_m+1))}{q_n^2 \cdots q_m^2}$. Notice that $\{l_{n,m}\}_{m \geq n}$ is a decreasing sequence, then it converges to some $\alpha_n \in [0,1]$, and so,

$$\lim_{m\to\infty} A_n \cdots A_m u_{m+1}^{(j)} = u^{(n,j)} = \frac{1}{qp_n^2}\left[(1-\alpha_n)\begin{pmatrix} 1 \\ \vdots \\ \vdots \\ \vdots \\ 1 \end{pmatrix} + q\alpha_n e_j\right]$$

The points $u^{(n,1)}, \cdots, u^{(n,q)}$ generate the convex $\bigcap_{m\geq n} A_n \cdots A_m \triangle_{m+1}$. By choosing $q_n^> \frac{4q}{1-\delta^{\frac{1}{2^n}}}$ for all $n \geq 0$, where δ is some point in $(0,1)$, we have that $\alpha_n > 0$ for all $n \geq 0$. This implies that for all $n \geq 0$, $u^{(n,1)}, \cdots, u^{(n,q)}$ are linearly independent vectors and then, they are the extreme points of $\bigcap_{m\geq n} A_n \cdots A_m \triangle_{m+1}$. Since $A_n u^{(n+1,j)} = u^{(n,j)}$ for all $j \in \{1,..,q\}$, we have that $u^{(1)} = \{u^{(n,1)}\}_{n\geq 0}, \cdots, u^{(q)} = \{u^{(n,q)}\}_{n\geq 0}$ are the extreme points of $\lim_{\leftarrow n}(\triangle_n, A_n)$. Thus $(\Omega_{\mathbb{Z}^2}(\bar{x}), \mathbb{Z}^2)$ has exactly q ergodic measures. If the sequence $\{q_n\}_{n\geq 0}$ is constant then $\alpha_n = 0$ for all $n \geq 0$, which implies that in this case $(\Omega_{\mathbb{Z}^2}(\bar{x}), \mathbb{Z}^2)$ is uniquely ergodic.

8.2. **An example of a uniquely ergodic $\mathbb{Z}^2$-Toeplitz system with positive entropy.** We take $k_0 = q$, $q_0 = k_0 + 2$, $k_n = f(k_{n-1})$ and $q_n = k_n + 2$ for $n > 0$, where $f : \mathbb{N} \to \mathbb{N}$ is the function defined by

$$f(n) = \frac{(n^2-1)!}{(n+1)!^{n-1}}, \text{ for all } n \in \mathbb{N}.$$

Remark 31. Observe that $f(n)$ is the number of partitions $\mathcal{P} = \{A_i\}_{i=1}^{n-1}$ of a set A with $|A| = n^2 - 1$ such that $|A_i| = n+1$ for all $i \in \{1,..,n-1\}$.

Given the alphabet $\Sigma = \Sigma_0$, consider the subset Σ_n of Σ^{D_n} such that $B \in \Sigma_n$ if and only if:

(1) $B(D_{n-1,0}) = B_{n-1,1}$,
(2) $B(D_{n-1,v}) \in \Sigma_{n-1} \setminus \{B_{n-1,1}\}$ for $v \in S_n \setminus (\{0\} \cup \partial S_n)$,
(3) $B(D_{n-1,v}) = B_{n-1,k_{n-1}}$ for all $v \in \partial S_n$,
(4) $|\{v \in S_n \setminus (\{0\} \cup \partial S_n) : B(D_{n-1,v}) = B_{n-1,l}\}| = k_{n-1} + 1$, for all $l \in \{2,..,k_{n-1}\}$

From remark 31, we easily see that $|\Sigma_n| = k_n$.

The point (3) from the construction insures that statement (2) of Lemma 29 is satisfied. We have that $B_{n,k}(D_{n-1,v}) = B_{n-1,1}$ if and only if $v = 0$. This and (1) from the construction imply that statement (1) of Lemma 29 is satisfied. Thus $\{\mathcal{P}_n\}_{n\geq 0}$ is a sequence of partitions which spans the topology of $\Omega_{\mathbb{Z}^2}(\bar{x})$, and by Lemma 30, the set $\mathcal{M}(\Omega_{\mathbb{Z}^2}(\bar{x}))$ is given by $\lim_{\leftarrow n}(\triangle_n, A_n)$.

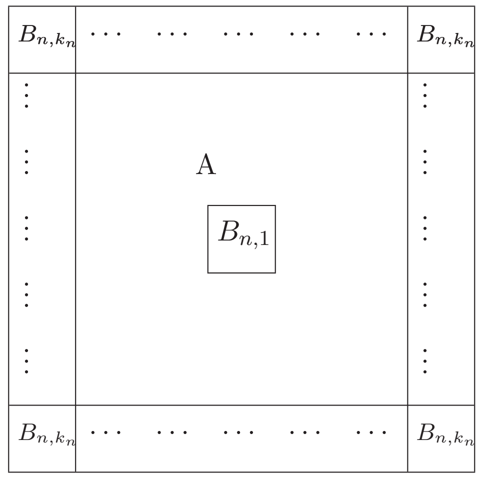

FIGURE 3. The blank region A is filled by a concatenation $k_n^2 - 1$ blocks from Σ^{D_n}. The block shown above belongs to Σ_{n+1} if the concatenation filling A uses exactly $k_n + 1$ copies of every block from $\Sigma_n \setminus \{B_{n,1}\}$

Let $n \geq 0$. The incidence matrix $A_n \in \mathcal{M}_{k_n \times k_{n+1}}(\mathbb{N})$ between $\mathcal{P}_n$ and $\mathcal{P}_{n+1}$ is given by

$$A_n(i,j) = \begin{cases} 1 & \text{if } i = 1 \\ k_n + 1 & \text{if } i = 2, \cdots, k_n - 1 \\ 5k_n + 5 & \text{if } i = k_n \end{cases}$$

For $j \in \{1, \cdots, k_{n+1}\}$, since $A_n \triangle_{n+1} = \{\frac{1}{p_{n+1}^2}(1, k_n + 1, \cdots, k_n + 1, 5k_n + 5)^T\}$, we have

$$\lim_{\leftarrow n}(\triangle_n, A_n) = \{(\frac{1}{p_{n+1}^2}(1, k_n + 1, \cdots, k_n + 1, 5k_n + 5)^T)_{n \geq 0}\}.$$

This implies that $(\Omega_{\mathbb{Z}^2}(\bar{x}), \mathbb{Z}^2)$ is uniquely ergodic with unique invariant probability measure $\mu \in \mathcal{M}(\Omega_{\mathbb{Z}^2}(\bar{x}))$ defined by

$$\mu(C_{i,j}) = \begin{cases} \frac{1}{p_{i+1}^2} & \text{if } j = 1 \\ \frac{k_i+1}{p_{i+1}^2} & \text{if } j = 2, \cdots, k_i - 1 \\ \frac{5k_i+5}{p_{i+1}^2} & \text{if } j = k_i \end{cases}$$

For every $i \geq 0$.

Consider $\mathcal{U}$ and $\mathcal{V}$, two open coverings of $\Omega_{\bar{x}}$. We define $N(\mathcal{U}) = \min\{|\mathcal{U}'| : \mathcal{U}' \text{ is a subcovering of } \mathcal{U}\}$ and $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$. The topological entropy of $(\Omega_{\mathbb{Z}^2}(\bar{x}), \mathbb{Z}^2)$ is defined by

$$h_{top}(\Omega_{\mathbb{Z}^2}(\bar{x}), \mathbb{Z}^2) = \sup_{\mathcal{U}} \limsup_{n \to \infty} \frac{1}{|L_n|} \ln N(\bigvee_{v \in L_n} \mathcal{U} - v),$$

where $L_n=\{0,..,n-1\}^2$.
Notice that for $n\geq 0$ and $i\geq n$ we have that every $C_{i,j}$ is a different atom of the partition $\bigvee_{v\in L_{p_i}}\mathcal{P}_n-v$. This implies that

$$N(\bigvee_{v\in L_{p_i}}\mathcal{P}_n-v)\geq k_i$$

and then

$$h_{top}(\Omega_{\mathbb{Z}^2}(\bar{x}),\mathbb{Z}^2)\geq\limsup_{n\to\infty}\frac{\ln k_n}{p_n^2}=\limsup_{n\to\infty}\frac{\ln k_n}{((k_0+2)(k_1+2)\cdots(k_{n-1}+2))^2}. \tag{8.3}$$

We will prove that by choosing an appropriate $k_0=q$, $h_{top}(\Omega_{\mathbb{Z}^2}(\bar{x}),\mathbb{Z}^2)>0$.
In the next Lemma we skip "+2" in each term of the equation (8.3). We will take care of this inaccuracy after the proof of the lemma.

Lemma 32. *If $q>5$ then there exists $a>0$ such that for all $n>0$,*

$$\frac{\ln(f^n(q))}{(qf(q)f^2(q)\cdots f^{n-1}(q))^2}>a.$$

Proof. Let $n>1$. By using Stirling's inequality we get

$$f(n)=\frac{(n^2-1)!}{(n+1)!^{n-1}}>(\frac{n^2-1}{e})^{n^2-1}\frac{(2\pi(n^2-1))^{\frac{1}{2}}}{(n+1)!^{n-1}}.$$

Since $(n+1)!<(n+1)^{n+1}$,

$$f(n)>(\frac{n-1}{e})^{n^2-1}(2\pi(n^2-1))^{\frac{1}{2}}. \tag{8.4}$$

By using that $(n^2-1)^{\frac{1}{2}}>n-1$ in (8.4) we obtain

$$f(n)>(\frac{n-1}{e})^{n^2}e(2\pi)^{\frac{1}{2}}>(\frac{n-1}{e})^{n^2}. \tag{8.5}$$

Let $q>2$ be an integer number. We call $q_n=f^n(q)$ and $S_{n+1}=\frac{\ln(q_{n+1})}{(q_0q_1q_2\cdots q_n)^2}$ for all $n\geq 0$. From (8.5) we deduce

$$S_n>\frac{\ln(\frac{q_{n-1}-1}{e})^{q_{n-1}^2}}{(q_0\cdots q_{n-1})^2}=\frac{\ln(q_{n-1}-1)}{(q_0\cdots q_{n-2})^2}-\frac{1}{(q_0\cdots q_{n-2})^2},$$

and finally

$$S_n>\frac{\ln(q_{n-1})}{(q_0\cdots q_{n-2})^2}-\frac{\ln(\frac{q_{n-1}}{q_{n-1}-1})}{(q_0\cdots q_{n-2})^2}-\frac{1}{(q_0\cdots q_{n-2})^2}. \tag{8.6}$$

We can use recursively (8.6) to get

$$S_n>\ln(q_0)-\left(\ln(\frac{q_0}{q_0-1})+\sum_{i=1}^{n-1}\frac{\ln(\frac{q_i}{q_i-1})}{(q_0\cdots q_{i-1})^2}\right)-\left(1+\sum_{i=1}^{n-1}\frac{1}{(q_0\cdots q_{i-1})^2}\right) \tag{8.7}$$

Since for every $i > 0$ it holds $\ln(\frac{q_0}{q_0-1}) > \ln(\frac{q_i}{q_i-1})$, then

$$-\left(\ln(\frac{q_0}{q_0-1}) + \sum_{i=1}^{n-1} \frac{\ln(\frac{q_i}{q_i-1})}{(q_0\cdots q_{i-1})^2}\right) > -\ln(\frac{q_0}{q_0-1})\left(1+\sum_{i=1}^{n-1}\frac{1}{(q_0\cdots q_{i-1})^2}\right) \tag{8.8}$$

On the other hand $-\frac{1}{q_0^2} < -\frac{1}{q_i^2}$ for $i > 0$, which implies that

$$-\sum_{i=1}^{n-1}\frac{1}{(q_0\cdots q_{i-1})^2} > -\sum_{i=1}^{n-1}\frac{1}{(q_0^2)^i}. \tag{8.9}$$

From (8.7), (8.8) and (8.9) we get

$$S_n > \ln q_0 - \left(\ln(\frac{q_0}{q_0-1})+1\right)\sum_{i=0}^{n-1}\left(\frac{1}{q_0^2}\right)^i$$
$$= \ln q_0 - \left(\ln(\frac{q_0}{q_0-1})+1\right)\left(\frac{q_0^2}{q_0^2-1}\right)\left(1-\left(\frac{1}{q_0^2}\right)^n\right)$$

Finally we have

$$\lim_{n\to\infty}\frac{\ln(q_n)}{(q_0\cdots q_{n-1})^2} \geq \ln q_0 - \left(\ln(\frac{q_0}{q_0-1})+1\right)\frac{q_0^2}{q_0^2-1}. \tag{8.10}$$

Not hard computations show that $\ln q_0 - (\ln(\frac{q_0}{q_0-1})+1)\frac{q_0^2}{q_0^2-1} > 0$ if $q_0 > 5$ which proves the lemma. □

For $n > 0$ we set

$$x_n = \frac{\ln k_n}{(k_0+2)(k_1+2)\cdots(k_{n-1}+2))^2},\ x'_n = \frac{\ln k_n}{(k_0\cdots k_{n-1})^2},$$

and $y_n = \frac{x_n}{x'_n} = \Pi_{i=0}^{n-1}\left(\frac{k_i}{(k_i+2)}\right)^2$. From the proof of Lemma 32 we deduce that, for all $n \geq 3$

$$\frac{f(n)}{f(n)+2} > \frac{1}{1+2(\frac{e}{n-1})^{n^2}}. \tag{8.11}$$

For $n \geq 7$ we have

$$\frac{1}{1+2(\frac{e}{n-1})^{n^2}} > \frac{2^{n^2}}{2^{n^2}+1}. \tag{8.12}$$

We can prove by induction that for $n \geq 2$

$$\frac{2^{n^2}}{2^{n^2}+1} > \left(\frac{1}{2}\right)^{\frac{1}{2^n}}. \tag{8.13}$$

From (8.11),(8.12) and (8.13), if $k_0 \geq 7$ then $\frac{k_n}{k_n+2} \geq (\frac{1}{2})^{\frac{1}{2^{k_{n-1}}}}$ for all $n > 0$. So,

$$y_n^{\frac{1}{2}} > \left(\frac{1}{2}\right)^{\sum_{j=0}^{n-1}\frac{1}{2^{k_j}}} \geq \left(\frac{1}{2}\right)^{\sum_{j=0}^{k_{n-1}}\frac{1}{2^j}} \geq \frac{1}{4}.$$

Thus, by Lemma 32, there exists $a > 0$ such that for all $n > 0$

$$\frac{\ln k_n}{(k_0+2)(k_1+2)\cdots(k_{n-1}+2))^2} > a.$$

Which implies, by (8.3), that $h_{top}(\Omega_{\mathbb{Z}^2}(\bar{x}), \mathbb{Z}^2) > 0$ if we take $q \geq 7$.

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