Max-Min Relations in Combinatorial Optimization

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Finding a needle in a haystack.

Needle/Haystack
Bipartite Matching
König
Dilworth
Menger
Menger(2)
LP
Integrality
Symmetry
TUM and weights
Egerváry
Matching/Cover(2)
General TDI
More examples
Perfect graphs
Perfect graphs(2)
Lovász ϑ-function
Dualities
Conclusions
Finding a needle in a haystack.

- Search for an **Optimum** (Needle) in a huge collection of objects (Haystack).
- e.g. Maximum Matching in a Graph, minimum weight spanning trees.
- Want an algorithm that runs in **polynomial time** in the representation of the input.
- First problem... Once we found the object, how can we check if it is indeed the Needle?
E.g. Matching in bipartite graphs

- Find a Max. Matching.
E.g. Matching in bipartite graphs

- Find a **Max.** Matching.
- Matching of size 3. (optimum?)
E.g. Matching in bipartite graphs

- Find a Max. Matching.
- Matching of size 3. (optimum?)
- Matching \( \leq \) Vertex Cover.
E.g. Matching in bipartite graphs

- Find a Max. Matching.
- Matching of size 3. (optimum?)
- Matching ≤ Vertex Cover.
- Vertex Cover of size 3.

Optimum!
In fact, **König’s Theorem [König 1931]**

For every bipartite graph,

\[
\text{Maximum matching} = \text{Minimum vertex cover}.
\]

This **Max-Min** relation, guarantees us that the decision problem is in \(\text{NP} \cap \text{co-NP}\).

This is we can certificate if an element is the optimum or not. “Well-characterized” problems (Edmonds).

Can often give hints on how to find a polynomial algorithm.
Find maximum size antichain of a poset.
Dilworth’s Theorem [Dilworth 1950]

Maximum size of an antichain is equal to
Minimum number of chains needed to cover $P$. 
Edge disjoint \( s-t \) paths

**Max.** number of edge-disjoint paths between red vertices?
Edge disjoint $s-t$ paths

Max. number of edge-disjoint paths between red vertices?
Edge disjoint $s$-$t$ paths

Menger’s Theorem (edge version) [Menger 1927]

Maximum number of edge-disjoint paths between $s$ and $t$

is equal to

Minimum number of edges we need to cut to separate $s$ and $t$. 
**Vertex disjoint $s$-$t$ paths**

**Menger’s Theorem (vertex version)** [Menger 1927]

Maximum number of **vertex-disjoint paths** between $s$ and $t$

is equal to

Minimum number of **vertices we need to cut** to separate $s$ and $t$. 

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Linear programming helps

\[
\max |N| \quad \Rightarrow \quad \max \sum_{i} x_i
\]

\(N\) in the haystack.

\[
Ax \leq 1
\]

\[
x \geq 0.
\]

**Fractional** version of the problem.

**Relaxation:** Integral solutions are feasible for original problem.
Linear programming helps

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\max |N| \implies \max \sum_i x_i
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\[N\text{ in the haystack.}\]
\[Ax \leq 1\]
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**Fractional** version of the problem.

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**Linear program duality:**

Maximum Fractional Primal = Minimum Fractional Dual.

\[
\max \sum_i x_i = \min \sum_j y_j
\]

\[Ax \leq 1\]
\[A^T y \geq 1\]
\[x \geq 0\]
\[y \geq 0.\]
Linear programming helps

\[
\begin{align*}
\max |N| & \implies \max \sum_{i} x_i \\
\text{Ax} & \leq 1 \\
x & \geq 0.
\end{align*}
\]

\(N\) in the haystack.

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\begin{align*}
\max \sum_{i} x_i & = \min \sum_{j} y_j \\
Ax & \leq 1 \\
x & \geq 0 \\
A^{T}y & \geq 1 \\
y & \geq 0.
\end{align*}
\]

Solvable in polytime!*

[*]: Provided the description is polynomial or that we can solve the separation problem associated.
The previous examples are “nice”

\[
\text{Max. Integral Primal} \leq \text{Max. Fractional Primal} = \text{Min. Fractional Dual} \leq \text{Min. Integral Dual}.
\]
The previous examples are “nice”

Max. Integral Primal
\[ \leq \] Max. Fractional Primal
\[ = \] Min. Fractional Dual
\[ \leq \] Min. Integral Dual.

For the previous examples of packing “matchings and paths” or covering a poset with “chains” we get equality.
The previous examples are “nice”

Max. Integral Primal
\[ \leq \text{Max. Fractional Primal} \]
\[ = \text{Min. Fractional Dual} \]
\[ \leq \text{Min. Integral Dual}. \]

For the previous examples of **packing** “matchings and paths” or **covering** a poset with “chains” we get equality.

**Sufficient condition for equality**: Both polytopes are integral.
i.e. the vertices of
\[ \{Ax \leq 1, x \geq 0\} \text{ and } \{A^ty \leq 1, y \geq 0\} \]
are 0-1 vectors.
For the previous cases, not only we have equality, but we also have complementary, antiblocking or antiblocking duality. This means that, (in a way), we can interchange the role of what we are packing and covering to get new max-min relations.
Max. Number of independent edges

= 

Min. Number of vertices needed to cover all edges.
Max. Number of independent vertices

=  

Min. Number of edges needed to cover all vertices.
Aside- Complementary/Antiblocking/Blocking pairs

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Max. Size of an antichain

=\n
Min. Number of chains needed to cover $P$
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Max. Size of a chain

\[\text{Max. Size of a chain} = \text{Min. Number of antichains needed to cover } P.\]
Maximum number of disjoint paths between $s$ and $t$ is equal to

Minimum size of an cut between $s$ and $t$. 
The maximum number of disjoint cuts between $s$ and $t$ is equal to the minimum size of an path between $s$ and $t$. 
Usually we are not so lucky.

**Max. Integral Primal**

\[
\max \sum_i x_i = \min \sum_j y_j
\]

\[
Ax \leq 1 \quad \quad A^T y \geq 1
\]

\[
x \geq 0 \quad \quad y \geq 0.
\]

**Max. Fractional Primal**

\[
\leq \quad = \quad \leq
\]

**Min. Fractional Dual**

**Min. Integral Dual.**
Usually we are not so lucky.

Max. Integral Primal
\[ \leq \text{Max. Fractional Primal} \]
\[ = \text{Min. Fractional Dual} \]
\[ \leq \text{Min. Integral Dual}. \]

\[
\max \sum_i x_i = \min \sum_j y_j
\]
\[
A x \leq 1 \\
A^T y \geq 1
\]
\[
x \geq 0 \\
y \geq 0.
\]

To have equality through we need both to be integral...
This happens for instance when $A$ is a totally unimodular matrix
(i.e. all square subdeterminants are in $\{-1, 0, 1\}$).
Usually we are not so lucky.

Max. Integral Primal
\[ \leq \text{Max. Fractional Primal} \]
\[ = \text{Min. Fractional Dual} \]
\[ \leq \text{Min. Integral Dual}. \]

\[
\max \sum_i x_i = \min \sum_j y_j
\]
\[ Ax \leq 1 \quad \quad \quad A^T y \geq 1 \]
\[ x \geq 0 \quad \quad \quad \quad \quad y \geq 0. \]

In general, when the problem is written as above, \( A \) is not even a 0-1 matrix!
Usually we are not so lucky.

**Max. Integral Primal**
\[ \max \sum_i c_i x_i \]

\[ \leq \max \text{ Fractional Primal} \]
\[ = \min \sum_j b_j y_j \]

\[ \leq \min \text{ Fractional Dual} \]

\[ \leq \min \text{ Integral Dual}. \]

\[ Ax \leq b \]
\[ A^T y \geq c \]
\[ x \geq 0 \]
\[ y \geq 0. \]

In general, when the problem is written as above, \( A \) is not even a 0-1 matrix!

Go to general systems and allow costs/weights too.
In some cases, we still have both equalities.
Weighted versions with equality.

- (Egerváry 1931) **Max.** weight bipartite matching = **Min.** sum of potential in the nodes, such that each edge $e$ contains at least its weight in potential. [Integral weights implies integral potentials]

- **Max.** weight of an independent set in a connected bipartite matching = **Min.** sum of potential in the edges, such that the "potential degree of each node" is at least the weight of the node. [Integral weights implies integral potentials]

\[
\max \sum_{e \in E} w_e x_e = \min \sum_{v \in V} y_v
\]

\[
x(\delta(v)) \leq 1 \quad y_u + y_v \geq w_{uv}
\]

\[
x \geq 0, \quad y \geq 0.
\]
Weighted versions with equality.

- (Egerváry 1931) **Max.** weight bipartite matching = **Min.** sum of potential in the nodes, such that each edge $e$ contains at least its weight in potential. [Integral weights implies integral potentials]

- **Max.** weight of an independent set in a connected bipartite matching = **Min.** sum of potential in the edges, such that the "potential degree of each node" is at least the weight of the node. [Integral weights implies integral potentials]

$$\max \sum_{v \in V} w_v x_v = \min \sum_{e \in E} y_e$$

$x_u + x_v \leq 1$

$x \geq 0.$

$y(\delta(v)) \geq w_v$

$y \geq 0.$
(Egerváry 1931) \textbf{Max.} weight bipartite matching = \textbf{Min.} sum of potential in the nodes, such that each edge $e$ contains at least its weight in potential. [Integral weights implies integral potentials]

\textbf{Max.} weight of an independent set in a connected bipartite matching = \textbf{Min.} sum of potential in the edges, such that the "potential degree of each node" is at least the weight of the node. [Integral weights implies integral potentials] (Ford-Fulkerson 1956)

\textbf{Max.} flow = \textbf{Min.} cut. [Integral capacities implies integral flow/cut variables]
Matching vs. Cover in general graph

**Max. Matching**

\(< \text{Max.} \) “Fractional packing” of edges in a general graph

\(= \text{Min.} \) “Fractional Vertex Cover”.

\(< \text{Min.} \) Vertex Cover.

\[
\max \sum_{e} x_e = \min \sum_{v \in V} y_v
\]

\(x(\delta(v)) \leq 1\)

\(y_u + y_v \geq 1\)

\(x \geq 0.\)

\(y \geq 0.\)

Integrality Gap!
Matching vs. Cover in general graph

**Max. Matching**

< Max. “Fractional packing” of edges in a general graph

= Min. “Fractional Vertex Cover”.

< Min. Vertex Cover.

\[
\begin{align*}
\max \sum_{e} x_e &= \min \sum_{v \in V} y_v + \sum_{|U| \text{ odd}} z_U (|U| - 1)/2 \\
x(\delta(v)) &\leq 1 \\
x(E(U)) &\leq \frac{|U| - 1}{2}, (|U| \text{ odd}) \\
x &\geq 0.
\end{align*}
\]

\[
\begin{align*}
y_u + y_v + \sum_{U : uv \in E(U)} z_U &\geq 1 \\
y, z &\geq 0.
\end{align*}
\]

We can add inequalities to LP, to make it integral.
Max. Matching

= Max. “Fractional Primal”

= Min. “Fractional Dual”.

< Min. Vertex Cover.

\[
\max \sum_{e} x_e = \min \sum_{v \in V} y_v + \sum_{|U| \text{ odd}} z_U (|U| - 1)/2
\]

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x(\delta(v)) \leq 1
\]

\[
x(E(U)) \leq \frac{|U| - 1}{2}, (|U| \text{ odd})
\]

\[
x \geq 0.
\]

We can add inequalities to LP, to make it integral.
Max. Matching

= \text{Max. “Fractional Primal”}

= \text{Min. “Fractional Dual”}.

< \text{Min. Vertex Cover.}

Get the following \textbf{Max-Min} relation
\begin{equation}
\text{Max Matching} = \min_{U \subseteq V} \left( \frac{1}{2} |U| + |V| - \text{odd}(G - U) \right).
\end{equation}

What about Vertex Cover?
We can’t express the convex hull of vertex covers succinctly.

Vertex Cover is **NP-hard**

Still we have **Max. Matching ≤ Min. Vertex Cover.**

We can get a 2-approximation using this! (choose endpoints of a max. matching)

**Max.-Min.** relations can help to obtain approximate solutions too.
To find the **Maximum** object of a family, consider the polytope given by the convex-hull of the incidence vectors of the elements.

Express (if possible) the polytope as a succinct* collection of inequalities.

Use techniques as Total-Dual Integrality to prove that the polytope is integral.

If successful, LP duality gives **Max-Min** relation.
More examples

1. (Edmonds) In a digraph: \textbf{Min.} cardinality \( r \)-cut = \textbf{Max.} number of pairwise disjoint \( r \)-arborescences.
2. (Luchesi-Younger) In a planar digraph \( D \): \textbf{Min.} Number of edges we need to remove to make \( D \) acyclic (feedback set) = \textbf{Max.} number of pairwise arc disjoint directed cycles.
3. Matroid Intersection: \textbf{Max.} cardinality of a set independent in two matroids = \( \min_{U \subseteq S} r_1(U) + r_2(S \setminus U) \).

Other examples where we can characterize the “primal” or “dual” polytopes (yields \textbf{max-min} relationship)

1. \textbf{Max.} Packing of bases of a matroid. (e.g. spanning trees, invertible submatrices)
2. \textbf{Min.} Cover by bases of a matroid.
3. \textbf{Max.} \( b \)-matching.
4. \textbf{Min.} \( b \)-edge cover.
A graph is **perfect** if for every subgraph:

\[
\text{Max. Clique} = \text{Min. Chromatic Number.}
\]
A graph is **perfect** if for every subgraph:

\[
\text{Max. Clique} = \text{Min. Chromatic Number.}
\]

**Perfect Graph Theorem:** A graph is perfect iff it contains no odd holes and no odd antiholes.
If we are given a perfect graph. Is the \textit{max.-min.} relation enough to find a maximum clique efficiently?
If we are given a perfect graph. Is the \textbf{max.-min.} relation enough to find a maximum clique efficiently? Try the LP. In a perfect graph:

\textbf{Max. Clique = } \omega(G)

\[ \max \sum_{v} x_v = \min \sum_{S \text{ stable}} y_S \]

\[ x(S) \leq 1, \forall \text{ stable } S \]

\[ \sum_{S \ni v} y_S \geq 1 \]

\[ x \geq 0. \quad y \geq 0. \]

\[ = \chi(G) = \text{Min. Chromatic number.} \]
If we are given a perfect graph. Is the max.-min. relation enough to find a maximum clique efficiently? Try the LP. In a perfect graph:

**Max. Clique** = \( \omega(G) \)

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\max \sum_e x_v = \min \sum_{S \text{ stable}} y_S
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\[
x \geq 0.
\]

\[
y \geq 0.
\]

\[
= \chi(G) = \text{Min. Chromatic number.}
\]

**Problem:** LP has exponential inequalities and the separation problem is not easy (in fact, it reduces to compute a max. clique in the complement... which is perfect). What can we do?
Lovász $\varphi$-function

Lovász introduced the quantity:

$$\varphi(G) = \max \{ 1^T M 1 : M \in \mathcal{M}(G), \text{ positive semidefinite} \},$$

where

$$\mathcal{M}(G) = \{ M \in \mathbb{R}^{V \times V}, \text{ symmetric, } M_{u,v} = 0 \text{ if } uv \in E(G), \text{ Tr}(M) = 1 \}.$$  

This number can be found in polynomial time by solving a semidefinite program. And it is such that,

$$\omega(G) \leq \varphi(\bar{G}) \leq \chi(G).$$

For perfect graphs, we can use it to find a maximum clique.
We can use stronger tools to get \textbf{max-min} relations. (Useful for approximation algorithms)

**LP**

\[
\begin{align*}
\max c \cdot x & = \min \sum_j b_j y_j \\
a_i \cdot x & = b_i \\
\sum_{i=1}^m y_j a_j & \geq c \\
x & \geq 0.
\end{align*}
\]
We can use stronger tools to get max-min relations. (Useful for approximation algorithms)

**SDP**

\[
\begin{align*}
\max C \cdot X &= * \min \sum_{j} b_j y_j \\
a_i \cdot X &= b_i \quad \sum_{i=1}^{m} y_j A_j \geq c \\
X &\geq 0.
\end{align*}
\]

e.g: Problems about partitioning a set into two groups can be formulated in this fashion. Max-cut, Max-sat, etc.
We can use stronger tools to get \textbf{max-min} relations. (Useful for approximation algorithms)

**Conic Program**

\[
\begin{align*}
\max \langle c, x \rangle &= \star \min \sum_j b_j y_j \\
\langle a_i, x_i \rangle &= b_i \quad \sum_{i=1}^{m} y_j a_j \preceq K^* c \\
x &\succeq K 0.
\end{align*}
\]

Still solvable* in polynomial time.
We can use stronger tools to get max-min relations. (Useful for approximation algorithms)

What else?

- Under mild assumption can solve Convex Programs too.
- In general we only have weak duality. Can still certificate approximated solutions.
- In some (very particular) cases we can even approximate nonconvex programs.
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