A simple PTAS for Weighted Matroid Matching on Strongly Base Orderable Matroids

José A. Soto

Department of Mathematics
M.I.T.

2011
∅ \neq \mathcal{B} \subseteq 2^V is the basis system of a matroid if

- Every \( B \in \mathcal{B} \) has the same size.
- basis exchange property: For all \( A, B \in \mathcal{B} \) distinct there are \( a \in A \setminus B, b \in B \setminus A \) s.t. \( A - a + b \in \mathcal{B} \).
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**Independent sets** \( \mathcal{I} \).

**Independent Sets** = Basis subset.

Examples of matroid bases
- **(Free)** Only \( V \).
- **(Uniform)** Sets of size \( k \).
- **(Graphic)** Spanning forests.
- **(Linear)** Vector space bases.
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Strongly Base Orderable (SBO) Matroids.

\[ \emptyset \neq B \subseteq 2^V \] is the basis system of a SBO matroid

- Every \( B \in B \) has the same size.
- (SBO) basis exchange property: For all \( A, B \in B \) distinct
  \[ \exists \text{ bijection } \pi : A \setminus B \rightarrow B \setminus A \text{ s.t. } \forall X \subseteq A, A \setminus X \cup \pi(X) \in B. \]

Examples of SBO matroid bases:
- (Uniform) Sets of size \( k \).
- (Gammoid) Maximum sets of clients connected to servers by edge-disjoint paths.
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- **(Uniform) Sets of size k.**
- **(Gammoid)**

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Weighted Matroid Matching Problem

**Problem**

- Weighted graph $G = (V, E)$, $w: E \rightarrow \mathbb{R}^+$. 
- Matroid $\mathcal{M} = (V, \mathcal{I})$.

A matching $M \subseteq E$ is **feasible** for $\mathcal{M}$ if $V(M)$ is independent.

**Goal**: Find a maximum weight feasible matching.
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Weighted Matching

Free matroid.

Weighted Matroid Intersection

$\mathcal{M}_1 \oplus \mathcal{M}_2$
Complexity of WMM

- Not in **oracle coNP** even for unweighted case.
- **NP-hard** even for unweighted case.
- Special subproblems in $\mathbf{P}$:
  - Weighted matching / Weighted Matroid Intersection.
  - (Lovász 1981) Unweighted case in $\mathbf{P}$ for linear matroids.
  - (Tong et al. 1982) Weighted case in $\mathbf{P}$ for gammoids.
Approximation algorithms

Unweighted

- **Greedy** gives 2-approximation.

(Fujito 1993) 3/2-approximation using local search.

(Lee et al. 2010) PTAS using local search.

(S. 2011) PTAS for SBO-matroids.
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WMM on SBO-matroids

Hardness
Still outside oracle coNP and NP-hard.

Weighted Parity on SBO-matroids
Find maximum weight paired basis of a SBO-matroid $M$. (with dummy pairs)

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Weighted Parity on SBO-matroids
Find maximum weight set of pairs \textit{feasible} for a SBO-matroid \(M\).
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Weighted Parity on SBO-matroids
Find maximum weight \textit{paired basis} of a SBO-matroid $\mathcal{M}$. (with dummy pairs)
Local moves

\textbf{t}-swap: For a current paired basis $A$

Swap at most $t$ pairs to obtain paired basis $B$.

\textbf{Gain:} $w(B) - w(A)$. \textbf{High gain:} $w(B) - w(A) \geq w(A)/n^2$. 
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Algorithm: For constant $1 \leq t \leq n$,

- Start with greedy solution.
- Do $t$-swaps with high gain until local optimum is found.
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- Start with greedy solution.
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At most $O(\log_{(1+1/n^2)} 1/2) = O(n^2)$ moves suffice.

Can find an \textit{t-local optimum} in polynomial time ($O(n^{2t+2})$).
Main result for WMM on SBO-matroids.

**Theorem**

*If paired basis $A$ is a $t$-local optimum and $B = \text{OPT}$ then*

$$w(B) \leq \left( 1 + \frac{2}{t-1} \right) w(A).$$

**PTAS:** To get $(1 + \varepsilon)$-approx set $t = 1 + 2/\varepsilon$.

Running time $n^{O(1/\varepsilon)}$. 
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**Auxiliar construction**
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- Matroid *with dummy pairs* is still SBO...
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- (multi)graph $H$ of degrees 0 and 2.
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- Matroid *with dummy pairs* is still SBO...
- (multi)graph $H$ of degrees 0 and 2.
- $\vec{H}$: Union of directed cycles.
Proof of $w(B) \leq (1 + 2/(t - 1)) w(A)$.

For a pair $p$ of $A$,

$H_p$: Reachable from $p$ using $\leq 2(t - 1)$ edges.
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Since we swap at most $t$ pairs

$$\frac{w(A)}{n^2} > \text{Gain}(\text{swap}(p)) = w(H_p \cap B) - w(H_p \cap A).$$
Proof of $w(B) \leq (1 + 2/(t - 1)) w(A)$. (cont.)

$$\sum_{p\in A_{\text{long}}} \frac{w(A)}{n^2} > \sum_{p\in A_{\text{long}}} w(H_p \cap B) - w(H_p \cap A)$$
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\sum_{p \in A_{\text{rest}}} \frac{w(A)}{n^2} > \sum_{p \in A_{\text{rest}}} w(H_p \cap B) - w(H_p \cap A)
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Proof of \( w(B) \leq \frac{1}{t} + \frac{2}{t-1} \) \( w(A) \). (cont.)

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\[
(t - 1) w(B) - t w(A) \leq ((t - 1) w(B_{\text{long}}) - t w(A_{\text{long}})) + t (w(B_{\text{rest}}) - w(A_{\text{rest}})) < \frac{w(A)}{n^2} (|A_{\text{long}}| + t |A_{\text{rest}}|) \leq w(A).
\]
Conclusions

First PTAS for Weighted Matroid Matching on Strongly Orderable Matroids.

Open Problems

- Can we get a PTAS for general matroids?
- Can we get a FPTAS for this class?