Secretary Problems

October 21, 2010

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SPAMS

A little history

50's:	Problem appeared.
60's:	Simple solutions: Lindley, Dynkin.
70-80:	Generalizations. It became a "field". ("Toy problem" for Theory of Optimal Stopping)
80-now:	Generalizations to Game Theory, Decision Theory, Combinatorial Optimization, etc...
Other:	Cayley's problem (1875). Gardner's Googol Problem (1960)

Rules.

1. Want to choose **one** object.
2. Number $\mathbf{n}$ of objects is known.
3. Objects appear sequentially in **uniform random order**.
4. Objects are **rankable**.
5. Accepted or rejected **before the next object appears**.
6. Decision depend only on the **relative ranks.**
7. Rejected objects **can not be recalled**.
8. Payoff: Only win if the **best** is selected.

Toy Examples.

Secretary Problem:
Hire a secretary among n candidates.

Marriage Problem:
Getting married with one of n possible people.

Sultan's Dowry Problem:
A Sultan granted a commoner a chance to marry one of his n daughters, each one has a different dowry.

Random auction:
Sell an item to one of n agents.

Variations / Break the rules.

Normal	Variation
Select **one**	Combinatorial restrictions.
Known $\mathbf{n}$	Unknown $\mathbf{n}$.
Unif. Random order	**Dependent** order.
Object **rankable**	Limited **Confidence**.
Immediate decision	**Deferred** decision.
Rank-dependent decision	**Value** or **time** dependent decision.
Can not **recall** rejected	They accept with certain **prob.**
Win if **best** selected	Other **payoff functions**.

Classic variations:

- Cayley's problem: Urn with n balls labeled $1, \ldots, n$. Select at random, k chances. When to stop?
- Googol problem: Two player version.

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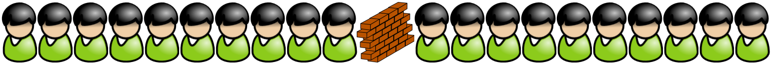

- Observe half and find X=best.
- Select first element $\mathbf{Y}$ better than X.

Win with probability at least $1/4$.

Can show (e.g. by Backwards Induction, or using properties of Markov Chains) that the optimum rule is of the form:

- For some $r(n)$, reject $r-1$ elements.
- Accept the first element of relative rank 1 (**RECORD**) after that.

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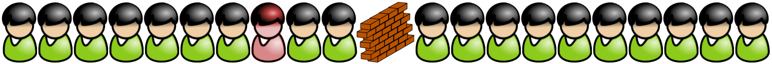

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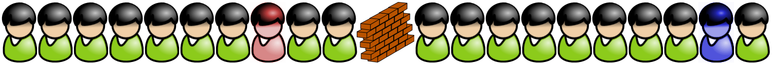

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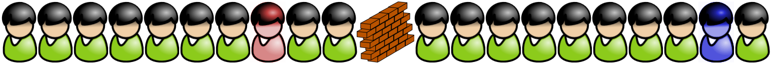

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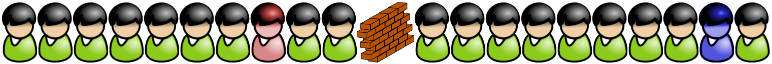

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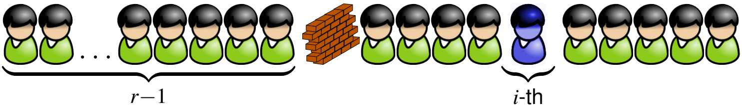

P_i = Probability that BEST is in i-th position AND we select it.
(We need that no element with relative rank 1 is after the wall.)

$$P_i = \begin{cases} 0 & \text{if } i \le r-1 \\ \frac{1}{n} \cdot \frac{r-1}{i-1} & \text{if } i \ge r. \end{cases}$$

$$\text{Prob. to WIN} = \sum_{i=r}^{n} P_i = \frac{r-1}{n} \sum_{i=r}^{n} \frac{1}{i-1}.$$

Maximized on the the maximum value of r such that

$$\frac{1}{n-1} + \cdots + \frac{1}{r-1} \ge 1 \quad \text{(roughly } r = n/e \text{ and Pr} = 1/e \approx 36.7\%\text{)}$$

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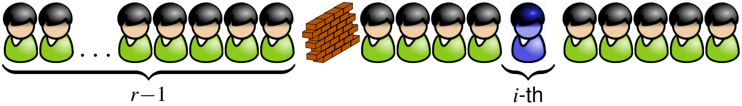

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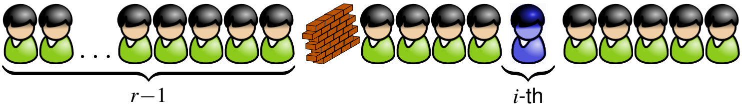

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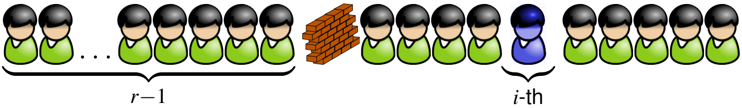

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Example, 100 daughter Sultan.

Reject first 37, accept the first **RECORD** you see.

What to do if n is unknown.

 $\cdots$???

- If n is random w/known distribution. Can we WIN?
- Example: n uniform from $\{1, 2, \ldots, N\}$.
- Surprisingly, BEST STRATEGY is still to wait until a fixed position $r(N)$ and select first **RECORD** you see.
- For large N, $r \approx N \cdot e^{-1/2}$ and Pr(WIN)$\approx 2e^{-2} \approx 27.1\%$.
- For general distributions the best strategy is of the form: Select the first **RECORD** in positions

$$[a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_k, N].$$

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- Can we WIN? (Optimal strategy is not simple.)
- Next one is almost optimal: Every time you see a **RECORD**, accept it with prob. p.
- What p should we choose?
- $\mathbb{E}[\#Records(n)] = \sum_{i=1}^{n} 1/i = H_n \leq H_N$.
- By choosing $p = 1/H_N$ we get

$$\text{Pr(WIN)} \geq 1/(eH_N).$$

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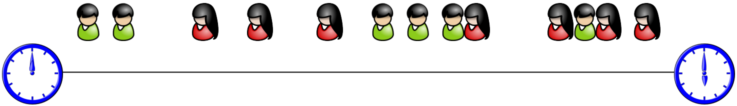
Example, selecting from at most 100 secretaries.

The previous strategy has prob. $\geq 1/(eH_{100}) \approx 7\%$ to win.
The best strategy has prob. $\geq 1/H_{100} \approx 19\%$ to win.

Another case: unknown n arriving in a fix time period.



- Assume people arrive in interval $[0, 1]$ independently.
- May assume also uniformly.
- Can **NOT** beat probability $1/e$ to win. Can we achieve it?

Fix a wall at time T and select first **RECORD** after T.

Let t be the time at which **BEST** arrives.

$$\Pr(\text{Win}) \geq \mathbb{E}_t \left[1_{\{t > T\}} \frac{T}{t} \right] = \int_T^1 \frac{T}{t} dt = -T \ln T.$$

Optimum at $T = 1/e$, with $\Pr(\text{Win}) = 1/e \approx 36.7\%$.

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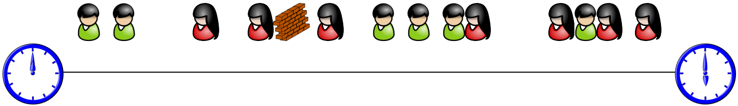

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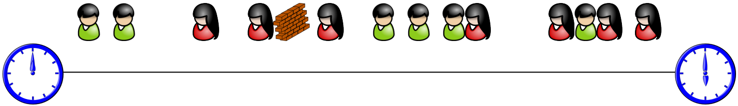

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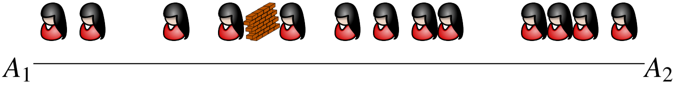

- Looking from age A_1 to age A_2, roughly uniform.
- Only getting married once, and only happy with the best choice.
- Must decide over a candidate before the next arrive.
- Best strategy: "Sample" until $A_1 + (A_2 - A_1)/e$ and then choose first **Record**. Win with prob. $1/e \approx 36.7\%$.
- Example: If $A_1 = 18$, $A_2 = 40$, then wall is at $18 + 22/e \approx 26$.

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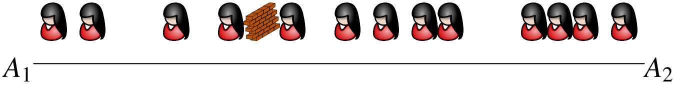

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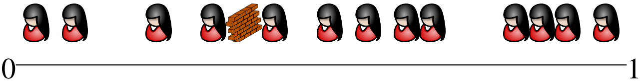

$$\text{Utility} = \begin{cases} a & \text{marry the BEST} \\ b & \text{stay single} \\ c & \text{marry other.} \end{cases} \qquad \text{with } a \geq b \geq c.$$

Where should we put the wall T?

$$\Pr(\text{marry the BEST}) = -T \ln T.$$

$$\Pr(\text{stay single}) = T.$$

$$\Pr(\text{marry other}) = 1 - (T - T \ln T).$$

Hence $\mathbb{E}[\text{Utility}] = -aT \ln T + bT - c(1 - T + T \ln T)$

Optimum at $T = \exp(-(a-b)/(a-c))$.

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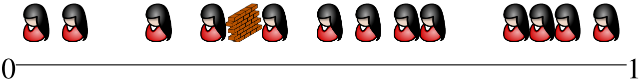

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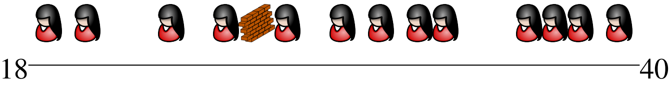

$$\text{Utility} = \begin{cases} 1 & \text{marry the BEST} \\ 0 & \text{stay single} \\ -1 & \text{marry other.} \end{cases} \qquad \text{with } a \geq b \geq c.$$

Optimum: $A = 18 + 22/\sqrt{e} \approx 31.3$.

$$\Pr(\text{marry the BEST}) = 1/(2\sqrt{e}) \approx 30.33\%.$$
$$\Pr(\text{stay single}) = 1/\sqrt{e} \approx 60.65\%.$$
$$\Pr(\text{marry other}) = 1 - 3/(2\sqrt{e}) \approx 9.02\%.$$

And $\mathbb{E}[\text{Utility}] = -1 + 2/\sqrt{e} \approx 0.213$.

Other utility functions.

Second Best

- Suppose (for some reason) that we WIN only if we select the 2nd one.
- Optimal strategy: "Sample" first $\lfloor (n+1)/2 \rfloor$ candidates, and then choose first element with relative rank 2.
- $\Pr(Win) \to 1/4$ as $n \to \infty$. Note that $1/4 < 1/e$, hence getting second one is HARDER.

Other utility functions.

Minimum Rank.

- Want to Minimize the expected rank of the element selected.
- Optimal strategy is complex. For each position i, define a threshold $H_n(i)$ a priori and select i if and only if the relative rank of it is at least $H_n(i)$
- For the best choices of $H_n(i)$, $\mathbb{E}[\text{rank}] \to 3.8695$ as $n \to \infty$.

$$H_n(i) = \left\lfloor \frac{i-1}{n+1} R_n(i+1) \right\rfloor .$$

$$R_n(i) = \frac{i - H_n(i)}{i} R_n(i+1) + \frac{n+1}{i(i+1)} \cdot \frac{H_n(i)(H_n(i)+1)}{2} .$$

$$H_n(n) = n, \qquad R_n(n) = \frac{n+1}{2} .$$

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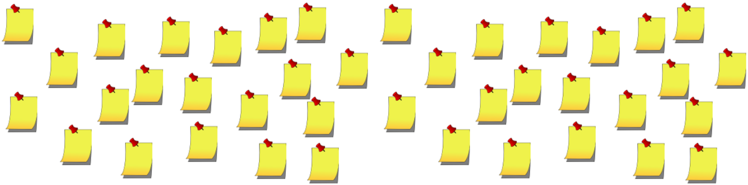
Game of Googol.



1. Player I writes n diff. numbers (in $[0, 1]$ or $[0, 10^{100}]$).
2. Player II uncovers them at random.
3. Player II wins if he stops on biggest. O.w, Player I wins.

- Player II has $\sim 1/e$ prob of winning for any strategy of Player I.
- Player II can now SEE the numbers, can he do better?

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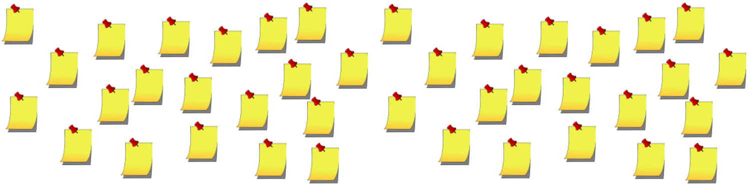

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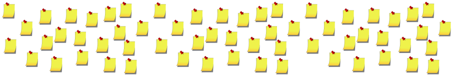
Game of Googol (for Player II and n large)



What if P.II. knows that...

- P.I. selects the numbers $\{1, \ldots, n\}$? He WINS with prob. 1
- P.I. selects numbers uniformly in $[0, 1]$ i.i.d.?
  Simple strategy: Select any number better than $1 - t/n$.
  $t \approx 1.5$ leads to $\Pr(\text{WIN}) \approx 51.7\%$.
  Best strategy: For every i, selects a threshold $t(i)$,
  Best selection gives $\Pr(\text{WIN}) \approx 58\%$.
- P.I. selects number i.i.d. from known distribution?
  Same as for uniform.

Can Player I decrease the probability of winning of P.II?

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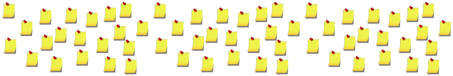

What if P.II. knows that...

- P.I. selects the numbers $\{1, \ldots, n\}$? He WINS with prob. 1
- P.I. selects numbers uniformly in $[0, 1]$ i.i.d.?
  Simple strategy: Select any number better than $1 - t/n$.
  $t \approx 1.5$ leads to $\Pr(\text{WIN}) \approx 51.7\%$.
  Best strategy: For every i, selects a threshold $t(i)$,
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- P.I. selects number i.i.d. from known distribution?
  Same as for uniform.

Can Player I decrease the probability of winning of P.II?

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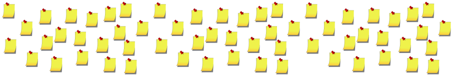

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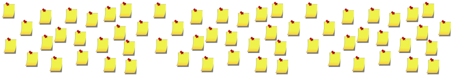

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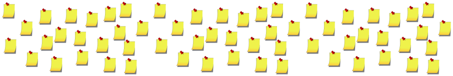

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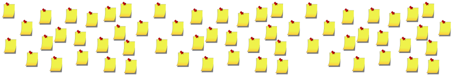

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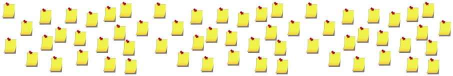

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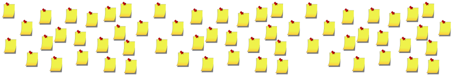

What if P.II. knows that...

- P.I. selects i.i.d. from an UNKNOWN distribution?
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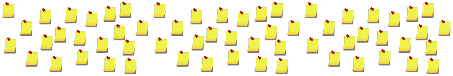

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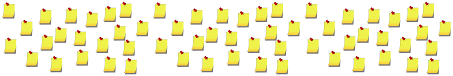

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- P.I. selects i.i.d. from a PARAMETRIC FAMILY of dist?
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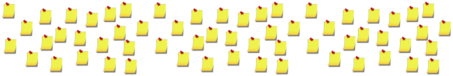

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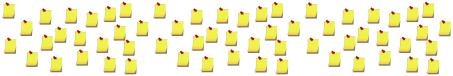

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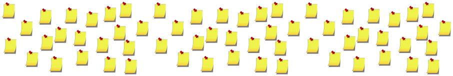

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- Combined distribution: Select $\theta \sim \mathrm{Pareto}(\alpha, 1)$, and n numbers from Uniform$[0, \theta]$.
  $\forall \varepsilon > 0, n$, there is α such that

$$\Pr(\text{P.II. wins}) \leq 1/e + \varepsilon.$$

Selecting more than one object

- Suppose we want to select more than one object.
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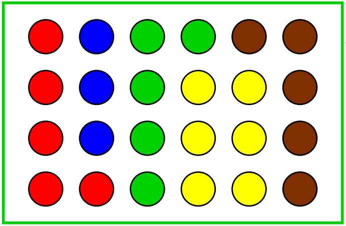
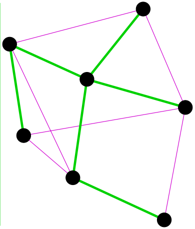
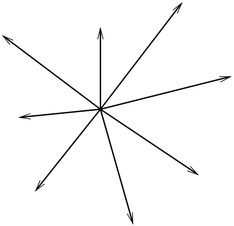
  Best strategy: Can recover $1 - \theta(1/\sqrt{k})$ fraction of the optimum.

Extension: Combinatorial objects.

Want to select a maximum weight...

1. ... set of balls of the same color.
2. ... spanning tree.
3. ... of linearly independent vectors.

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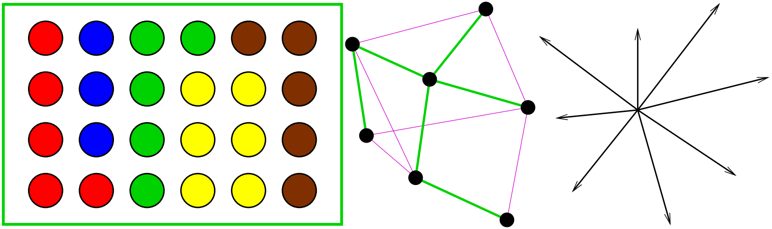

Want to select a maximum weight...

1. ... set of balls of the same color. $\omega(\log n / \log\log n)$-comp.
2. ... spanning tree. $O(1)$-comp.
3. ... of linearly independent vectors. $O(\log \text{dim.})$-comp

   OPEN: Find an $O(1)$-comp. algorithm.