p-Box: A “new” graph model

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Wireless Sensor Networks
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Related Graphs Classes

- **Intersection Graphs:** $G = (V, E)$

  $V \rightarrow \mathcal{F}$ (Interval, Disk, Boxes)

  $u \rightarrow F_u$

  $uv \in E \iff F_u \cap F_v \neq \emptyset$

- **Tolerance Graphs:**

  $u \rightarrow (l_u, t_u)$

  $uv \in E \iff |l_u \cap l_v| \geq \min\{t_u, t_v\}$. 
Related Graphs Classes

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  \[ V \rightarrow \mathcal{F} \text{ (Interval, Disk, Boxes)} \]
  \[ u \mapsto F_u \]
  \[ uv \in E \iff F_u \cap F_v \neq \emptyset \]

- **Max-Tolerance Graphs:**
  \[ u \mapsto (l_u, t_u) \]
  \[ uv \in E \iff |l_u \cap l_v| \geq \max\{t_u, t_v\}. \]
The Model

- **p-Box (1):**

\[
\nu \quad \leftrightarrow \quad (I_v, p_v)
\]

\[uv \in E \iff (p_v \in B_u) \land (p_u \in B_v)\]
The Model

- **p-Box (1):**
  \[ v \leftrightarrow (l_v, p_v) \]
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- **c-p-Box:**
  p-Box model where \( p_v \) is the center of its interval \( l_v \).
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- c-p-Box:
  p-Box model where \( p_v \) is the center of its interval \( l_v \).

The set \( \{(l_v, p_v)\}_{v \in V} \) is called \((c-)p-Box\) (1) realization of \( G \).
Today

Boxicity(2) → p-Box(1) → c-p-Box(1) → Block

Cyclic segment → Max-Tolerance → Outerplanar

2DORG

Rooted directed path → Interval
Related Work

- H. Maehara ['84]: **Interval catch digraph** $vu \in D \iff p_v \in l_u$
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\[ L_1 \]
\[ L_2 \]
\[ L_3 \]
\[ L_4 \]
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Point-Tolerance $= p - Box(1) \subsetneq DCS \subsetneq SRG$
Related Work

- **Max-Weighted-Independent-Set**
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  - NP-complete for **Box(2)**

Hixon ['13]: \( G \) is NP-complete.

Correa et al ['13]: MHS/MIS^2 [3, 2, 2]. (Wegner conjecture)
Related Work

- **Max-Weighted-Independent-Set**
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  - Linear for Interval Graph (Chordal Graph)
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- Hixon['13]: $\chi(G)$ is NP-complete.
- Correa et al['13]: MHS/MIS $\in [3/2, 2]$. (Wegner conjecture)
Intersection Model

\[ (p_1; p_1) \]
\[ (p_2; p_2) \]
\[ (p_3; p_3) \]
\[ (p_4; p_4) \]

Kaufmann et al. [SODA'06]:

Max-tolerance = Intersection of isosceles triangles

\[ c_{-p-Box}(1) \]
Intersection Model

\[ B_1 \quad B_2 \quad B_3 \quad B_4 \]

\[ (p_1, -p_1) \]

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\[ c - p - \text{Box} (1) \]
Intersection Model

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Max-tolerance $c$-p-Box (1).

$(p_1, -p_1)$

Graph:
1 -- 2
  
1 -- 3
  
3 -- 4

Line segments:

- $B_1$
- $B_2$
- $B_3$
- $B_4$
Intersection Model

Kaufmann et al. [SODA'06]:

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Max-tolerance = p-box (1).
Kaufmann et al. [SODA'06]:

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Max-tolerance $c$-p-Box (1).

Intersection Model
Intersection Model

- Kaufmann et al. [SODA’06]: \textbf{Max-tolerance $=$ Intersection of isosceles triangles}

\textbf{Max-tolerance $\supset c$-p-Box(1).}
Combinatorial Characterization

Positions of representative points $p_v$ induce an order of $V$.

\[ x \rightarrow u \rightarrow v \rightarrow y \]
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Positions of representative points $p_v$ induce an order of $V$.

- Interval graphs: [S. Olariu '91]

\[
\begin{align*}
\text{Original Order:} & \quad x \rightarrow u \rightarrow v \rightarrow y \\
\text{Reordered Order:} & \quad x \rightarrow u \rightarrow v \rightarrow y
\end{align*}
\]
Combinatorial Characterization

Positions of representative points $p_v$ induce an order of $V$.

- Interval graphs: [S. Olariu '91]

- Outerplanar: [T. Bilski '92] Page number 1.
Combinatorial Characterization

Positions of representative points $p_v$ induce an order of $V$.

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- Outerplanar: [T. Bilski '92] Page number 1.

We will prove: \{Interval, Outerplanar\} $\in$ c-p-Box (1)
Combinatorial Characterization cont.

**Rooted directed path graph**: intersection graphs of directed paths in a rooted directed tree
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**Rooted directed path graph**: intersection graphs of directed paths in a rooted directed tree

An inverse DFS on the tree is \((j \ i \ h \ g \ f \ e \ d \ c \ b \ a)\) inducing the order of the vertices of the graph: \((t \ z \ y \ w \ v \ x \ u)\)
The $c$-$p$-$Box$ (1) class
Some Definitions

- Given an order $\pi$ of the vertex, we note by:
  - $\ell_\pi(v)$ the most left neighbor of $v$.
  - $\rho_\pi(v)$ the most right neighbor of $v$. 

Given a realization of $G$:
- $L(v)$ denotes the left extreme of $I_v$
- $R(v)$ denotes the right extreme of $I_v$
- $v$ is safe if its position $p_v$ belongs only to its neighbors' intervals.
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Theorem

\text{Interval} \subset \text{c-p-Box (1)}.
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[S. Olariu 1991]
Interval Graphs and \( c-p \)-Box (1)

**Theorem**

\( \text{INTERVAL} \subset \text{c-p-Box} \ (1) \).

[S. Olariu 1991]

We greedily construct a realization according to order such that at step \( i \):

1. \( p_{k-1} < p_k \)
2. \( \rho(j) <_{\pi} \rho(k) \Rightarrow R(j) < R(k) \)
3. \( L(j) < p_{\ell(j)} \)
4. \( \rho(k) <_{\pi} j \Rightarrow R(k) < p_j \)
Theorem

\textbf{Interval} $\subset c$-$p$-Box (1).

[S. Olariu 1991]

We greedily construct a realization according to order s.t. at step $i$:

1. $p_{k-1} < p_k$
2. $\rho(j) <_\pi \rho(k) \Rightarrow R(j) < R(k)$
3. $L(j) < p_{\ell(j)}$
4. $\rho(k) <_\pi j \Leftrightarrow R(k) < p_j$

\textbf{First,} set position $p_i$ after $p_{i1}$ and s.t. is contained only by intervals associated to its previous neighbors.

\textbf{Second,} set the interval $I_i$ s.t. it contains all its previous neighbors.

\textbf{Finally,} we modify, if necessary, the interval of previous vertices in order to satisfy conditions 2.
Interval Graphs and $c$-$p$-Box (1)

**Theorem**

**Interval $\subset c$-$p$-Box (1).**

[S. Olariu 1991]

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**First**, set position $p_i$ after $p_{i_1}$ and s.t. is contained only by intervals associated to its previous neighbors.

**Second**, set the interval $l_i$ s.t. it contains all its previous neighbors.

**Finally**, we modify, if necessary, the interval of previous vertices in order to satisfy conditions 2.

All vertices are safe with respect to previous neighbors.
Outerplanar Graphs and $c$-$p$-Box (1)

Theorem

Outerplanar $\subseteq c$-$p$-Box (1).

- Non trivial biconnected components are dissections of polygons.

- Cycles are in $c$-$p$-Box (1)
- How to “glue” two cycles by an edge
- How to “glue” two biconnected components by a vertex
$C_n \in \text{c-p-Box}(1)$. Moreover, if $\pi$ denotes the permutation induced by a realization Then, there exists a clockwise (or anticlockwise) labeling $l : V \rightarrow \{1, 2, \ldots, n\}$ such that:

1. $\pi(l^{-1}(1)) = 1 \land \pi(l^{-1}(n)) = n$. 
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2. p-Box (1): $\forall u \in V, |l(u) - \pi(u)| \leq 1$.
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Extremes vertices are safe!
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Extremes vertices are safe!
Outerplanar and $c$-$p$-$Box$ (1)

We can “glue” cycles by an edge in A DFS of weak dual
We construct a realization according to a BFS on the Block-tree of $G$ scaling biconnected component.
p-Box (1) \ c-p-Box (1)
Any $H^{x, y, z}$ graph such that $l_z > l_y - 1$ does not belong to $c$-p-Box (1).
Any $H_{l_x, l_y, l_z}$ graph such that $l_z > 3$ does not belong to $p$-Box $(1)$. 
• Any $H^{l_x,l_y,l_z}$ graph such that $l_z \geq l_y \geq l_x \geq 2$ does not belong to $c$-$p$-$\text{Box}$ (1).
• Any $H^{l_x, l_y, l_z}$ graph such that $l_z \geq l_y \geq l_x \geq 2$ does not belong to $c$-p-Box (1).

• Any $H^{l_x, l_y, l_z}$ graph such that $l_z \geq l_y \geq l_x > 3$ does not belong to p-Box (1).
Future Work and Open Questions

- Combinatorial characterization for $c$-$p$-Box (1).
Future Work and Open Questions

- Combinatorial characterization for \( c-p-\text{Box} \) (1).
  - Given an order?
Future Work and Open Questions

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  - Unit-p-Box
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- Related graph classes
  - Unit-$p$-Box
  - $p$-Box with arcs
Duality gap p-Box (1), Correa et al. ['13]

- $\text{MIS}(G) \geq \max\{|I_x|, |I_y|\} = \max\{|H_x|, |H_y|\}$
- $\text{MHS}(G) \leq H \leq |H_x| + |H_y| \leq 2\text{MIS}(G)$
WMIS in p-Box (1), Cantazaro et al.[’13]

- $opt[u, v] = \max_{B_i \in [a, b]} \{ opt[a, B_i] + w(B_i) + opt[i, b] \}$
- $O(n^2)$ pairs computed in $O(n)$
c-p-Box with given order

\[
\begin{align*}
\text{min} & \quad \sum_{i \in V} r_i \\
\text{s.t.} & \quad x_i - x_{\ell_\pi(i)} \leq r_i - \varepsilon_1 \quad 1 \leq i \leq n \\
& \quad x_{\rho_\pi(i)} - x_i \leq r_i - \varepsilon_1 \quad 1 \leq i \leq n
\end{align*}
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& \quad x_{\rho_{\pi}(i)} - x_i \leq r_i - \varepsilon_1 \quad 1 \leq i \leq n \quad (3) \\
& \quad x_j - x_i \geq r_j \quad 1 \leq i < \pi_j \leq n, \ ij \not\in E, \ i \leftrightsquigarrow j \quad (4) \\
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& \quad x_j - x_i \leq r_j & 1 \leq i < \pi j \leq n, \ ij \notin E, \ i \not\sim j \quad (7) \\
& \quad x_{i+1} - x_i \geq \varepsilon_2 & 1 \leq i \leq n - 1 \quad (8) \\
& \quad x_1 = 0, \ x_n = L 
\end{align*}\]
Containment Relations

- Cyclic segment
- Max-Tolerance
- Outerplanar
- Boxcity(2) → p-Box(1) → c-p-Box(1) → Block
- 2DORG
- Rooted directed path → Interval