TSP in cubic graphs. Beyond 4/3.

J. Correa, O. Larré, **J.A. Soto**

U. de Chile, TU-Berlin

Ljubljana, Sept. 12th , 2012.

A motivation from graph theory: Barnette's conjecture.

Barnette's conjecture (1969/70):

$\underbrace{\text{Cubic, 3-conn., planar, bipartite}}_{\text{Barnette}}$ graphs are Hamiltonian.

- Open for more than 40 years.
- Rich history (sequence of conjectures).
- Good evidence in favor.
  - Minimal counterexample has $\geq$ 84 vertices (Holton et al. 85, McKay et al. 00).
  - True if faces are square and hexagons (Goodey 75).
  - Equivalent/stronger formulations.

TSP?

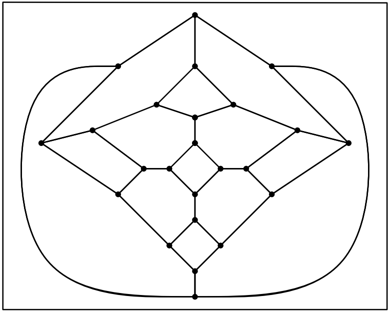
We do not know if Barnette graphs are Hamiltonian.
What about (short) tours?

Tour: closed walk visiting all vertices at least once.

Can we find a tour of length n? $\ldots(1+\varepsilon)n$? $\ldots cn$?

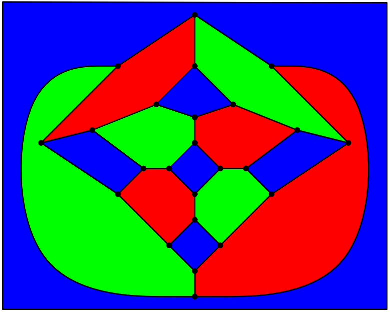
$4n/3$ is easy for Barnette's graphs.

Barnette (cubic, 3-conn, planar, bipartite) graphs are **3-face-colorable**.



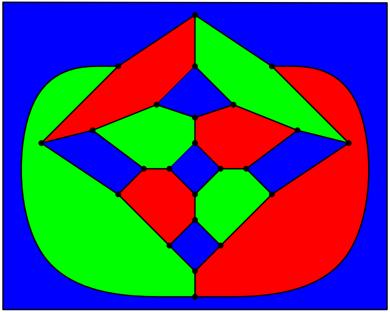
$4n/3$ is easy for Barnette's graphs.

Barnette (cubic, 3-conn, planar, bipartite) graphs are **3-face-colorable**.



$4n/3$ is easy for Barnette's graphs.

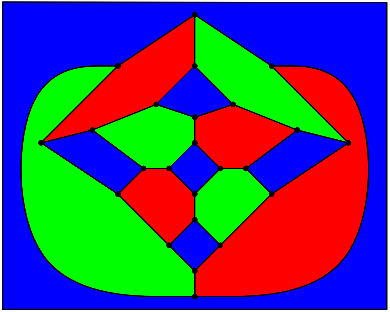
Barnette (cubic, 3-conn, planar, bipartite) graphs are **3-face-colorable**.



$$|F| \underset{\text{Euler}}{=} 2 + |E| - |V|$$
$$\underset{\text{cubic}}{=} 2 + 3n/2 - n$$
$$= (n+4)/2.$$

$4n/3$ is easy for Barnette's graphs.

Barnette (cubic, 3-conn, planar, bipartite) graphs are **3-face-colorable**.

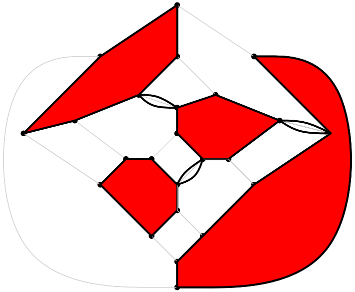


$$|F| \underset{\text{Euler}}{=} 2 + |E| - |V| \underset{\text{cubic}}{=} 2 + 3n/2 - n = (n+4)/2.$$

- One color class has $\leq (n+4)/6$ faces.

$4n/3$ is easy for Barnette's graphs.

Barnette (cubic, 3-conn, planar, bipartite) graphs are **3-face-colorable**.



$$|F| \underset{\text{Euler}}{=} 2 + |E| - |V| \underset{\text{cubic}}{=} 2 + 3n/2 - n = (n+4)/2.$$

- One color class has $\leq (n+4)/6$ faces.
- Connect them by a (doubled) spanning tree of faces.
  Get a tour of length $\leq n + 2((n+4)/6 - 1) = 4n/3 - 2/3$.

But $4n/3$ is achievable for many superclasses.

- Barnette graphs (3-conn., cubic, bipartite, planar).
- 3-conn., cubic (Aggarwal, Garg, Gupta, 2011)
- 2-conn., cubic (Boyd, Sitters, van der Ster, Stougie, 2011)
- 2-conn., subcubic (Mömke, Svensson, 2011)
- 3-conn., planar (Kawarabayashi, Ozeki, 2012)

Can we find tours of length $<< (4/3)n$ in some of these cases?

But $4n/3$ is achievable for many superclasses.

- Barnette graphs (3-conn., cubic, bipartite, planar).
- 3-conn., cubic (Aggarwal, Garg, Gupta, 2011)
- 2-conn., cubic (Boyd, Sitters, van der Ster, Stougie, 2011)
- 2-conn., subcubic (Mömke, Svensson, 2011) No.
- 3-conn., planar (Kawarabayashi, Ozeki, 2012) No.

Can we find tours of length $<< (4/3)n$ in some of these cases?

But $4n/3$ is achievable for many superclasses.

- Barnette graphs (3-conn., cubic, bipartite, planar).
- 3-conn., cubic (Aggarwal, Garg, Gupta, 2011)
- 2-conn., cubic (Boyd, Sitters, van der Ster, Stougie, 2011)
- 2-conn., subcubic (Mömke, Svensson, 2011) No.
- 3-conn., planar (Kawarabayashi, Ozeki, 2012) No.

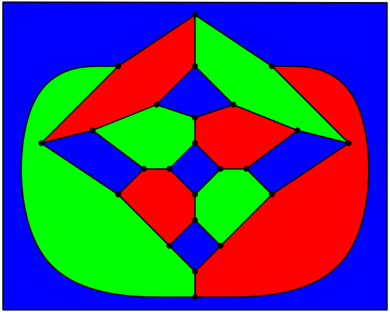
Can we find tours of length $<< (4/3)n$ in some of these cases?

Yes! [CLS12]:

2-connected, cubic graphs have tours of size $\leq (4/3 - \varepsilon)n$

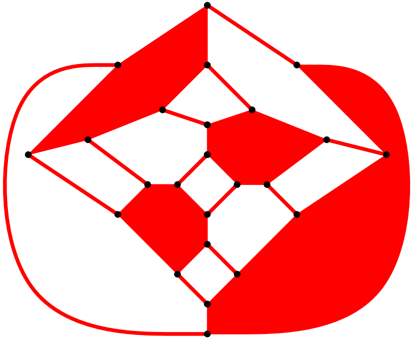
Warmout: Barnette Graphs

- **3-face-colorable:** There is a cycle-cover with $\leq n/6$ cycles.
- Idea: Find cycle-cover with $\leq \alpha n$ cycles.
  to get tour of length $\leq n(1 + 2\alpha)$.



Warmout: Barnette Graphs (cont.)

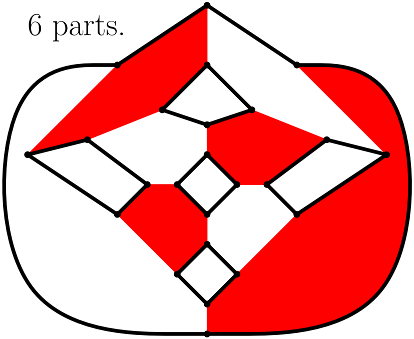
Want a cycle-cover containing the edges outside red faces with small number of cycles.



Warmout: Barnette Graphs (cont.)

Want a cycle-cover containing the edges outside red faces with small number of cycles.

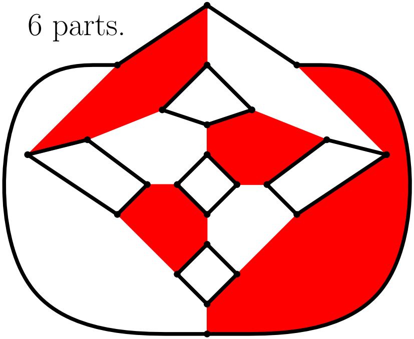
- Start from such a cycle cover.



Warmout: Barnette Graphs (cont.)

Want a cycle-cover containing the edges outside red faces with small number of cycles.

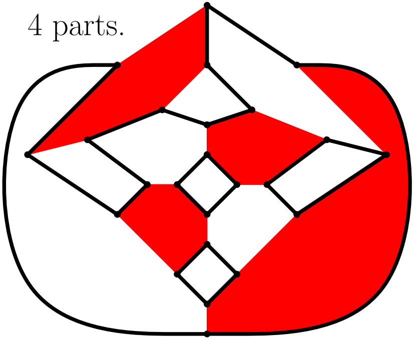
- Start from such a cycle cover.
- For each red face f:
  If we can improve $\mathcal{C}$ by alternating f's boundary. Do it.



Warmout: Barnette Graphs (cont.)

Want a cycle-cover containing the edges outside red faces with small number of cycles.

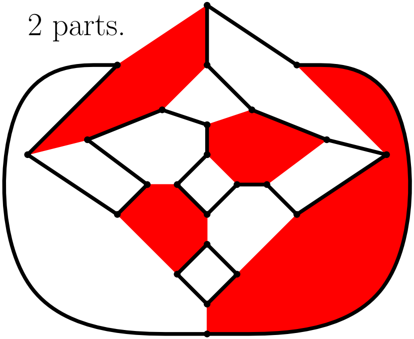
- Start from such a cycle cover.
- For each red face f:
  If we can improve $\mathcal{C}$ by alternating f's boundary. Do it.



Warmout: Barnette Graphs (cont.)

Want a cycle-cover containing the edges outside red faces with small number of cycles.

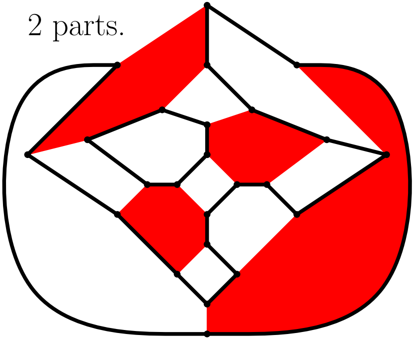
- Start from such a cycle cover.
- For each red face f:
  If we can improve $\mathcal{C}$ by alternating f's boundary. Do it.



Warmout: Barnette Graphs (cont.)

Want a cycle-cover containing the edges outside red faces with small number of cycles.

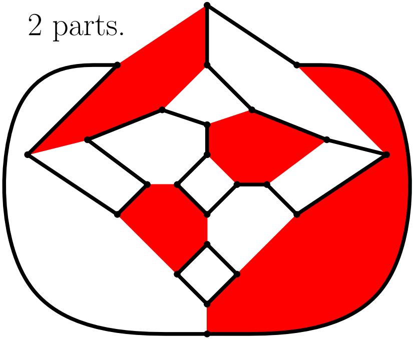
- Start from such a cycle cover.
- For each red face f:
  If we can improve $\mathcal{C}$ by alternating f's boundary. Do it.



Warmout: Barnette Graphs (cont.)

Want a cycle-cover containing the edges outside red faces with small number of cycles.

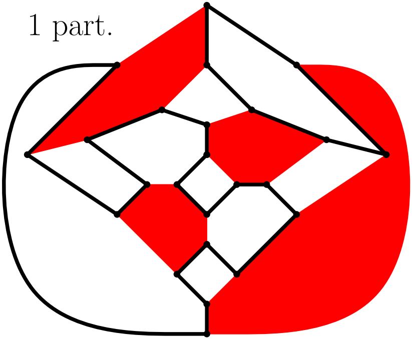
- Start from such a cycle cover.
- For each red face f:
  If we can improve $\mathcal{C}$ by alternating f's boundary. Do it.



Warmout: Barnette Graphs (cont.)

Want a cycle-cover containing the edges outside red faces with small number of cycles.

- Start from such a cycle cover.
- For each red face f:
  If we can improve $\mathcal{C}$ by alternating f's boundary. Do it.



Warmout: Barnette Graphs (cont.)

Local search: 3 cycle covers $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$. Select $\mathcal{C}$ as the one with fewer cycles.

Theorem (CLS12+)

We can find a cycle-cover $\mathcal{C}$ with $|\mathcal{C}| \leq 5n/36$.
Tour of length $\leq n(1 + 5/18) = (4/3 - 1/18)n$.

Analysis uses Euler's formula and average argument.

More general graphs.

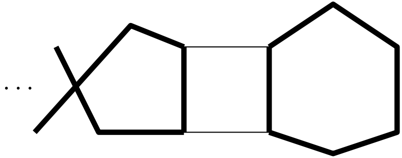
- Barnette (cubic,3-conn,bipartite,planar):
  Local search achieves $(4/3 - 1/18)n$.

More general graphs.

- Barnette (cubic,3-conn,bipartite,planar):
  Local search achieves $(4/3 - 1/18)n$.
- Cubic, 3-conn, bipartite:
  They admit three disjoint perfect matchings E_1, E_2, E_3.
  and by excluding each of them, three cycle covers.
- Local improvements?

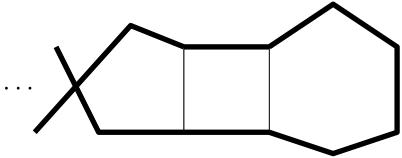
More general graphs.

- Barnette (cubic,3-conn,bipartite,planar):
  Local search achieves $(4/3 - 1/18)n$.
- Cubic, 3-conn, bipartite:
  They admit three disjoint perfect matchings E_1, E_2, E_3.
  and by excluding each of them, three cycle covers.
- Local improvements?
- If there is a (short) "alternating cycle" reducing the number of cycles, use it.



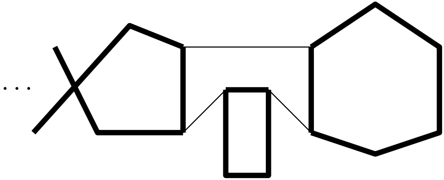
More general graphs.

- Barnette (cubic,3-conn,bipartite,planar):
  Local search achieves $(4/3 - 1/18)n$.
- Cubic, 3-conn, bipartite:
  They admit three disjoint perfect matchings E_1, E_2, E_3.
  and by excluding each of them, three cycle covers.
- Local improvements?
- If there is a (short) “alternating cycle” reducing the number of cycles, use it.



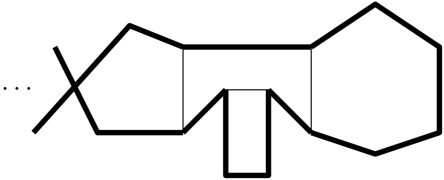
More general graphs.

- Barnette (cubic,3-conn,bipartite,planar):
  Local search achieves $(4/3 - 1/18)n$.
- Cubic, 3-conn, bipartite:
  They admit three disjoint perfect matchings E_1, E_2, E_3.
  and by excluding each of them, three cycle covers.
- Local improvements?
- If there is a (short) “alternating cycle” reducing the number of cycles, use it.



More general graphs.

- Barnette (cubic,3-conn,bipartite,planar):
  Local search achieves $(4/3 - 1/18)n$.
- Cubic, 3-conn, bipartite:
  They admit three disjoint perfect matchings E_1, E_2, E_3.
  and by excluding each of them, three cycle covers.
- Local improvements?
- If there is a (short) "alternating cycle" reducing the number of cycles, use it.



More general graphs.

- Barnette (cubic,3-conn,bipartite,planar):
  Local search achieves $(4/3 - 1/18)n$.
- Cubic, 3-conn, bipartite:
  They admit three disjoint perfect matchings E_1, E_2, E_3.
  and by excluding each of them, three cycle covers.
- Local improvements?
- If there is a (short) "alternating cycle" reducing the number of cycles, use it.
- Not planar anymore, so analysis is different (follow Boyd et al.)

Cubic, 3-conn, bipartite

- Improve each $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ locally (squares and hexagons).
  Label $\ell(v) = (\ell_1(v), \ell_2(v), \ell_3(v))$, where
  $\ell_i(v) :=$ length of cycle in $\mathcal{C}_i$ containing v.

Cubic, 3-conn, bipartite

- Improve each $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ locally (squares and hexagons).
  Label $\ell(v) = (\ell_1(v), \ell_2(v), \ell_3(v))$, where
  $\ell_i(v) :=$ length of cycle in $\mathcal{C}_i$ containing v.

- Define $z_i(v) = (1 + \frac{2}{\ell_i(v)})$, then

$$\sum_{v \in V} z_i(v) = n + 2(\#\text{cycles}) = \text{length of tour } i + 2.$$

Cubic, 3-conn, bipartite

- Improve each $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ locally (squares and hexagons).
  Label $\ell(v) = (\ell_1(v), \ell_2(v), \ell_3(v))$, where
  $\ell_i(v) :=$ length of cycle in $\mathcal{C}_i$ containing v.

- Define $z_i(v) = (1 + \frac{2}{\ell_i(v)})$, then

$$\sum_{v \in V} z_i(v) = n + 2(\#\text{cycles}) = \text{length of tour } i + 2.$$

- Contribution of v: $z(v) = \frac{1}{3} \sum_{i=1}^{3} z_i(v)$, then

$$\sum_{v \in V} z(v) \geq \text{length of best tour.}$$

Cubic, 3-conn, bipartite

- Improve each $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ locally (squares and hexagons).
  Label $\ell(v) = (\ell_1(v), \ell_2(v), \ell_3(v))$, where
  $\ell_i(v) :=$ length of cycle in $\mathcal{C}_i$ containing v.

- Define $z_i(v) = (1 + \frac{2}{\ell_i(v)})$, then

$$\sum_{v \in V} z_i(v) = n + 2(\#\text{cycles}) = \text{length of tour } i + 2.$$

- Contribution of v: $z(v) = \frac{1}{3}\sum_{i=1}^{3} z_i(v)$, then

$$\sum_{v \in V} z(v) \geq \text{length of best tour.}$$

- Want $z(v) < 4/3$ in average. (essentially we win if $\ell_i(v) > 6$)

$z(v) < 4/3$ in average for cubic 3-conn. bipartite

Lemma (After processing squares and hexagons)

- *If* $\ell(v) = (4, j, k)$*, then* $j, k \geq 10$*.*
- *No 6-cycle is completely* $(6, 6, 6)$*.*

$z(v) < 4/3$ in average for cubic 3-conn. bipartite

Lemma (After processing squares and hexagons)

- *If* $\ell(v) = (4, j, k)$*, then* $j, k \geq 10$*.*
- *No 6-cycle is completely* $(6, 6, 6)$*.*

Using the first part of lemma.

- If $\ell_i(v) = 4$, then $z(v) \leq \left(\frac{6}{4} + \frac{12}{10} + \frac{12}{10}\right)/3 = \frac{13}{10} \leq \frac{4}{3} - \epsilon$.

$z(v) < 4/3$ in average for cubic 3-conn. bipartite

Lemma (After processing squares and hexagons)

- *If $\ell(v) = (4, j, k)$, then $j, k \geq 10$.*
- *No 6-cycle is completely $(6, 6, 6)$.*

Using the first part of lemma.

- If $\ell_i(v) = 4$, then $z(v) \leq \left(\frac{6}{4} + \frac{12}{10} + \frac{12}{10}\right)/3 = \frac{13}{10} \leq \frac{4}{3} - \epsilon$.
- $$\text{If } \ell(v) \neq (6,6,6) \text{ then } z(v) \leq \frac{4}{3} - \epsilon.$$
  $$\text{If } \ell(v) = (6,6,6) \text{ then } z(v) = \frac{4}{3}.$$

$z(v) < 4/3$ in average for cubic 3-conn. bipartite

Lemma (After processing squares and hexagons)

- *If $\ell(v) = (4, j, k)$, then $j, k \geq 10$.*
- *No 6-cycle is completely $(6, 6, 6)$.*

Using the first part of lemma.

- If $\ell_i(v) = 4$, then $z(v) \leq \left(\frac{6}{4} + \frac{12}{10} + \frac{12}{10}\right)/3 = \frac{13}{10} \leq \frac{4}{3} - \epsilon$.
- $$\text{If } \ell(v) \neq (6,6,6) \text{ then } z(v) \leq \frac{4}{3} - \epsilon.$$
  $$\text{If } \ell(v) = (6,6,6) \text{ then } z(v) = \frac{4}{3}.$$

Second part of lemma + averaging argument:

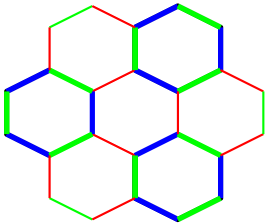
Theorem (Theorem CLS12+)

In average, $z(v) < \frac{4}{3} - \frac{1}{108}$. Tour of length $\leq (\frac{4}{3} - \frac{1}{108})n$.

Proof flavour of the lemma.

No hexagon has all its vertices labeled $(6, 6, 6)$.

Assume otherwise. Since $\ell_i(v)$ are increasing, the hexagons exists from the beginning (i.e. coming from 3-colored matchings).

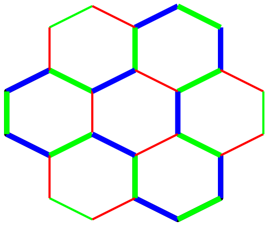


At the end, the hexagon is an improving alternating cycle. □

Proof flavour of the lemma.

No hexagon has all its vertices labeled $(6, 6, 6)$.

Assume otherwise. Since $\ell_i(v)$ are increasing, the hexagons exists from the beginning (i.e. coming from 3-colored matchings).



At the end, the hexagon is an improving alternating cycle. □

Towards cubic, 2-connected

In cubic, 3-conn., bip., we get $\leq (4/3 - 1/108)n$.

- What about cubic, 2-connected? Not 3-edge colorable but...

Towards cubic, 2-connected

In cubic, 3-conn., bip., we get $\leq (4/3 - 1/108)n$.

- What about cubic, 2-connected? Not 3-edge colorable but...
- (Boyd et al.) Can find collection of special cycle covers (complements of 3-cut perfect matchings) $\{\mathcal{C}_i\}_i$ such that

$$(2/3)\chi(E) = \sum_i \lambda_i \chi(\mathcal{C}_i), \quad \sum_i \lambda_i = 1, \quad \lambda_i \geq 0.$$

Towards cubic, 2-connected

In cubic, 3-conn., bip., we get $\leq (4/3 - 1/108)n$.

- What about cubic, 2-connected? Not 3-edge colorable but...
- (Boyd et al.) Can find collection of special cycle covers (complements of 3-cut perfect matchings) $\{\mathcal{C}_i\}_i$ such that

$$(2/3)\chi(E) = \sum_i \lambda_i \chi(\mathcal{C}_i), \quad \sum_i \lambda_i = 1, \quad \lambda_i \geq 0.$$

- "Improving" each cover separately by finding augmenting 4 and 6 cycles is not enough.

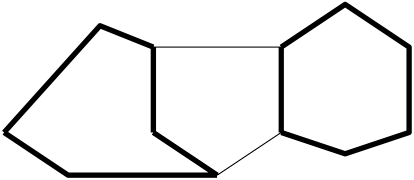
Towards cubic, 2-connected

In cubic, 3-conn., bip., we get $\leq (4/3 - 1/108)n$.

- What about cubic, 2-connected? Not 3-edge colorable but...
- (Boyd et al.) Can find collection of special cycle covers (complements of 3-cut perfect matchings) $\{\mathcal{C}_i\}_i$ such that

$$(2/3)\chi(E) = \sum_i \lambda_i \chi(\mathcal{C}_i), \quad \sum_i \lambda_i = 1, \quad \lambda_i \geq 0.$$

- "Improving" each cover separately by finding augmenting 4 and 6 cycles is not enough. As Boyd et al., we also need to use pentagons to improve a cycle-cover.



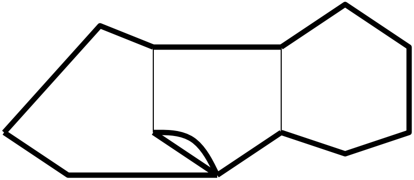
Towards cubic, 2-connected

In cubic, 3-conn., bip., we get $\leq (4/3 - 1/108)n$.

- What about cubic, 2-connected? Not 3-edge colorable but...
- (Boyd et al.) Can find collection of special cycle covers (complements of 3-cut perfect matchings) $\{\mathcal{C}_i\}_i$ such that

$$(2/3)\chi(E) = \sum_i \lambda_i \chi(\mathcal{C}_i), \quad \sum_i \lambda_i = 1, \quad \lambda_i \geq 0.$$

- "Improving" each cover separately by finding augmenting 4 and 6 cycles is not enough. As Boyd et al., we also need to use pentagons to improve a cycle-cover.



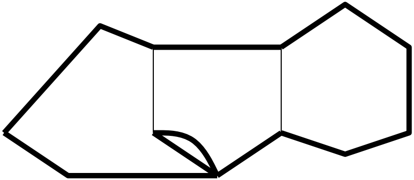
Towards cubic, 2-connected

In cubic, 3-conn., bip., we get $\leq (4/3 - 1/108)n$.

- What about cubic, 2-connected? Not 3-edge colorable but...
- (Boyd et al.) Can find collection of special cycle covers (complements of 3-cut perfect matchings) $\{\mathcal{C}_i\}_i$ such that

$$(2/3)\chi(E) = \sum_i \lambda_i \chi(\mathcal{C}_i), \quad \sum_i \lambda_i = 1, \quad \lambda_i \geq 0.$$

- "Improving" each cover separately by finding augmenting 4 and 6 cycles is not enough. As Boyd et al., we also need to use pentagons to improve a cycle-cover.



- We need Eulerian subgraphs covers instead of cycle-covers.

Cubic, 2-connected

Algorithm: Based on Boyd et al.'s algorithm.

- Find a decomposition of $(2/3)\chi(E)$ as convex comb. of complements of 3-cut perfect matchings, $\lambda_1\chi(\mathcal{C}_1), \ldots, \lambda_k\chi(\mathcal{C}_k)$
- For each i, do the following local moves:
  - Use squares and hexagons to improve $\mathcal{C}_i$ as possible.
  - Use pentagons to improve $\mathcal{C}_i$.
- Return the final cover of the list yielding the smallest tour.

Cubic, 2-connected

Algorithm: Based on Boyd et al.'s algorithm.

- Find a decomposition of $(2/3)\chi(E)$ as convex comb. of complements of 3-cut perfect matchings, $\lambda_1\chi(\mathcal{C}_1), \ldots, \lambda_k\chi(\mathcal{C}_k)$
- For each i, do the following local moves:
  - Use squares and hexagons to improve $\mathcal{C}_i$ as possible.
  - Use pentagons to improve $\mathcal{C}_i$.
- Return the final cover of the list yielding the smallest tour.

Analysis:

- $z_i(v) = \frac{e(C)+2}{v(C)}$.
- $z(v) = \sum_{i \in I} \lambda_i z_i(v)$. $\leftarrow$ contribution of v.
- Tour length $\leq \sum_{v \in V} z(v)$.

We win if $z(v) < 4/3$ in average.

Cubic, 2-connected

Algorithm: Based on Boyd et al.'s algorithm.

- Find a decomposition of $(2/3)\chi(E)$ as convex comb. of complements of 3-cut perfect matchings, $\lambda_1\chi(\mathcal{C}_1), \ldots, \lambda_k\chi(\mathcal{C}_k)$
- For each i, do the following local moves:
  - Use squares and hexagons to improve $\mathcal{C}_i$ as possible.
  - Use pentagons to improve $\mathcal{C}_i$.
- Return the final cover of the list yielding the smallest tour.

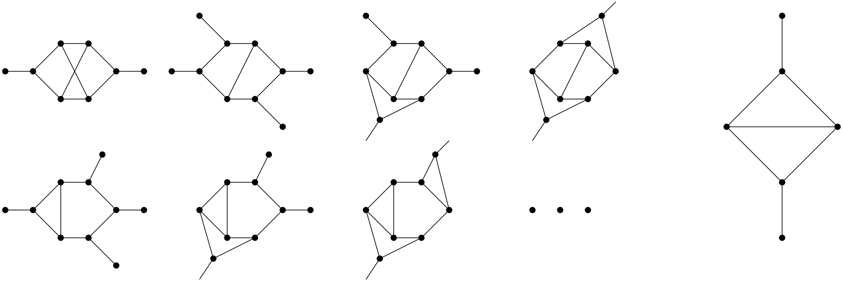
Analysis:

- $z_i(v) = \frac{e(\mathrm{C})+2}{v(\mathrm{C})}$.
- $z(v) = \sum_{i \in I} \lambda_i z_i(v)$. $\leftarrow$ contribution of v.
- Tour length $\leq \sum_{v \in V} z(v)$.

We win if $z(v) < 4/3$ in average. **It does not work...**

Cubic, 2-connected

- Bad subgraphs: Chorded hexagons and squares

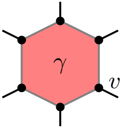


- Can get rid of chorded hexagons via a reduction:
  Tour in the reduced graph of length in $[(5/4)n', \alpha n']$ yields a tour in original graph of length in $[(5/4)n, \alpha n]$.
- But chorded squares (diamonds) remain.

Analysis: After the reduction.

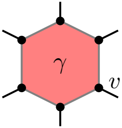
- If v is neither in a diamond nor in a 6-cycle belonging to some cover then $z(v) \leq 4/3 - \epsilon$.

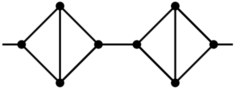
Analysis: After the reduction.

- If v is neither in a diamond nor in a 6-cycle belonging to some cover then $z(v) \leq 4/3 - \epsilon$.
- 

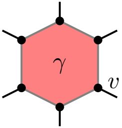
If v is in a 6-cycle γ then $z(v) \leq 4/3$ and $\sum_{w \in \gamma} z(w) \leq 6(4/3 - \epsilon')$

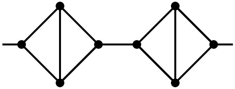
Analysis: After the reduction.

- If v is neither in a diamond nor in a 6-cycle belonging to some cover then $z(v) \leq 4/3 - \epsilon$.
- 

If v is in a 6-cycle γ then $z(v) \leq 4/3$ and $\sum_{w \in \gamma} z(w) \leq 6(4/3 - \epsilon')$
- 
If v is in a diamond, then $z(v) \leq 4/3$. If the diamond is not isolated, then $z(v) \leq 4/3 - \epsilon''$.

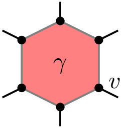
Analysis: After the reduction.

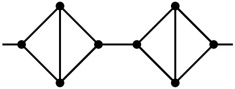
- If v is neither in a diamond nor in a 6-cycle belonging to some cover then $z(v) \leq 4/3 - \epsilon$.
- 

If v is in a 6-cycle γ then $z(v) \leq 4/3$ and $\sum_{w \in \gamma} z(w) \leq 6(4/3 - \epsilon')$
- 
If v is in a diamond, then $z(v) \leq 4/3$.
If the diamond is not isolated, then $z(v) \leq 4/3 - \epsilon''$.

A constant fraction of vertices are outside isolated diamonds

Analysis: After the reduction.

- If v is neither in a diamond nor in a 6-cycle belonging to some cover then $z(v) \leq 4/3 - \epsilon$.
- 

If v is in a 6-cycle γ then $z(v) \leq 4/3$ and $\sum_{w \in \gamma} z(w) \leq 6(4/3 - \epsilon')$
- 
If v is in a diamond, then $z(v) \leq 4/3$. If the diamond is not isolated, then $z(v) \leq 4/3 - \epsilon''$.

A constant fraction of vertices are outside isolated diamonds

Theorem (CLS12)

In average, $z(v) \leq 4/3 - \epsilon^*$, for $\epsilon^* = 1/61236$. Every cubic 2-conn. graph has a tour of at most $(4/3 - \epsilon^*)n - 2$.

Summary

Theorem (CLS12+)

Every cubic 2-conn. graph has a tour of length at most $(4/3 - \epsilon^*)n - 2$.

In particular, Held and Karp's integrality gap in this class is $< 4/3$.

Summary

Theorem (CLS12+)

Every cubic 2-conn. graph has a tour of length at most $(4/3 - \epsilon^*)n - 2$.

In particular, Held and Karp's integrality gap in this class is $< 4/3$.

Open:

What is the "gap" (with respect to n) for this class?

	Lower bound	Upper bound
Barnette	$(9/9)n$	$(4/3 - 1/18)n$
Cubic, 2-conn, bipartite	$(10/9)n$	$(4/3 - 1/108)n$
Cubic, 2-conn.	$(11/9)n$	$(4/3 - 1/61236)n$
Subcubic, 2-conn.	$(12/9)n$	$(4/3)n$

Summary

Theorem (CLS12+)

Every cubic 2-conn. graph has a tour of length at most $(4/3 - \epsilon^*)n - 2$.

In particular, Held and Karp's integrality gap in this class is $< 4/3$.

Open:

What is the "gap" (with respect to *n*) for this class?

	Lower bound	Upper bound
Barnette	$(9/9)n$	$(4/3 - 1/18)n$
Cubic, 2-conn, bipartite	$(10/9)n$	$(4/3 - 1/108)n$
Cubic, 2-conn.	$(11/9)n$	$(4/3 - 1/61236)n$
Subcubic, 2-conn.	$(12/9)n$	$(4/3)n$

Related (and more important: Gap with respect to LP?).