Matroid Secretary Problem in the Random Assignment Model

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Classical / multiple-choice Secretary Problem

Rules

1. Given a set E of elements with **hidden nonnegative weights.**

2. Each element reveals its weight in **uniform random order.**
3. We accept or reject **before the next weight is revealed.**
4. Maintain a feasible set: Set of size at most r.

5. Goal: Maximize the sum of weights of selected set.

Matroid secretary problem

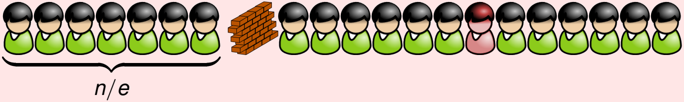
Babaioff, Immorlica, Kleinberg [SODA07]

Rules

1. Given a set E of elements with **hidden nonnegative weights.**
   E is the ground set of a known matroid $\mathcal{M} = (E, \mathcal{I})$.
2. Each element reveals its weight in **uniform random order.**
3. We accept or reject **before the next weight is revealed.**
4. Maintain a feasible set: Set of size at most r.
   Feasible set = Independent Set in $\mathcal{I}$.
5. Goal: Maximize the sum of weights of selected set.

Algorithms for classical problem (uniform matroid).

For $r = 1$: Dynkin's Algorithm



- Observe n/e objects. Accept the first record after that.

Top weight is selected w.p. $\geq 1/e$.

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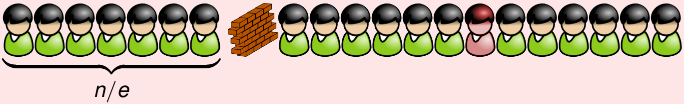

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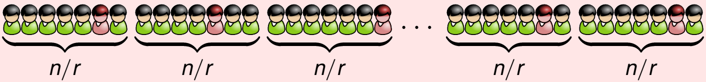

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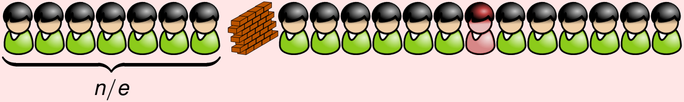

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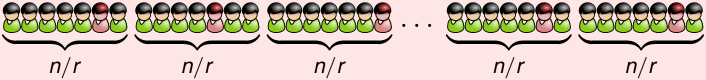

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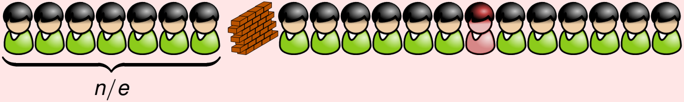

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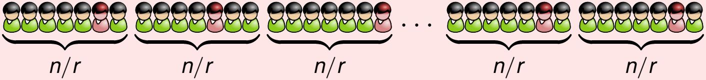

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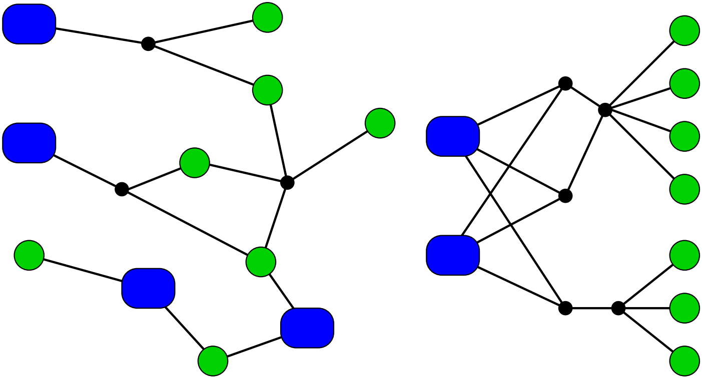
- e/C (constant) competitive algorithm.

Harder example: Gammoid



- Servers
- Clients ← Elements.
- Connections

Independent Sets:
Clients that can be simultaneously connected to Servers using edge-disjoint paths.

Models of Weight Assignment:

1. Adversarial Assignment:
   Hidden weights are arbitrary.
2. Random Assignment:
   A hidden (adversarial) list of weights is assigned uniformly.
3. Unknown distribution:
   Weights selected i.i.d. from unknown distribution.
4. Known Distribution:
   Weights selected i.i.d. from known distribution.

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   [L61,D63,K05,BIK07,DP08,KP09,BDGIT09,IW11]
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Random Assignment.

Data:

- Known matroid $\mathcal{M} = (\mathbf{E}, \mathcal{I})$ on $\mathbf{n}$ elements.
- Hidden list of weights: $\mathbf{W}\colon \mathbf{w_1} \geq \mathbf{w_2} \geq \mathbf{w_3} \geq \cdots \geq \mathbf{w_n} \geq \mathbf{0}$.
- Random assignment. $\sigma\colon W \to E$.
- Random order. $\pi\colon E \to \{1, \ldots, n\}$.

Objective

Return an independent set $\mathrm{ALG} \in \mathcal{I}$ such that:

$$\mathbb{E}_{\pi,\sigma}[w(\mathrm{ALG})] \geq \alpha \cdot \mathbb{E}_{\sigma}[w(\mathrm{OPT})], \text{ where}$$

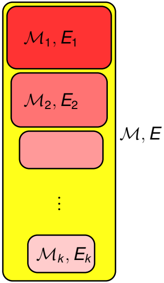
- $w(S) = \sum_{e \in S} \sigma^{-1}(e)$.
- OPT is the optimum base of $\mathcal{M}$ under assignment σ. (Greedy)
- α: Competitive Factor.

Divide and Conquer to get $O(1)$-competitive algorithm.

For a general matroid $\mathcal{M} = (E, \mathcal{I})$:

Find matroids $\mathcal{M}_i = (E_i, \mathcal{I}_i)$ with $E = \bigcup_{i=1}^k E_i$.

1. $\mathcal{M}_i$ admits $O(1)$-competitive algorithm (Easy parts).
2. Union of independent sets in each $\mathcal{M}_i$ is independent in $\mathcal{M}$. $\mathcal{I}(\bigoplus_{i=1}^k \mathcal{M}_i) \subseteq \mathcal{I}(\mathcal{M})$. (Combine nicely).
3. Optimum in $\bigoplus_{i=1}^k \mathcal{M}_i$ is comparable with Optimum in $\mathcal{M}$. (Don't lose much).



(Easy matroids): Uniformly dense matroids are like Uniform

Definition (Uniformly dense)

A loopless matroid $\mathcal{M} = (E, \mathcal{I})$ is uniformly dense if

$$\frac{|F|}{\mathrm{rk}(F)} \leq \frac{|E|}{\mathrm{rk}(E)}, \text{ for all } F \neq \emptyset.$$

e.g. Uniform $(\mathrm{rk}(F) = \min(|F|, r))$. Graphic K_n. Projective Spaces.

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Property: Sets of $\mathrm{rk}(E)$ elements have almost full rank.

$$\mathbb{E}_{(X:|X|=\mathrm{rk}(E))}[\mathrm{rk}(X)] \geq \mathrm{rk}(E)(1 - 1/e).$$

Uniformly dense matroid: Simple algorithm

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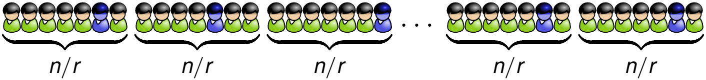
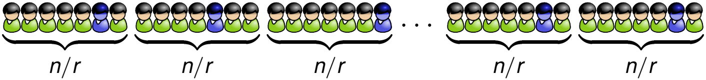


- Simulate e/C-comp. alg. for Uniform Matroids with $r = \mathrm{rk}_{\mathcal{M}}(E)$.
- Try to add each selected weight to the independent set.
- Selected elements have expected rank $\geq r(1 - 1/e)$.

Uniformly dense matroid: Simple algorithm



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Lemma: Constant competitive algorithm for Uniformly Dense.

$$\mathbb{E}_{\pi,\sigma}[w(\mathrm{ALG})] \geq \underbrace{\frac{C}{e}\left(1 - \frac{1}{e}\right)}_{K} \sum_{i=1}^{r} w_i \geq K \cdot \mathbb{E}_{\pi}[w(\mathrm{OPT}_{\mathcal{M}})].$$

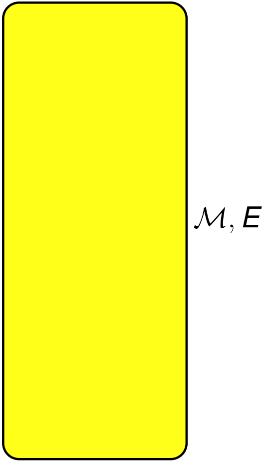
In fact: $\mathbb{E}_{\pi,\sigma}[w(\mathrm{ALG})] \geq K \cdot \mathbb{E}_{\sigma}[w(\mathrm{OPT}_{\mathcal{P}})]$,

where $\mathcal{P}$ is the uniform matroid in E with bound $r = \mathrm{rk}_{\mathcal{M}}(E)$.

Uniformly Dense (sub)matroids That combine nicely

Densest Submatroid

- Let $\mathcal{M} = (E, \mathcal{I})$ be a loopless matroid.

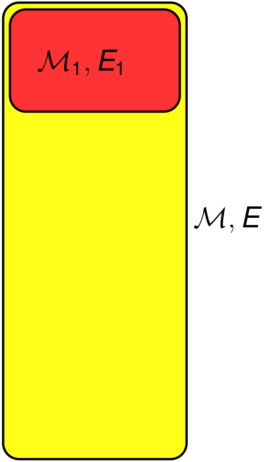


Densest Submatroid

- Let $\mathcal{M} = (E, \mathcal{I})$ be a loopless matroid.
- Let E_1 be the densest set of $\mathcal{M}$ of maximum cardinality.

$$\gamma(\mathcal{M}) := \max_{F \subseteq E} \frac{|F|}{\mathrm{rk}_{\mathcal{M}}(F)} = \frac{|E_1|}{\mathrm{rk}_{\mathcal{M}}(E_1)}.$$

- $\mathcal{M}_1 = \mathcal{M}|_{E_1}$ is uniformly dense.



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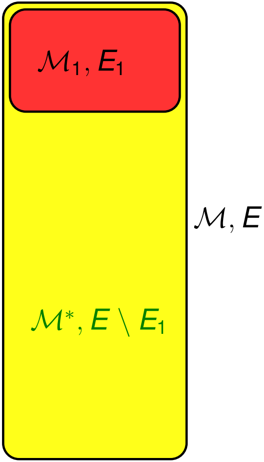
- $\mathcal{M}_1 = \mathcal{M}|_{E_1}$ is uniformly dense.
- $\mathcal{M}^* = \mathcal{M}/E_1$ is loopless and

$$\gamma(\mathcal{M}^*) := \max_{F \subseteq E \setminus E_1} \frac{|F|}{\mathrm{rk}_{\mathcal{M}^*}(F)} < \gamma(\mathcal{M}).$$

- $I_1 \in \mathcal{I}_1$, $I^* \in \mathcal{I}^*$ implies $I_1 \cup I^* \in \mathcal{I}$.



Uniformly Dense (sub)matroids That combine nicely

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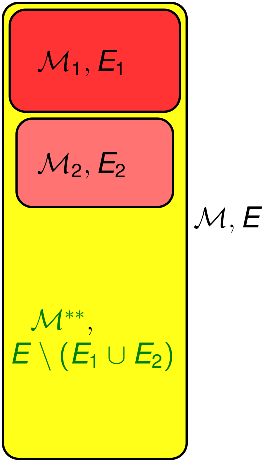
- Let E_2 be the densest set of $\mathcal{M}^*$ of maximum cardinality.
- $\mathcal{M}_2 = \mathcal{M}^*|_{E_2}$ is uniformly dense.
- $\mathcal{M}^{**} = \mathcal{M}/(E_1 \cup E_2)$ is loopless and

$$\gamma(\mathcal{M}^{**}) < \gamma(\mathcal{M}_2) < \gamma(\mathcal{M}_1) = \gamma(\mathcal{M}).$$



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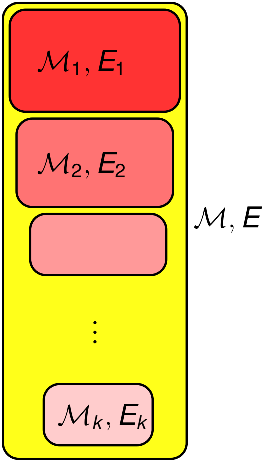
- Iterate...



Principal Partition of a matroid

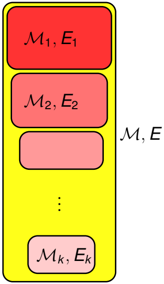
Theorem (Principal Partition)

Given $\mathcal{M} = (E, \mathcal{I})$ loopless, there exists a partition $E = \bigcup_{i=1}^{k} E_i$ such that

1. *The principal minor $\mathcal{M}_i = (\mathcal{M}/E_{i-1})|_{E_i}$ is a uniformly dense matroid with density*

$$\lambda_i = \gamma(\mathcal{M}_i) = \frac{|E_i|}{r_i}.$$

2. $\lambda_1 > \lambda_2 > \cdots > \lambda_k \geq 1.$



Note:

- If $I_i \in \mathcal{I}(\mathcal{M}_i)$, then $I_1 \cup I_2 \cup \cdots \cup I_k \in \mathcal{I}(\mathcal{M})$.
- Can compute the partition in polynomial time.

Algorithm for a General Matroid $\mathcal{M}$

Algorithm

1. Remove the loops from $\mathcal{M}$.
2. Let $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k$ be the principal minors.
3. In each $\mathcal{M}_i$ use the K-competitive algorithm for uniformly dense matroids to obtain an independent set I_i.
4. Return $\mathrm{ALG} = I_1 \cup I_2 \cup \cdots \cup I_k$.

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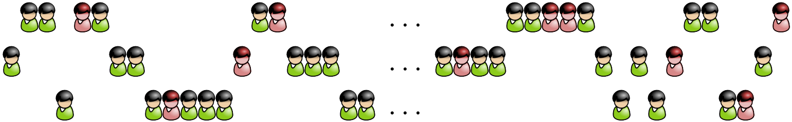

Analysis.

From Uniformly Dense Matroids to Uniform Matroids

- Each $\mathcal{M}_i$ is uniformly dense.
- Let $\mathcal{P}_i$ be the uniform matroid on E_i with bounds $r_i = \mathrm{rk}_{\mathcal{M}_i}(E_i)$.
- Let $\mathcal{P} = \bigoplus_{i=1}^k \mathcal{P}_i$ be the corresponding partition matroid.

By Lemma: $\mathbb{E}_{\pi,\sigma}[w(\mathrm{ALG} \cap E_i)] \geq K \cdot \mathbb{E}_\sigma[w(\mathrm{OPT}_{\mathcal{P}_i})]$.

Hence: $\mathbb{E}_{\pi,\sigma}[w(\mathrm{ALG})] \geq K \cdot \mathbb{E}_\sigma[w(\mathrm{OPT}_{\mathcal{P}})]$.

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Hence: $\mathbb{E}_{\pi,\sigma}[w(\mathrm{ALG})] \geq K \cdot \mathbb{E}_\sigma[w(\mathrm{OPT}_{\mathcal{P}})]$.

To conclude we show:

$$(*):\ \mathbb{E}_\sigma[w(\mathrm{OPT}_{\mathcal{P}})] \geq (1 - 1/e) \cdot \mathbb{E}_\sigma[w(\mathrm{OPT}_{\mathcal{M}})].$$

$$\Leftrightarrow (**):\ \mathbb{E}[\mathrm{rk}_{\mathcal{P}}(X_j)] \geq (1 - 1/e) \cdot \mathbb{E}[\mathrm{rk}_{\mathcal{M}}(X_j)],\ \text{for all } j,$$

where X_j is a uniform random set of j elements in E.

Analysis II: Proof of $\mathbb{E}[\text{rk}_{\mathcal{P}}(X_j)] \geq (1 - 1/e) \cdot \mathbb{E}[\text{rk}_{\mathcal{M}}(X_j)]$.

$$\begin{aligned}
\mathbb{E}[\text{rk}_{\mathcal{P}}(X_j)] &= \sum_{i=1}^{k} \mathbb{E}[\text{rk}_{\mathcal{P}}(X_j \cap E_i)] = \sum_{i=1}^{k} \mathbb{E}[\min(|X_j \cap E_i|, r_i)] \\
&\geq \sum_{i=1}^{k} (1 - 1/e) \cdot \min(\mathbb{E}[|X_j \cap E_i|], r_i) \\
&= \sum_{i=1}^{k} (1 - 1/e) \cdot \min(|E_i| \frac{j}{n}, r_i).
\end{aligned}$$

Since $\lambda_i = |E_i|/r_i$ is a decreasing sequence, there is an index $i^* = i^*(j)$ such that:

$$\mathbb{E}[\text{rk}_P(X_j)] \geq (1 - 1/e) \cdot \left(\sum_{i=1}^{i^*} r_i + \sum_{i=i^*+1}^{k} |E_i| \frac{j}{n} \right).$$

Analysis III: Proof of $\mathbb{E}[\mathrm{rk}_{\mathcal{P}}(X_j)] \geq (1-1/e)\cdot\mathbb{E}[\mathrm{rk}_{\mathcal{M}}(X_j)]$.

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$$\geq (1-1/e)\cdot\left(\mathrm{rk}_{\mathcal{M}}(\underbrace{E_1\cup\cdots\cup E_{i^*}}_{E^*}) + |(E_{i^*+1}\cup\cdots\cup E_k)|\frac{j}{n}\right)$$

$$= (1-1/e)\cdot\left(\mathrm{rk}_{\mathcal{M}}(E^*) + \mathbb{E}[|X_j\cap(E\setminus E^*)|]\right)$$

$$\geq (1-1/e)\cdot\mathbb{E}[\mathrm{rk}_{\mathcal{M}}(X_j\cap E^*) + rk_{\mathcal{M}}(X_j\cap(E\setminus E^*)]$$

$$\geq (1-1/e)\cdot\mathbb{E}[\mathrm{rk}_{\mathcal{M}}(X_j)]. \qquad \square$$

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$$\geq (1-1/e)\cdot\mathbb{E}[\mathrm{rk}_{\mathcal{M}}(X_j)]. \qquad \square$$

Therefore:

$$\mathbb{E}_{\pi,\sigma}[w(\mathrm{ALG})] \geq K(1-1/e)\cdot\mathbb{E}_{\sigma}[w(\mathrm{OPT}_{\mathcal{M}})].$$

Conclusions and Open Problems.

Summary

- First constant competitive algorithm for Matroid Secretary Problem in Random Assignment Model.
- Corollary: Also holds for i.i.d. weights from known or unknown distributions.
- Algorithm does not use hidden weights (only relative ranks).

Open

- Find constant competitive algorithm for General Matroids under Adversarial Assignment.
- Extend to other independent systems:
  Note[BIK07]: $\Omega(\log(n)/\log\log(n))$ lower bound.