A law of large numbers for random partitions of the interval and the limiting transient search-cost of the move-to-front strategy

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Abstract

We prove a law of large numbers for certain finite random partitions of $[0,1]$, when the number of fragments go to ∞. Then, we apply it to compute the limiting distribution of the transient search-cost of the move-to-front rule for general classes of random and deterministic request probabilities, when the list size goes to ∞.

Keywords: law of large numbers, move-to-front rule, search-cost, propagation of chaos.

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1 Introduction

In this work, we shall prove a law of large numbers for some finite random partitions of $[0,1]$,

$$\mathbf{p}^{(n)} = (p_1^{(n)}, \dots, p_n^{(n)}) \quad p_i^{(n)} \geq 0, \quad \sum_{i=1}^{n} p_i^{(n)} = 1$$

when the number n of fragments goes to ∞. Based on it, we shall introduce a strategy to compute the asymptotic law of some functionals of such partitions. Our results concern various types of random partitions, the central example of which is the following: consider a sequence (ω_i) of positive i.i.d. random variables of given law P and the partition of $[0,1]$ defined by

$$p_i^{(n)} = \frac{\omega_i}{\sum_{j=1}^{n} \omega_j}. \tag{1}$$

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For instance, if each ω_i follows a Gamma law of parameters $(\alpha, 1)$, then $\mathbf{p}^{(n)}$ is the widely studied symmetric Dirichlet random partition of parameter α (see [14], Ch. 9).
For partitions of the form (1), no "macroscopic" fragments arise in the limit $n \to \infty$: we have $p_i^{(n)} \to 0$ for all i almost surely, as a consequence of the strong law of large numbers. Moreover, as we shall point out in Proposition 2.2 below, when suitably normalized, the sizes of any fixed finite number of fragments of (1) become asymptotically independent as $n \to \infty$, and their limiting laws are equal to P. Such a property is known as "propagation of chaos", and arises in the asymptotic behavior of mean field interacting particle systems, or in filtering theory (we refer to Sznitman [15] for general background on propagation of chaos).
In the case of partitions (1) (or more generally of exchangeable random vectors) the propagation of chaos property is equivalent to the convergence of their empirical measures to a deterministic probability measure. We will deduce from this a natural way of studying the asymptotic law of general functionals of the partition which depend only on the empirical measure of their (normalized) fragments. Moreover, we shall extend this argument in order to include partitions of type (1) induced by more general random or deterministic w_i's.
To illustrate the potential applications of this result, we will study the limiting distribution of the search-cost process in the so-called move-to-front Markov chain. A file of n items ("records") is dynamically maintained as a serial list, and items are requested at random instants, each of them with a given probability. In the move-to-front heuristic, an item is moved upon request to the front of the list (leaving the relative order of the other items unchanged). The search-cost at a given instant is defined as the position in the list of the requested item. The exact and the limiting behavior of the move-to-front rule have received much attention in the computer science and discrete probability literature since the 1960's (see Fill, [10] for historical references). For a fixed finite number of objects, the search-cost distribution has been studied by Fill, [9, 10], Fill and Holst [11], and Flajolet *et al.* [12] for deterministic request probability vectors. The limiting (large objects number) search-cost distribution is known so far only for the stationary regime. This problem was studied by Fill [9] for several deterministic request probabilities, and by Barrera and Paroissin in [3] as well as Barrera *et al.* in [1],[2] for random request probabilities, defined by partitions of the interval given by (1).
In the present work, we will explicitly compute the limiting law of the transient search-cost, when the number of objects tends to ∞. We introduce a unified approach that covers a large class of deterministic and random request probability vectors (or partitions of the interval). In particular, we will show that the limiting transient law can be decomposed in one part "in equilibrium", depending only on the large numbers limit of the partitions, and a second part "out of equilibrium", which also depends on considerations of ordering of the partitions. Moreover, there exists an explicit deterministic threshold, such that the limiting transient and stationary search-costs have the same distribution in the event of being below it. We will also give a natural explanation, in light of our results, of the coincidence of the limiting stationary search-cost for some random partitions of the type of (1) and certain deterministic partitions, a question posed in Barrera *et al.* [2].

The rest of this work is organized as follows. In Section 2 we recall the notion of propagation of chaos. We deduce from it a law of large number for random partitions as defined in (1), and prove an extension of this result to more general arrays of random variables w_i's. In Section 3 we define the move-to-front Markov chain in continuous time and the search-cost process as presented by Fill and Holst in [11]. In Section 4 we apply the concepts developed in Section 2 to compute the limiting law of the transient search-cost for random and deterministic request probabilities. We consider three different general assumptions of "a priori knowledge" about the initial ordering of the list. Namely: 1) no information at all about request probabilities is available at the beginning; 2) objects are known to be initially ranged in decreasing order of "popularity"; and 3) objects are known to be initially ranged in increasing order of "popularity". In Section 5 we prove a stochastic order relation between the limiting search-cost distributions in those three cases. In Section 6 we give some examples, with explicit expressions of the search-cost law when possible.
Let us point out that the stationary search-cost for the Dirichlet partition, and its asymptotic law in the Kingman limit (see [14] Ch. 9), were studied by Barrera *et al.* in [1]. Unfortunately, by the moment we cannot treat through our techniques the limiting Poisson-Dirichlet partition arising in that case, the scaling we consider being different. The limits of empirical functionals of partitions converging to the Poisson-Dirichlet partition were computed by Joyce and Tavaré [13], but only for a restricted class of linear functionals of the empirical measures. We cannot use that result to obtain the limiting search-cost distribution since, as we shall see, the computation of the latter involves highly nonlinear functionals. We expect however that the ideas we develop here could be extended in order to provide in the future a unified treatment of that example and others of partitions.

2 A law of large numbers for random partitions

We start by recalling the notion of "propagation of chaos", fundamental in the mathematical study of mean field models. Here and in the sequel, the notation "$\Longrightarrow$" means weak convergence of probability measures. By $\mathcal{P}(E)$ we denote the space of Borell probability measures on a metric space E.

Proposition 2.1 *Let E be a polish space and for each $n \in I\!N$ let $X^{(n)} = (X_1^{(n)}, \ldots, X_n^{(n)})$ be a random variable taking values in E^n with law P_n. Assume that the law P_n is exchangeable, that is, for all permutation π of $\{1, \ldots, n\}$,*

$$law(X_{\pi(1)}^{(n)}, \ldots, X_{\pi(n)}^{(n)}) = P_n.$$

Then, the following assertions are equivalent:

i) There exists a Borell probability measure P in E such that for all $k \in I\!N$, when $n \to \infty$,

$$law(X_1^{(n)}, \ldots, X_k^{(n)})) \Longrightarrow P^{\otimes k}.$$

ii) The sequence of random variables $Y_n := \sum_{i=1}^n \delta_{X_i^{(n)}}$ with values in the (polish) space $\mathcal{P}(E)$ converges in law to the deterministic random variable $P \in \mathcal{P}(E)$.

This result is owed to Tanaka. A proof can be found in Sznitmann [15] along with complete background on propagation of chaos. A sequence of P_n of probability measures satisfying condition *i)* of Proposition 2.1 is said to be $P-$chaotic.

Remark 2.1 *Condition ii) is equivalent to the fact that* $law(Y_n) \Longrightarrow \delta_P$ *as probability measures on the polish space* $\mathcal{P}(E)$.

We deduce a natural method for computing the asymptotic law of some functionals of random partitions (1).

Proposition 2.2 *Let* $(\omega_i)_{i\in\mathbb{N}}$ *be i.i.d. random variables on* $\mathbb{R}_+$ *of law* P *with finite mean* $\mu > 0$, *and* $\mathbf{p}^{(n)} = (p_1^{(n)}, \dots, p_n^{(n)})$ *be the random probability vector given by (1). Then, the empirical measure*

$$\nu^n := \frac{1}{n}\sum_{i=1}^{n} \delta_{n\mu p_i^{(n)}}$$

converges in law to the deterministic probability measure P. *In particular, if* $f_n : \mathbb{R}_+^n \to \mathbb{R}$, $n \in \mathbb{N}$, *are functions such that for all* n, $f_n(\mathbf{p}^{(n)}) = F(\nu^n)$ *for some bounded continuous function* $F : \mathcal{P}(\mathbb{R}_+) \to \mathbb{R}$, *then*

$$E(f_n(\mathbf{p}^{(n)})) \to F(P) \text{ when } n \to \infty.$$

Proof In view of Proposition 2.1, it is enough to prove for each $k \in \mathbb{N}$ that

$$law(n\mu p_1^{(n)}, \dots, n\mu p_k^{(n)}) \Longrightarrow P^{\otimes k} \text{ when } n \to \infty.$$

This is a simple consequence of the strong law of large numbers: we have

$$(n\mu p_1^{(n)}, \dots, n\mu p_k^{(n)}) = \frac{n\mu}{\sum_{j=1}^{n} \omega_j}(w_1, \dots, w_k) \longrightarrow (w_1, \dots, w_k)$$

almost surely when $n \to \infty$. □

We now generalize the previous result to triangular arrays $(w_i^{(n)})_{i=1}^n, n \in \mathbb{N}$ satisfying weaker laws of large numbers. The proof is self-contained using the Wasserstein distance for probability measures.

Theorem 2.1 *For each* $n \in \mathbb{N}$ *let* $(w_i^{(n)})_{i=1}^n$ *be nonnegative random variables and assume there exist positive random variables* α_n *such that the empirical measure*

$$\hat{\nu}^{(n)} := \frac{1}{n}\sum_{i=1}^{n} \delta_{\alpha_n w_i^{(n)}}$$

converges in law to the non random probability measure $P \in \mathcal{P}(\mathbb{R}_+)\backslash\{\delta_0\}$. *Assume moreover that the empirical mean* $\frac{1}{n}\sum_{i=1}^{n} \alpha_n w_i^{(n)}$ *converges in law to a finite deterministic value* $\mu > 0$. *Let* $p_i^{(n)} := \frac{w_i^{(n)}}{\sum_{j=1}^{n} w_j^{(n)}}$, $i = 1 \dots n$. *Then, the empirical measure*

$$\nu^n := \frac{1}{n}\sum_{i=1}^{n} \delta_{n\mu p_i^{(n)}}$$

converges in law to the non random probability measure P.

Proof We recall the definition of the Wasserstein distance d_1 in $\mathcal{P}(\mathbb{R}_+)$. Let

$$d_1(m, m') := \inf_Q \int_{\mathbb{R}^2} |x-y| Q(dx, dy),$$

where the inf is taken over all Borell probability measures Q on $\mathbb{R}^2$ having finite first moment and with first and second marginal respectively equal to m and m' (that is, couplings of m and m'). Then, d_1 is a distance inducing the weak topology $\mathcal{P}(\mathbb{R}_+)$, refined with the convergence of first order moments (see for instance Theorem 7.12 in Villani [16]). Let us define

$$Q^n := \frac{1}{n} \sum_{i=1}^{n} \delta_{\left(n\mu p_i^{(n)}, \alpha_n w_i^{(n)}\right)}$$

which is a coupling of $\hat{\nu}^{(n)}$ and ν^n. Then, we have

$$\begin{aligned} \int_{\mathbb{R}^2} |x-y| Q^n(dx, dy) =& \frac{1}{n} \sum_{i=1}^{n} \left| n\mu p_i^{(n)} - \alpha_n w_i^{(n)} \right| \\ =& \frac{1}{n} \sum_{i=1}^{n} \alpha_n w_i^{(n)} \left| \frac{n\mu}{\sum_{j=1}^{n} \alpha_n w_j^{(n)}} - 1 \right| \\ =& \left| \mu - \frac{\sum_{i=1}^{n} \alpha_n w_i^{(n)}}{n} \right|, \end{aligned} \tag{2}$$

from where

$$d_1(\nu^n, \hat{\nu}^{(n)}) \leq \left| \frac{\sum_{i=1}^{n} \alpha_n w_i^{(n)}}{n} - \mu \right|.$$

We conclude that

$$d_1(\nu^n, P) \leq \left| \frac{\sum_{i=1}^{n} \alpha_n w_i^{(n)}}{n} - \mu \right| + d_1(\hat{\nu}^{(n)}, P). \tag{3}$$

The result follows since $\frac{\sum_{i=1}^{n} \alpha_n w_i^{(n)}}{n} \to \mu$ and $\hat{\nu}^{(n)} \to P$ both in probability. □

Examples: In each of the following general cases, Theorem 2.1 applies, *if the convergence of the empirical mean holds*:

a) $(w_1^{(n)}, \ldots, w_n^{(n)})$, $n \in \mathbb{N}$ is itself exchangeable and chaotic ($\alpha_n = 1$).

b) $w_i^{(n)} = w_i$ for all $n \in \mathbb{N}$, with $(w_i)_{i \in \mathbb{N}}$ an ergodic process ($\alpha_n = 1$).

c) $w_i^{(n)} = i^\gamma$ for all $n \in \mathbb{N}$, for some $\gamma \in \mathbb{R}$, and $\alpha_n = n^{-\gamma}$.

Let us discuss in details Example c). (The corresponding deterministic partitions of the interval were considered in the study of the so-called move-to-root rule by Fill and Dobrow in [7] and of the move-to-front rule by Fill in [9]). We notice that in this case, the probability measure

$$\hat{\nu}^{(n)} := \frac{1}{n} \sum_{i=1}^{n} \delta_{\alpha_n w_i^{(n)}}$$

converges weakly to

$$P(dx) = \begin{cases} \frac{1}{\gamma} x^{\frac{1}{\gamma}-1} \mathbf{1}_{[0,1]}(x)dx & \text{if } \gamma > 0, \\ \delta_1(dx) & \text{if } \gamma = 0, \\ \frac{1}{|\gamma|} x^{\frac{1}{\gamma}-1} \mathbf{1}_{[1,\infty)}(x)dx & \text{if } \gamma < 0. \end{cases} \tag{4}$$

Indeed, for any continuous and bounded $f : I\!R \to I\!R$ we have for all γ that $\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i^\gamma}{n^\gamma}\right) \to \int_0^1 f(x^\gamma)dx$, using the fact that $x \mapsto x^\gamma$ is continuous in $(0,1]$ for all γ. The obvious change of variable gives us the expression of P. We deduce the following result:

Corollary 2.1 *For each $n \in I\!N$, let $p_i^{(n)}, i = 1,\ldots,n$ be proportional to i^γ. Then,*

$$\frac{1}{n}\sum_{i=1}^{n} \delta_{np_i^{(n)}}$$

converges weakly to a non degenerate limit if and only $\gamma > -1$. The limit is equal to $P \circ (x \mapsto \frac{1}{\mu}x)^{-1}$, where P is given by (4) and $\mu = \int_0^1 x^\gamma dx$.

Proof We notice that the mean $\langle \hat{\nu}^{(n)}, x\rangle$, converges to a finite value (equal to $\int_0^1 x^\gamma dx$) if and only if $\gamma > -1$. Then we apply the previous observations and Theorem 2.1.

□

(In Section 6, the previous deterministic example will be compared to some random partitions in the context of the move-to-front search-cost distribution.)

Notice that if $f : I\!R_+ \to I\!R$ is Lipschitz continuous with constant $L_f \leq 1$, the identities (2) imply that $|\langle \nu^n, f\rangle - \langle \hat{\nu}^{(n)}, f\rangle| \leq \left| \frac{\sum_{i=1}^{n} \alpha_n w_i^{(n)}}{n} - \mu \right|$, from where convergence estimates for $\langle \nu^n, f\rangle$ can be obtained. For instance, if the $w_i^{(n)} = w_i$ are non negative i.i.d random variables with law P and finite second moment, we obtain

$$E|\langle P, f\rangle - \langle \nu^n, f\rangle| \leq \frac{1}{\sqrt{n}} \left[Var(w_1) + Var(f(w_1)) \right].$$

Moreover, under stronger moment assumptions on P, one can use concentration results for the empirical measure $\hat{\nu}^{(n)}$ of i.i.d random variables proved in Bolley *et al.* [6], and inequality (3) to deduce concentration estimates for ν^n in $(\mathcal{P}(I\!R_+), d_1)$. We shall not go into details in that direction.

3 The move to front rule for deterministic popularities and its search-cost

Consider a list of n objects labelled $\{1,\ldots,n\}$. Assume that at time $t = 0$, object i is at position $\pi(i)$ of the list, for a given permutation π of $\{1,\ldots,n\}$. On later instants $t > 0$, objects are requested randomly, and the list is instantaneously modified by placing the requested objet on its top. This is the so-called Move-to-front (MTF) rule.

As in the usual analysis of the MTF rule in continuous time ([11], [5], etc), we assume that each file is requested independently, at random instants given by standard Poisson processes in the line. Plainly, let $\mathbf{w} = (w_1, \ldots, w_n)$ be a *deterministic* nonnegative vector and consider a Poisson point process in $\mathbb{R}_+ \times \{1, \ldots, n\}$ with intensity measure $dt \otimes \mathbf{w}$. The request instants of object i are given by the restriction of the point measure to $\mathbb{R}_+ \times \{i\}$ and follows a Poisson process of rate w_i. We denote by N_t the total number of requests, which is standard Poisson process of rate $\sum_{i=1}^n w_i$.
It is well known (and easily checked by the strong Markov property) that at each arrival, the probability that object i be requested is

$$p_i = \frac{w_i}{\sum_{j=1}^n w_j}.$$

We call p_i the "popularity" of object i.
For simplicity the convention is made in the MTF rule that the list is updated "instantaneously after t", if a request is made at time t. We denote

$$S_i^n(t)$$

the position of item i in the list at time t, and by $I_k \in \{1, \ldots, n\}$ the $k-$th requested file (in chronological order). Thus, $I_{N_{t-}+1}$ is the label of the first object requested in the time interval $[t, +\infty)$.

We are interested in the search-cost of the next requested item, defined by the random variable

$$S^n(t) := \sum_{i=1}^n S_i^n(t) \mathbf{1}_{\{I_{N_{t-}+1}=i\}}.$$

(Notice that $S^n(t)$ is right continuous, whereas the $S_i^n(t)$ are left continuous, by the above mentioned convention.) We will need the following notation:

- C_t is the subset of $\{1, \ldots, n\}$ of objets we have been requested at least once in $[0, t[$.
- We decompose the search-cost into two random variables,

  $$S^n(t) = S_{eq}^n(t) + S_{neq}^n(t)$$

  where

  $$S_{eq}^n(t) := S^n(t) \mathbf{1}_{\{I_{N_{t-}+1} \in C_t\}}$$

  and

  $$S_{neq}^n(t) := S^n(t) \mathbf{1}_{\{I_{N_{t-}+1} \notin C_t\}}.$$

Hence, $S_{eq}^n(t)$ is the search-cost of an object that has been requested at least once in $[0, t[$, or 0 otherwise, and $S_{neq}^n(t)$ is defined conversely. We use the subscripts eq and neq to denote "equilibrium" and "non equilibrium". This is inspired in an argument of Fill [10], who constructed a coupling between one list in its stationary regime and an arbitrary second list, both updated according to the same requests. The search-cost of an object is the same in both lists after it has been requested at least once.

Remark 3.1 *Notice that a file has has been requested before time t if and only if it lies in one of the first $|C_t|$ positions in the list. Therefore, we have*

$$\{S^n_{eq}(t) > 0\} = \{S^n(t) \leq |C_t|\}.$$

The next result will be used in the sequel:

Proposition 3.1 *Let $\pi(i)$ be the position in the list of item i at time 0. We set $\mathbf{1}_{ij} = \mathbf{1}_{\pi(i)<\pi(j)}$. Let $B_1(q_1)\ldots, B_n(q_n)$ independent Bernoulli random variables with given parameters $q_1, \ldots, q_n \in [0,1]$.*

a) For all $k, i \in \{1, \ldots, n\}$,

$$I\!P\{S^n_i(t) = k, i \in C_t\} = \int_0^t p_i e^{-p_i u} I\!P\{J^n_{eq}(u) = k\} du$$

where $J^n_{eq}(u) =^d \sum_{j=1, j\neq i}^{n} B_j(1 - e^{-p_j u})$.

b) For all $k, i \in \{1, \ldots, n\}$,

$$I\!P\{S^n_i(t) = k, i \notin C_t\} = I\!P\{J^n_{neq}(t) = k\} e^{-p_i t}$$

where $J^n_{neq}(t) =^d \sum_{j=1, j\neq i}^{n} B_j\left(1 - e^{-p_j t}\mathbf{1}_{ij}\right)$.

c) For all $k \in \{1, \ldots, n\}$,

$$I\!P\{S^n_{eq}(t) = k\} = \sum_{i=1}^{n} \int_0^t p_i^2 e^{-p_i u} I\!P\{J^n_{eq}(u) = k\} du.$$

d) For all $k \in \{1, \ldots, n\}$,

$$I\!P\{S^n_{neq}(t) = k\} = \sum_{i=1}^{n} p_i I\!P\{J^n_{neq}(t) = k\} e^{-p_i t}.$$

Proof The proof of relations *a)* and *b)* can be deduced from Proposition 2.1 in [11]. The basic ideas are to condition in the last instant $u \in]0, t]$ where object i has been requested, to consider the Poisson point process in reversed time starting from t, and use the fact that $\mathbf{1}_{ij} = 1$ if and only if object i precedes j in the initial permutation. (See Theorem 2.3.1.3. and Corollary 2.3.1.6 in [5] for a complete proof.)
Relation *c)* (resp. *d)*) follows easily from *a)* (resp. *b)*), thanks to independence of the events $\{S^n_i(t) = k, i \in C_t\}$ (resp. $\{S^n_i(t) = k, i \notin C_t\}$) and $\{I_{N_{t^-}+1} = i\}$. □

4 The limiting transient search-cost distribution

In this section, we shall use results of Section 2. to compute the limiting search-cost distribution for the MTF rule when the number n of objects tends to ∞. We will define a unified setting that includes several random or deterministic popularities. In the sequel the notation $\to^d$ means "converges in distribution". For each n we consider a random or deterministic vector of intensities $\mathbf{w}^{(n)} = (w_1^{(n)}, \dots, w_n^{(n)})$, a the Poisson process $N_t = N_t^{(n)}$ and the search-cost $S^{(n)}(t)$, both defined *conditionally* on $\mathbf{w}^{(n)}$ as in Section 3.
Let $\mathbf{p}^{(n)} = (p_1^{(n)}, \dots, p_n^{(n)})$ be the vector of popularities,

$$p_i^{(n)} = \frac{w_i^{(n)}}{\sum_{j=1}^n w_j^{(n)}}.$$

Remark 4.1 *By Proposition 3.1, the law of $S^{(n)}(t)$ conditional on $\mathbf{w}^{(n)}$ depends only on $\mathbf{p}^{(n)}$. We shall therefore need assumptions concerning only the asymptotic of behavior $\mathbf{p}^{(n)}$.*

We fix in the sequel a probability measure $P \in \mathcal{P}(\mathbb{R}_+)\backslash\{\delta_0\}$ and we make the following hypothesis

(LLN): There exists $\mu \in]0, \infty[$ such that

$$\text{the empirical measure } \nu^n := \frac{1}{n}\sum_{i=1}^n \delta_{n\mu p_i^{(n)}} \text{ converges in law to } P.$$

Remark 4.2 *Theorem 2.1 provides sufficient conditions on the sequence $\mathbf{w}^{(n)}$ for* **(LLN)** *to hold. The constant μ can be thought of as the asymptotic averaged intensity of the Poisson process $N_t^{(n)}$.*

In [2], the authors have computed the limiting stationary search-cost distribution for the MTF rule, for random popularities constructed as in (1) from independent random variables w_i of law P. That is, they considered for each n the limit in distribution $S^n(\infty)$ of the search-cost $S^n(t)$ when $t \to \infty$. Then, they proved that

$$\frac{S^n(\infty)}{n} \to^d S_\infty$$

when $n \to \infty$, where S_∞ has the density function

$$f_{S_\infty}(x) = -\frac{1}{\mu}\frac{\phi''\left(\phi^{-1}(1-x)\right)}{\phi'\left(\phi^{-1}(1-x)\right)}\mathbf{1}_{[0,1-\mathbf{p}_0]},$$

with $\phi(t)$ the Laplace transform of P and $\mathbf{p}_0 = P(\omega_i = 0)$. (The techniques used were based on Laplace integrals, see section 6.4 in [4].)
In order to study the asymptotic behavior of $S^n(t)$, we shall consider separately the random variables $S^n_{eq}(t)$ and $S^n_{neq}(t)$. It is important to notice from Proposition 3.1 that the law of $S^n_{eq}(t)$ does not depend on the initial permutation π of the list, whereas that of $S^n_{neq}(t)$ does. This is simply the fact that the cost of requesting an object, after it has been requested at least once, does not any more depend on its initial position. Accordingly, in the study of $S^n_{neq}(t)$, the permutation π will in turn play a role. In order to observe any coherent limiting behavior

of $S^n_{neq}(t)$, some assumptions on π (asymptotically in n) will thus be needed. Of course, we can always take $\pi = Id$ and translate any relevant assumption on π into properties of the vector of popularities, $\mathbf{p}^{(n)} = (p_1^{(n)}, \ldots, p_n^{(n)})$. Therefore, and for the sake of concreteness, when studying $S^n_{neq}(t)$ we assume either of the three following conditions:

(LLN)ex: (LLN) holds and the vector $\mathbf{p}^{(n)}$ is exchangeable for each $n \in \mathbb{N}$.

(LLN)$^-$: (LLN) holds, $\pi = Id$ and $\mathbf{p}^{(n)}$ is decreasing a.s. for each $n \in \mathbb{N}$.

(LLN)$^+$: (LLN) holds, $\pi = Id$ and $\mathbf{p}^{(n)}$ is increasing a.s. for each $n \in \mathbb{N}$.

Remark 4.3

- *Clearly, we may also assume that $\pi = Id$ in case* **(LLN)ex** *holds.*
- *The first 2 assumptions represent different levels of a priori knowledge on the positions of objects in the list, according to their popularities. More precisely,* **(LLN)ex** *means having "no a priori at all", whereas* **(LLN)$^-$** *is interpreted as "relative popularities are known, and objects are placed at the beginning so as to optimize the searching cost".*
- *The third assumption can be interpreted as the "worst possible initial ordering".*

Indeed, Fill and Holst proved in Corollary 4.2 in [11] that for a given finite partition, the transient search cost is stochastically larger than that of the same partition rearranged in decreasing order, and smaller than when rearranged in increasing order.
In what follows, we drop for notational simplicity the superscript (n) of the popularity $p_i^{(n)} = p_i$.

4.1 The transient search-cost "in equilibrium"

Theorem 4.1 *For $\lambda \geq 0$ and $t \geq 0$, define*

$$A_n(t,\lambda) := \mathbb{E}\left(\exp\{-\lambda S^n_{eq}(t)\}\mathbf{1}_{\{S^n_{eq}(t)>0\}}\right) = \mathbb{E}\left(\exp\{-\lambda S^n(t)\}\mathbf{1}_{\{S^n(t)\leq |C_t|\}}\right).$$

We then have

$$\lim_{n\to\infty} A_n(n\mu t, \frac{\lambda}{n}) = \frac{1}{\mu}\int_0^t \int_{\mathbb{R}^+} x^2 e^{-xu} P(dx) \exp\{-\lambda(1-\phi(u))\}\, du.$$

We shall need the following lemma on the size-biased picking of probability measures on $\mathbb{R}^+$. Recall that, given $\mathbf{m} \in \mathcal{P}(\mathbb{R}_+)$ with $0 < \langle \mathbf{m}, x\rangle < \infty$, the law of the *size-biased picking of* $\mathbf{m}$, is the law $\overline{\mathbf{m}}$ defined by

$$\overline{\mathbf{m}}(dy) = \frac{y}{\langle \mathbf{m}, x\rangle}\mathbf{m}(dy).$$

It is obtained from $\mathbf{m}$ as in the *waiting time paradox*, see Feller [8], Ch. VI.

Lemma 4.1 *Let* $(\mathbf{m}_n)$ *be a sequence of probability measures on* $\mathbb{R}^+$ *with finite means and weakly converging to a probability measure* $\mathbf{m} \neq \delta_0$. *Assume moreover that* $\langle \mathbf{m}, x\rangle < \infty$ *and that* $\langle \mathbf{m}_n, x\rangle \to \langle \mathbf{m}, x\rangle$ *when* n *goes to* ∞. *Then, we have*

$$\overline{\mathbf{m}}_n \Longrightarrow \overline{\mathbf{m}}.$$

Proof Since $\langle \overline{\mathbf{m}}_n, 1\rangle \longrightarrow \langle \overline{\mathbf{m}}, 1\rangle$ as n goes to ∞, it is enough to prove that $\langle \overline{\mathbf{m}}_n, f\rangle \longrightarrow \langle \overline{\mathbf{m}}, f\rangle$ for each continuous function f with compact support. Since for such f the function $xf(x)$ is continuous and bounded, this follows from the assumptions. □

Proof (Theorem 4.1) From Proposition 3.1, it holds that

$$A_n(n\mu t, \frac{\lambda}{n}) \quad = \quad \mathbb{E}\left(\int_0^{n\mu t} e^{-u} \sum_{i=1}^n p_i^2 \left(\prod_{j=1 j\neq i}^n \left(1 + (e^{p_j u} - 1)e^{-\lambda/n}\right)\right) du, \right).$$

Let $g_n, f_n : \mathbb{R}_+^2 \to \mathbb{R}$ be functions defined by

$$\begin{aligned} g_n(u,x) &= \frac{x^2 e^{-xu}}{1-(1-e^{-xu})(1-e^{-\lambda/n})} \\ h_n(u,x) &= n \log\left(1-(1-e^{-xu})(1-e^{-\lambda/n})\right). \end{aligned}$$

Making the right change of variable we can write $A_n(n\mu t, \frac{\lambda}{n})$ as

$$\begin{aligned} A_n(n\mu t, \frac{\lambda}{n}) &= \frac{1}{\mu}\mathbb{E}\left(\int_0^t \frac{1}{n}\sum_{i=1}^n g_n(u, np_i\mu) \exp\left(\frac{1}{n}\sum_{i=1}^n h_n(u, np_i\mu)\right) du\right) \\ &= \frac{1}{\mu}\mathbb{E}\left(\int_0^t < \nu^n, g_n(u,\cdot) > \exp\left(< \nu^n, h_n(u,\cdot) >\right) du\right). \end{aligned}$$

Now define $g(u,x) = x^2 e^{-xu}$ and $h(u,x) = -(1-e^{-xu})\lambda$. Then, if

$$\widetilde{A}_n(n\mu t, \frac{\lambda}{n}) := \frac{1}{\mu}\mathbb{E}\left(\int_0^t \langle \nu^n, g(u,\cdot)\rangle \exp\left(\langle \nu^n, h(u,\cdot)\rangle\right) du\right),$$

we see that

$$|\widetilde{A}_n(n\mu t, \frac{\lambda}{n}) - A_n(n\mu t, \frac{\lambda}{n})| \leq I_1(t,\lambda) + I_2(t,\lambda),$$

with $I_1(t,\lambda)$ and $I_2(t,\lambda)$ defined by

$$\begin{aligned} I_1(t,\lambda) &= \frac{1}{\mu}\mathbb{E}\left(\int_0^t < \nu^n, |g_n(u,\cdot) - g(u,\cdot)| > \exp\left(< \nu^n, h_n(u,\cdot) >\right) du\right) \\ I_2(t,\lambda) &= \frac{1}{\mu}\mathbb{E}\left(\int_0^t < \nu^n, g(u,\cdot) > |\exp\left(< \nu^n, h_n(u,\cdot) - h(u,\cdot) >\right) - 1|\, du\right). \end{aligned}$$

On the other hand, we have the following estimates for n large enough:

$$|g_n(u,x) - g(u,x)| \leq g(u,x)\frac{1-e^{-\lambda/n}}{1-(1-e^{-ux})(1-e^{-\lambda/n})} \leq 2g(u,x)\frac{\lambda}{n} \qquad (5)$$

(we use the bound $\frac{1-e^{-\alpha}}{1-c(1-e^{-\alpha})} \leq e^{\alpha}-1$ for $c \in [0,1], \alpha \geq 0$), and

$$
\begin{aligned}
|h_n(u,x) - h(u,x)| \leq & 2\lambda \left\{ \left(\frac{\log(1-(1-e^{-xu})(1-e^{-\lambda/n}))}{(1-e^{-xu})(1-e^{-\lambda/n})} + 1 \right) + \frac{\lambda}{n} \right\} \\
\leq & \frac{8\lambda^2}{n}.
\end{aligned} \tag{6}
$$

In the last line, we have used the bound

$$
\left| \frac{\log(1-c(1-e^{-\alpha}))}{c(1-e^{-\alpha})} + 1 \right| \leq 2(1-e^{-\alpha})
$$

for all $c \in [0,1]$ and $(1-e^{-\alpha}) \leq 1/2$.
Estimates (5) and (6) imply that for large enough n, we have

$$
I_1(t,\lambda) \leq \frac{2}{\mu} I\!E \left(\int_0^t < \nu^n, g(u,\cdot) > du \right) \frac{\lambda}{n} ,
$$

and

$$
\begin{aligned}
I_2(t,\lambda) & \leq \frac{1}{\mu} I\!E \left(\int_0^t < \nu^n, g(u,\cdot) >< \nu^n, |h_n(u,\cdot) - h(u,\cdot)| > du \right) \\
& \leq \frac{8}{\mu} I\!E \left(\int_0^t < \nu^n, g(u,\cdot) > du \right) \frac{\lambda^2}{n}.
\end{aligned}
$$

Since $\int_0^t < \nu^n, g(u,\cdot) > du = \int_{I\!R_+} x(1-e^{-xt})\nu^n(dx) \leq \mu$ by Fubini's theorem, we get from the previous estimates that

$$
|\widetilde{A}_n(n\mu t, \frac{\lambda}{n}) - A_n(n\mu t, \frac{\lambda}{n})| \leq \frac{C}{n}.
$$

for all n large enough. Consequently, we just need to prove that

$$
\lim_{n\to\infty} \widetilde{A}_n(t,s) = \frac{1}{\mu} I\!E \left(\int_0^t \langle P, g(u,\cdot)\rangle \exp\langle P, h(u,\cdot)\rangle \, du \right). \tag{7}
$$

Let us set

$$
\begin{aligned}
\triangle(t,n) := \Bigg| I\!E & \left(\int_0^t \langle \nu^n, g(u,\cdot)\rangle \exp\left(\langle \nu^n, h(u,\cdot)\rangle\right) \, du \right) \\
& - \int_0^t \langle P, g(u,\cdot)\rangle \exp\langle P, h(u,\cdot)\rangle \, du \Bigg|.
\end{aligned}
$$

For each $\delta > 0$, since $h(u,x) \leq 0$ we have the estimate

$$
\begin{aligned}
\triangle(t,n) \leq & \Bigg| I\!E \left(\int_\delta^t \langle \nu^n, g(u,\cdot)\rangle \exp\langle \nu^n, h(u,\cdot)\rangle \, du \right) \\
& - \int_\delta^t \langle P, g(u,\cdot)\rangle \exp\langle P, h(u,\cdot)\rangle \, du \Bigg| \\
& + \int_0^\delta I\!E\langle \nu^n, g(u,\cdot)\rangle du + \int_0^\delta \langle P, g(u,\cdot)\rangle du
\end{aligned}
$$

Observe that for each $u > 0$ the functions $g(u,\cdot)$ and $h(u,\cdot)$ are continuous and bounded. Moreover, for each $\delta > 0$, $g(u,\cdot)$ is bounded in x uniformly in $u \in]\delta, \infty]$. Thus, by using dominated convergence, the mapping

$$\nu \mapsto F(\nu) := \int_{\delta}^{t} \langle \nu, g(u,\cdot)\rangle \exp\langle \nu, h(u,\cdot)\rangle \, du$$

is seen to be continuous and bounded on $\in \mathcal{P}(\mathbb{R}_+)$. Thanks to **(LLN)**, we deduce that

$$\mathbb{E}(F(\nu^n)) \to F(P) \quad \text{when } n \text{ goes to } \infty$$

and, consequently, we get that for any $\delta > 0$

$$\limsup_{n\to\infty} \triangle(t,n) \leq \sup_{n\in\mathbb{N}} \int_0^{\delta} \mathbb{E}\langle \nu^n, g(u,\cdot)\rangle du + \int_0^{\delta} \langle P, g(u,\cdot)\rangle du. \tag{8}$$

In order to prove (7) it is therefore enough to establish that the two terms on the r.h.s. of inequality (8) go to 0 with δ. Notice that the second term is equal to

$$\begin{aligned}\int_{\mathbb{R}_+} \left(\int_0^{\delta} x^2 e^{-xu} du \right) P(dx) &= \int_{\mathbb{R}_+} xP(dx) - \int_{\mathbb{R}_+} xe^{-x\delta} P(dx) \\ &= \mu(\overline{\phi}(0) - \overline{\phi}(\delta)),\end{aligned}$$

where $\overline{\phi}(s) := \frac{1}{\mu} \int_{\mathbb{R}_+} xe^{-sx} P(dx)$ is the Laplace transform of the size-biased picking of P. Thus, that term goes to 0 with δ by continuity of $\bar{\phi}$.

To tackle the first term on the r.h.s in (8), we consider the "intensity measures" associated with the random measures ν^n. Namely, the (deterministic) probability measures defined for each $n \in \mathbb{N}$ by

$$\langle \mathbf{m}_n, f\rangle := \mathbb{E}\langle \nu^n, f\rangle$$

Notice that $\mathbf{m}_n$ has mean μ for all $n \in \mathbb{N}$. On the other hand, if we denote by $\overline{\mathbf{m}}_n$ the size-biased picking of $\mathbf{m}_n$, we get through similar computations as before that

$$\int_0^{\delta} \mathbb{E}\langle \nu^n, g(u,\cdot)\rangle du = \mu(\overline{\phi}_n(0) - \overline{\phi}_n(\delta)),$$

with $\overline{\phi}_n(s) := \frac{1}{\mu} \int_{\mathbb{R}_+} xe^{-sx} \mathbf{m}_n(dx)$ the Laplace transform of $\overline{\mathbf{m}}_n$.

Consequently, what we need to prove is that

$$\lim_{\delta\to 0} \sup_{n\in\mathbb{N}} |\overline{\phi}_n(\delta) - \overline{\phi}_n(0)| = 0. \tag{9}$$

But since the random measures ν^n converge in law to P, we know that

$$\langle \mathbf{m}_n, f\rangle = \mathbb{E}\langle \nu^n, f\rangle \to \langle P, f\rangle$$

if $f \in C_b(\mathbb{R})$, because the mapping $\nu \mapsto \langle \nu, f\rangle$ is continuous and bounded. In other words, the sequence $\mathbf{m}_n$ converges weakly to P. With Lemma 4.1 we deduce that the sequence $\overline{\mathbf{m}}_n$ is weakly convergent, and therefore, by standard properties of the Laplace transform the family of functions $(\overline{\phi}_n)_{n\in\mathbb{N}}$ is equicontinuous. Clearly, this implies that (9) holds, and the proof is finished. □

Remark 4.4 *Using similar techniques as in the previous theorem, it is also possible to recover the limiting stationary search-cost found in [2].*

4.2 The transient search-cost "out of equilibrium"

Theorem 4.2 *For $\lambda \geq 0$ and $t \geq 0$, define*

$$B_n(t, \lambda) := I\!\!E\left(\exp\{-\lambda S^n_{neq}(t)\}\mathbf{1}_{\{S^n_{neq}(t)>0\}}\right) = I\!\!E\left(\exp\{-\lambda S^n(t)\}\mathbf{1}_{\{S^n(t)>|C_t|\}}\right).$$

Then, $\lim_{n\to\infty} B_n(n\mu t, \frac{\lambda}{n}) = L(\mu, t, \lambda)$, where:

i) if **(LLN)ex** *holds,*

$$\begin{aligned} L(\mu, t, \lambda) =& \frac{|\phi'(t)|}{\mu}\left(\frac{e^{-\lambda(1-\phi(t))} - e^{-\lambda}}{\lambda\phi(t)}\right) \\ =& \frac{|\phi'(t)|}{\mu}\int_0^1 e^{-\lambda\phi(t)x}dx \exp\{-\lambda(1-\phi(t))\}; \end{aligned}$$

ii) if **(LLN)$^-$** *holds,*

$$L(\mu, t, \lambda) = \frac{1}{\mu}\int_0^\infty xe^{-xt}\exp\{-\lambda\int_{x^+}^\infty e^{-yt}P(dy)\}P(dx)\exp\{-\lambda(1-\phi(t))\};$$

iii) if **(LLN)$^+$** *holds,*

$$L(\mu, t, \lambda) = \frac{1}{\mu}\int_0^\infty xe^{-xt}\exp\{-\lambda\int_0^x e^{-yt}P(dy)\}P(dx)\exp\{-\lambda(1-\phi(t))\}.$$

We need the following elementary result, which we prove for completeness.

Lemma 4.2 *Let F_m denote the distribution function of $m \in \mathcal{P}(I\!\!R)$, and*

$$F_m^{-1}(x) := \inf\{t \geq 0 : F_m(t) \geq x\}$$

be its generalized inverse. Assume that $m_k \in \mathcal{P}(I\!\!R)$ converges weakly to m. Then, $F_{m_k}^{-1}(x)$ converges to $F_m^{-1}(x)$ for all $x \in [0, 1]$.

Proof If we had $\limsup_{k\to\infty} F_{m_k}^{-1}(x) > F_m^{-1}(x)$, up to subsequence we would have for some $\delta > 0$ and all $k \in I\!\!N$ large enough that $F_{m_k}(t) < x \quad \forall\, t \leq F_m^{-1}(x) + \delta$. Consequently, $m_k((-\infty, t)) < x$ and then $m((-\infty, t)) < x \leq F_m(F_m^{-1}(x))$. Taking $t = F_m^{-1}(x)$ gives a contradiction. On the other hand, if $\liminf_{k\to\infty} F_{m_k}^{-1}(x) < F_m^{-1}(x)$, then for some $\delta > 0$ and up to subsequence, we would have for each k a positive number $t_k < F_m^{-1}(x) - \delta$ such that $F_{m_k}(t_k) \geq x$. But then, $F_{m_k}(F_m^{-1}(x) - \delta) \geq x$. Since we can take δ so that $F_m^{-1}(x) - \delta$ is a point of continuity of F_m, we get that $F_m(F_m^{-1}(x) - \delta) \geq x$, again a contradiction.

□

Proof (Theorem 4.2) Recall that we always take $\pi = Id$. From Proposition 3.1 we have

$$B_n(n\mu t, \frac{\lambda}{n}) \quad = \quad I\!\!E\left(\sum_{i=1}^n p_i e^{-n\mu t}\prod_{j=1, j\neq i}^n \left[\mathbf{1}_{i<j} + (e^{n\mu p_j t} - \mathbf{1}_{i<j})e^{-\lambda/n}\right]\right).$$

Since $\sum_j p_j = 1$, we can rewrite

$$B_n(n\mu t, \tfrac{\lambda}{n}) = \mathbb{E}\left(\sum_{i=1}^{n} p_i e^{-n\mu p_i t} \prod_{j=1, j\neq i}^{n} \left[1 - (1 - e^{-\lambda/n})(1 - \mathbf{1}_{i<j} e^{-n\mu p_j t})\right] \right).$$

Let us define

$$\tilde{B}_n := E\left(\frac{1}{n\mu} \sum_{i=1}^{n} n\mu p_i e^{-n\mu p_i t} \exp\left\{ -\frac{\lambda}{n} \sum_{j=1, j\neq i}^{n} (1 - e^{-n\mu p_j t} \mathbf{1}_{i<j}) \right\} \right)$$

It is elementary to check that $|B_n(n\ \mu t, \frac{\lambda}{n}) - \tilde{B}_n| \leq \frac{C}{n}$, so we shall study the term $\tilde{B}_n$. We have that

$$\tilde{B}_n = E\Bigg(\exp\left\{ -\frac{\lambda}{n} \sum_{j=1}^{n} 1 - e^{-n\mu p_j t} \right\} \times \\ \frac{1}{n\mu} \sum_{i=1}^{n} n\mu p_i e^{-n\mu p_i t} \exp\left\{ -\frac{\lambda}{n} \sum_{j=1}^{i} e^{-n\mu p_j t} \right\} e^{-\frac{\lambda}{n}} \Bigg).$$

Therefore, we have

$$|e^{\frac{\lambda}{n}} \tilde{B}_n - L(\mu, t, \lambda)| \leq E\,|\Psi(\nu^n)| + \left| E(\hat{L}_n(\mu, t, \lambda)) - \hat{L}(\mu, t, \lambda) \right|,$$

with

$$\Psi(m) := \exp\left\{ -\lambda \int_{\mathbb{R}_+} 1 - e^{-xt} m(dx) \right\} - \exp\{-\lambda(1 - \phi(t))\},$$

$$\hat{L}_n(\mu, t, \lambda) := \frac{1}{n\mu} \sum_{i=1}^{n} n\mu p_i e^{-n\mu p_i t} \exp\left\{ -\frac{\lambda}{n} \sum_{j=1}^{i} e^{-n\mu p_j t} \right\}$$

and $\hat{L}(\mu, t, \lambda)$ defined as follows:

$$\hat{L}(\mu, t, \lambda) = \frac{-\phi'(t)}{\mu} \int_0^1 e^{-\lambda\phi(t)x} dx \quad \text{if } \mathbf{(LLN)ex} \text{ holds,}$$

$$\hat{L}(\mu, t, \lambda) = \frac{1}{\mu} \int_0^\infty x e^{-xt} \exp\{-\lambda \int_x^\infty e^{-yt} P(dy)\} P(dx) \quad \text{if } \mathbf{(LLN)}^- \text{ holds, or}$$

$$\hat{L}(\mu, t, \lambda) = \frac{1}{\mu} \int_0^\infty x e^{-xt} \exp\{-\lambda \int_0^x e^{-yt} P(dy)\} P(dx) \quad \text{if } \mathbf{(LLN)}^+ \text{ holds.}$$

Since Ψ is continuous and bounded in $\mathcal{P}(\mathbb{R}_+)$ and $\Psi(P) = 0$, we get by **(LLN)** that $E\,|\Psi(\nu^n)| \to 0$ when $n \to \infty$. Thus, we just have to prove that

$$E(\hat{L}_n(\mu, t, \lambda)) \longrightarrow \hat{L}(\mu, t, \lambda).$$

The exchangeable case. Notice that under **(LLN)ex**,

$$E(\hat{L}_n(\mu,t,\lambda)) = E\bigg(\frac{1}{n\mu}\sum_{i=1}^{n}\frac{1}{n!}\sum_{\sigma\in\Pi} n\mu p_{\sigma(i)}e^{-n\mu p_{\sigma(i)}t}\exp\Big\{-\frac{\lambda}{n}\sum_{j=1}^{i}e^{-n\mu p_{\sigma(j)}t}\Big\}\bigg)$$

$$= E\bigg(\frac{1}{n\mu}\sum_{i=1}^{n}\sum_{k=1}^{n}\frac{1}{n!}\sum_{\sigma\in\Pi,\sigma(i)=k} n\mu p_{\sigma(i)}e^{-n\mu p_{\sigma(i)}t}\exp\Big\{-\frac{\lambda}{n}\sum_{j=1}^{i}e^{-n\mu p_{\sigma(j)}t}\Big\}\bigg)$$

$$= E\bigg(\frac{1}{n\mu}\sum_{k=1}^{n} n\mu p_k e^{-n\mu p_k t}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{(n-1)!}\sum_{\sigma\in\Pi,\sigma(i)=k}\exp\Big\{-\frac{\lambda}{n}\sum_{j=1}^{i}e^{-n\mu p_{\sigma(j)}t}\Big\}\bigg)$$

Since by **(LLN)**,

$$E\bigg(\frac{1}{n\mu}\sum_{k=1}^{n} n\mu p_k e^{-n\mu p_k t}\bigg)\int_0^1 e^{-\lambda\phi(t)x}dx \longrightarrow \frac{-\phi'(t)}{\mu}\int_0^1 e^{-\lambda\phi(t)x}dx$$

when $n\to\infty$, it is enough to show that $\delta_n(\mu,t,\lambda)$ goes to 0 when $n\to\infty$, where

$$\begin{aligned}\delta_n(\mu,t,\lambda) :=& E(\hat{L}_n(\mu,t,\lambda)) - E\bigg(\frac{1}{n\mu}\sum_{k=1}^{n} n\mu p_k e^{-n\mu p_k t}\bigg)\int_0^1 e^{-\lambda\phi(t)x}dx\\ =& E(\hat{L}_n(\mu,t,\lambda)) - \frac{1}{\mu}E\bigg(\int_{I\!R_+} x e^{-xt}\nu^n(dx)\bigg)\int_0^1 e^{-\lambda\phi(t)x}dx.\end{aligned}$$

Let us write for $i = 1,\ldots,n-1$, and a permutation σ of $\{1,\ldots,n\}$,

$$\alpha_t^\sigma(i,n) := \sum_{j=1}^{i} e^{-n\mu p_{\sigma(j)}t},\ \text{ and }\ \alpha_t(n,n) := \sum_{j=1}^{n} e^{-n\mu p_j t}.$$

Define furthermore

$$\begin{aligned}I_n^k =& \frac{1}{n}\sum_{i=1}^{n}\Bigg(\exp\Big\{-\frac{\lambda}{n}\alpha_t(n,n)\frac{i}{n}\Big\}\times\\ &\frac{1}{(n-1)!}\sum_{\sigma\in\Pi,\sigma(i)=k}\bigg[\exp\Big\{-\frac{\lambda}{n}\Big[\alpha_t^\sigma(i,n)-\frac{i}{n}\alpha_t(n,n)\Big]\Big\}-1\bigg]\Bigg),\\ II_n =& \frac{1}{n}\sum_{i=1}^{n}\exp\Big\{-\frac{\lambda}{n}\alpha_t(n,n)\frac{i}{n}\Big\} - \exp\Big\{-\lambda\phi(t)\frac{i}{n}\Big\},\end{aligned}$$

and

$$III_n = \frac{1}{n}\sum_{i=1}^{n}\exp\Big\{-\lambda\phi(t)\frac{i}{n}\Big\} - \int_0^1 e^{-\lambda\phi(t)x}dx.$$

Then, we have

$$\begin{aligned}|\delta_n(\mu,t,\lambda)| \le& \bigg|E\bigg(\frac{1}{n\mu}\sum_{k=1}^{n} n\mu p_k e^{-n\mu p_k t}\left(I_n^k + II_n + III_n\right)\bigg)\bigg|\\ \le& \frac{1}{t\mu}\left[\frac{1}{n}\sum_{k=1}^{n}E|I_n^k| + E|II_n| + |III_n|\right]\end{aligned}$$

thanks to the bound $xe^{-xt} \leq \frac{1}{t}$. Term III_n clearly goes to 0 when $n \to \infty$. On the other hand, we have

$$E|II_n| \leq \frac{\lambda}{n} \sum_{i=1}^{n} \frac{i}{n} E\left|\frac{1}{n}\alpha_t(n,n) - \phi(t)\right| \leq \frac{\lambda}{2} E\left|\int_{I\!R_+} e^{-xt}\nu^n(dx) - \phi(t)\right|.$$

The mapping $\nu \mapsto |\int_{I\!R_+} e^{-xt}\nu(dx) - \int_{I\!R_+} e^{-xt}P(dx)|$ being continuous and bounded on $\mathcal{P}(I\!R_+)$, the latter term goes to 0 by **(LLN)**.
Now, by exchangeability $E|I_n^k|$ does not depend on k, and moreover, setting $\alpha_t(i,n) := \sum_{j=1}^{i} e^{-n\mu p_j t}$, we have

$$\begin{aligned}
E|I_n^k| &\leq \frac{1}{n}\sum_{i=1}^{n} E\left|\exp\left\{-\frac{\lambda}{n}\big[\alpha_t(i,n) - \frac{i}{n}\alpha_t(n,n)\big]\right\} - 1\right| \\
&\leq \frac{1}{n}\sum_{i=1}^{n} E\left|\frac{\lambda}{n}\left[\sum_{j=1}^{i} e^{-n\mu p_j t} - \frac{i}{n}\sum_{k=1}^{n} e^{-n\mu p_k t}\right]\right| \\
&= \frac{1}{n}\sum_{i=1}^{n} E\left|\frac{\lambda}{n}\sum_{j=1}^{i}\left(e^{-n\mu p_j t} - \int e^{-xt}\nu^n(dx)\right)\right|,
\end{aligned}$$

and so

$$E|I_n^k| \leq \frac{1}{n}\sum_{i=1}^{n} E\left|\frac{\lambda}{n}\sum_{j=1}^{i}\left(e^{-n\mu p_j t} - \phi(t)\right)\right| + \frac{\lambda}{2}E\left|\phi(t) - \int e^{-xt}\nu^n(dx)\right|.$$

Thus, we just have to check that $IV_n := \frac{\lambda}{n^2}\sum_{i=1}^{n} E\left|\sum_{j=1}^{i}\left(e^{-n\mu p_j t} - \phi(t)\right)\right|$ goes to 0. Indeed, we have

$$\begin{aligned}
IV_n &\leq \frac{\lambda}{n^2}\sum_{i=1}^{n}\left[E\Big(\sum_{j=1}^{i}\left(e^{-n\mu p_j t} - \phi(t)\right)\Big)^2\right]^{1/2} \\
&= \frac{\lambda}{n^2}\sum_{i=1}^{n}\Bigg[\sum_{j=1}^{i} E\left(e^{-n\mu p_j t} - \phi(t)\right)^2 \\
&\qquad + \sum_{k=1}^{i}\sum_{l=1, l\neq k}^{i} E(e^{-n\mu p_l t} - \phi(t))(e^{-n\mu p_k t} - \phi(t))\Bigg]^{1/2} \\
&\leq \frac{2\lambda}{\sqrt{n}} + \frac{\lambda}{n^2}\sum_{i=1}^{n}\left[i(i-1)\left|E(e^{-n\mu p_1 t} - \phi(t))(e^{-n\mu p_2 t} - \phi(t))\right|\right]^{1/2}.
\end{aligned}$$

Therefore,

$$IV_n \leq \frac{2\lambda}{\sqrt{n}} + \lambda\left|E(e^{-n\mu p_1 t} - \phi(t))(e^{-n\mu p_2 t} - \phi(t))\right|^{1/2}$$

By **(LLN)** and Proposition 2.1, *i)* with $k = 2$, we conclude that the latter term goes to 0. This finishes the proof in the exchangeable case.

The monotone cases. We consider the case when **(LLN)**$^+$ holds, the decreasing case being similar. Notice that if $F_n^{-1}(x) := \inf\{t \geq 0 : F_n(t) \geq x\}$ is the generalized inverse of $F_n(x) = \nu^n([0,x])$, we have that

$$\hat{L}_n(\mu,t,\lambda) := \frac{1}{\mu}\int_0^1 F_n^{-1}(x)e^{-F_n^{-1}(x)t}\exp\left\{-\lambda\int_0^{i_n(x)} e^{-F_n^{-1}(y)t}dy\right\}dx$$

where $i_n(x) = \frac{\lceil nx \rceil}{n}$ and $\lceil\cdot\rceil$ is the ceiling function. On the other hand, since the generalized inverse $F^{-1} : [0,1] \to I\!R_+$ of F is a random variable of law P under dx, we have

$$\hat{L}(\mu,t,\lambda) := \frac{1}{\mu}\int_0^1 F^{-1}(x)e^{-F^{-1}(x)t}\exp\left\{-\lambda\int_0^{x} e^{-F^{-1}(y)t}dy\right\}dx.$$

Thanks to this and the bound $xe^{-xt} \leq \frac{1}{t}$, we get that

$$\begin{aligned}
|E(\hat{L}_n(\mu,t,\lambda)) - \hat{L}(\mu,t,\lambda)| &\leq \frac{\lambda}{t\mu}E\left(\int_0^1\int_0^x |e^{-F_n^{-1}(y)t} - e^{-F^{-1}(y)t}|dy\ dx\right) \\
&\quad + \frac{\lambda}{t\mu}\int_0^1\int_x^{i_n(x)} e^{-F^{-1}(y)t}dy\ dx \\
&\quad + \frac{1}{\mu}E\left(\int_0^1 \left|F_n^{-1}(x)e^{-F_n^{-1}(x)t} - F^{-1}(x)e^{-F^{-1}(x)t}\right|dx\right) \\
&\leq \frac{\lambda}{t\mu}E\left(\int_0^1 |e^{-F_n^{-1}(y)t} - e^{-F^{-1}(y)t}|dy\right) + \frac{\lambda}{nt\mu} \\
&\quad + \frac{1}{\mu}E\left(\int_0^1 \left|F_n^{-1}(x)e^{-F_n^{-1}(x)t} - F^{-1}(x)e^{-F^{-1}(x)t}\right|dx\right)
\end{aligned}$$

Therefore, and thanks also to **(LLN)**, it is enough to prove that the bounded functionals on $\mathcal{P}(I\!R_+)$

$$\nu \mapsto \int_0^1 |e^{-F_\nu^{-1}(y)t} - e^{-F^{-1}(y)t}|dy$$

and

$$\nu \mapsto \int_0^1 \left|F_\nu^{-1}(x)e^{-F_\nu^{-1}(x)t} - F^{-1}(x)e^{-F^{-1}(x)t}\right|dx$$

are continuous, since they both vanish at $\nu = P$. This follows by dominated convergence and Lemma 4.2. The proof of the theorem is finished.

□

4.3 The limiting transient search-cost distribution

Let S_∞ be a random variable in $[0,1]$ with density function

$$f_{S_\infty}(x) = -\frac{1}{\mu}\frac{\phi''\left(\phi^{-1}(1-x)\right)}{\phi'\left(\phi^{-1}(1-x)\right)}\mathbf{1}_{[0,1-\mathbf{p}_0]},$$

and U be a uniform random variable in $[1-\phi(t),1]$ independent of S_∞.

Corollary 4.1 *If* **(LLN)ex** *holds, for each $t > 0$ we have*

$$\frac{S^n(n\mu t)}{n} \to^d S(t) \ ,$$

where $S(t)$ satisfies the following relation in distribution:

$$S(t) =^{(d)} S_\infty \mathbf{1}_{\{S_\infty \le 1-\phi(t)\}} + U\mathbf{1}_{\{S_\infty > 1-\phi(t)\}}. \tag{10}$$

Moreover, when $n \to \infty$ we have

$$I\!P\left(S^n_{neq}(n\mu t) > 0\right) \longrightarrow I\!P(S_\infty > 1 - \phi(t)) \ = \frac{|\phi'(t)|}{\mu}.$$

Finally, the random variable $S(t)$ has density

$$f_{S(t)}(x) = f_{S_\infty}(x)\mathbf{1}_{[0,1-\phi(t)]} + \frac{|\phi'(t)|}{\mu\phi(t)}\mathbf{1}_{[1-\phi(t),1]}$$

and with $\|\cdot\|_{TV}$ denoting the total variation distance, we have

$$\begin{aligned} \|law(S(t)) - law(S_\infty)\|_{TV} &= \int_{1-\phi(t)}^{1-\mathbf{p}_0} \left| f_{S_\infty}(x) + \frac{\phi'(t)}{\mu\phi(t)} \right| dx + \frac{\phi'(t)\mathbf{p}_0}{\mu\phi(t)} \\ &\le 2\frac{|\phi'(t)|}{\mu}. \end{aligned}$$

Proof The Laplace transform of S_∞ is

$$I\!E(\exp\{-\lambda S_\infty\}) = \int_0^\infty \phi''(u)\exp\{-\lambda(1-\phi(u))\}du$$

(see [2] for details). Now, from Theorem 4.1 we have

$$\lim_{n\to\infty} A_n(n\mu t, \frac{\lambda}{n}) = \frac{1}{\mu}\int_0^t \phi''(u)\exp\Big\{ -\lambda(1-\phi(u))\Big\}\, du.$$

Taking $\lambda = 0$ and using Remark 3.1, we obtain:

$$\begin{aligned} \lim_{n\to\infty} P\left(\frac{S^n(n\mu t)}{n} \le \frac{|C_{n\mu t}|}{n}\right) &= \lim_{n\to\infty} A_n(n\mu t, 0) \\ &= 1 - \frac{(-\phi'(t))}{\mu} \\ &= I\!P(S_\infty \le 1-\phi(t)) \ . \end{aligned}$$

On the other hand, since the Laplace transform of $\frac{S^n(n\mu t)}{n}$ conditional on the event $\frac{S^n(n\mu t)}{n} \le \frac{|C_{n\mu t}|}{n}$ is given by

$$I\!E\left(\exp\left\{-\lambda\frac{S^n(n\mu t)}{n}\right\} \ \Big| \ \frac{S^n(n\mu t)}{n} \le \frac{|C_{n\mu t}|}{n}\right) = \frac{A_n(n\mu t, \frac{\lambda}{n})}{A_n(n\mu t, 0)},$$

we obtain

$$\begin{aligned} &\lim_{n\to\infty} I\!E\left(\exp\left\{-\lambda\frac{S^n(n\mu t)}{n}\right\} \ \Big| \ \frac{S^n(n\mu t)}{n} \le \frac{|C_{n\mu t}|}{n}\right) = \\ &\qquad \frac{1}{\mu+\phi'(t)}\int_0^t \phi''(u)\exp\Big\{ -\lambda(1-\phi(u))\Big\}\, du \\ &\qquad\qquad = I\!E\left(\exp\{-\lambda S_\infty\} \ \big| \ S_\infty \le 1-\phi(t)\right) \ . \end{aligned}$$

Considering the limiting behavior of B_n we get in a similar way that

$$\lim_{n\to\infty} I\!P\left(\frac{S^n(n\mu t)}{n} > \frac{|C_{n\mu t}|}{n}\right) = -\frac{\phi'(t)}{\mu} = I\!P(S_\infty > 1-\phi(t)) \ .$$

and

$$\begin{aligned}\lim_{n\to\infty} I\!E\left(\exp\left\{-\lambda\frac{S^n(n\mu t)}{n}\right\} \,\middle|\, \frac{S^n(n\mu t)}{n} > \frac{|C_{n\mu t}|}{n}\right) &= \lim_{n\to\infty}\frac{B_n(n\mu t, \frac{\lambda}{n})}{B_n(n\mu t, 0)}\\ &= \left(\frac{e^{-\lambda(1-\phi(t))} - e^{-\lambda}}{\lambda\phi(t)}\right)\\ &= I\!E(\exp\{-\lambda U\}) \ .\end{aligned}$$

Combining the previous limits yields

$$\begin{aligned}\lim_{n\to\infty} I\!E\left(\exp\left\{-\lambda\frac{S^n(n\mu t)}{n}\right\}\right) &= \lim_{n\to\infty} A_n(n\mu t, \frac{\lambda}{n}) + B_n(n\mu t, \frac{\lambda}{n})\\ &= I\!E\left(\exp\{-\lambda S_\infty\}\mathbf{1}_{\{S_\infty\le 1-\phi(t)\}}\right)\\ &\quad + I\!E(\exp\{-\lambda U\})P(S_\infty > 1-\phi(t))\\ &= I\!E\left(\exp\{-\lambda S_\infty\}\mathbf{1}_{\{S_\infty\le 1-\phi(t)\}}\right)\\ &\quad + I\!E\left(\exp\{-\lambda U\}\mathbf{1}_{\{S_\infty> 1-\phi(t)\}}\right)\\ &= I\!E\left(\exp -\lambda\{S_\infty\mathbf{1}_{\{S_\infty\le 1-\phi(t)\}} + U\mathbf{1}_{\{S_\infty> 1-\phi(t)\}}\}\right) .\end{aligned}$$

From the last expression we obtain the density of $S(t)$, and then and the total variation distance to equilibrium. The last asserted inequality follows from the fact that $\|law(X) - law(Y)\|_{TV} \le 2I\!P\{X \ne Y\}$ for any coupling of random variables (X, Y).

□

Corollary 4.2 *Define*

$$g_t(y) = \int_0^y e^{-zt}P(dz), \quad \tilde{g}_t(y) = g_t^{-1}(1-y) \text{ if } \mathbf{(LLN)}^- \text{ holds,}$$

and

$$g_t(y) = \int_{y^+}^\infty e^{-zt}P(dz), \quad \tilde{g}_t(y) = (1-g_t)^{-1}(y) \text{ if } \mathbf{(LLN)}^+ \text{ holds.}$$

Here g^{-1} means the generalized inverse of a non decreasing right continuous function $g : I\!R_+ \to I\!R_+$. Then, under **(LLN)**$^-$ *or* **(LLN)**$^+$*, for each $t > 0$ we have*

$$\frac{S^n(n\mu t)}{n} \to^d S(t) \ ,$$

where $S(t)$ has the density

$$f_{S(t)}(x) = \mathbf{1}_{[0,1-\phi(t)]}(x)f_{S_\infty}(x) + \mathbf{1}_{[1-\phi(t),1]}(x)\frac{1}{\mu}\tilde{g}_t(x). \qquad (11)$$

Moreover, we have

$$I\!P\left(S^n_{neq}(n\mu t) > 0\right) \longrightarrow I\!P(S(t) > 1-\phi(t)) \ = I\!P(S_\infty > 1-\phi(t)) \ = \frac{|\phi'(t)|}{\mu},$$

when $n \to \infty$, and for all $t \ge 0$, $\|law(S(t)) - law(S_\infty)\|_{TV} \le 2\frac{|\phi'(t)|}{\mu}$.

Proof If **(LLN)**$^-$ holds, the result follows by using Theorem 4.2 and making the change of variable $z = 1 - g_t(x)$ to obtain

$$\begin{aligned} L(\mu, t, \lambda) =& \frac{1}{\mu} \int_0^\infty x e^{-xt} \exp\{-\lambda \int_{x^+}^\infty e^{-yt} P(dy)\} P(dx) \exp\{-\lambda(1 - \phi(t))\} \\ =& \frac{1}{\mu} \int_{1-\phi(t)}^1 \exp\{-\lambda z\} g_t^{-1}(1 - z) dz. \end{aligned}$$

The remaining case is similar. □

5 A stochastic order relation

Recall that given two real valued random variables X and Y, we say that X is stochastically smaller than Y, if for all $z \in \mathbb{R}$, one has

$$\mathbb{P}(X \leq z) \geq \mathbb{P}(Y \leq z).$$

We write in that case $X \preceq Y$. The following result can be naturally expected, as a consequence of Corollary 4.2 in [11]. We give here a proof based on the explicit expressions for $f_{S(t)}$ we have obtained:

Corollary 5.1 *Let $S^{ex}(t)$, $S^+(t)$ and $S^-(t)$ denote the limiting transient search-cost $S(t)$ respectively under the assumptions,* **(LLN)ex**, **(LLN)**$^+$ *and* **(LLN)**$^-$. *Then, we have*

$$S^-(t) \preceq S^{ex}(t) \preceq S^+(t)$$

Proof From Corollaries 4.1 and 4.2, we just need to prove that

$$\mathbb{P}\{1-\phi(t) \leq S^-(t) \leq x\} \geq \mathbb{P}\{1-\phi(t) \leq S^{ex}(t) \leq x\} \geq \mathbb{P}\{1-\phi(t) \leq S^+(t) \leq x\}$$

for all $x \in [1 - \phi(t), 1]$. The first inequality is equivalent to the following one,

$$\int_{g_t^{-1}(1-x)}^\infty z e^{-zt} P(dz) \geq \frac{|\phi'(t)|}{\phi(t)}(x - 1 + \phi(t)),$$

for all $x \in [1 - \phi(t), 1]$, where $g_t(y) = \int_0^y e^{-zt} P(dz)$. This will follow if we can prove that

$$\frac{\int_{y+}^\infty z e^{-zt} P(dz)}{|\phi'(t)|} \geq \frac{\int_{y+}^\infty e^{-zt} P(dz)}{\phi(t)}$$

for all $y \geq 0$ or, equivalently,

$$\frac{\int_0^y z e^{-zt} P(dz)}{|\phi'(t)|} \leq \frac{\int_0^y e^{-zt} P(dz)}{\phi(t)}. \quad (12)$$

Observe that both sides have the same points of discontinuity, as functions of y. Therefore, by suitably approximating P, we may assume that $P(dz)$ has a continuous density $f(z)$ which is strictly positive. Write $a(y) = \int_0^y z e^{-zt} f(z) dz$ and $b(y) = \int_0^y e^{-zt} f(z) dz$. We need to check that

$$h(y) := \frac{a(y)}{a(\infty)} - \frac{b(y)}{b(\infty)} \leq 0.$$

Since h is differentiable and $h(0) = h(\infty) = 0$, it is enough to prove that h has a unique critical point y_0 and that $h(y_0) \leq 0$. By the assumption on P, the condition $h'(y_0) = 0$ is satisfied if and only if $y_0 = \frac{a(\infty)}{b(\infty)}$. But then, $h(y_0) \leq 0$ is the same as

$$\frac{\int_0^{\frac{a(\infty)}{b(\infty)}} ze^{-zt}f(z)dz}{a(\infty)} \leq \frac{\int_0^{\frac{a(\infty)}{b(\infty)}} e^{-zt}f(z)dz}{b(\infty)},$$

which is trivially true. We conclude that $I\!P\{1 - \phi(t) \leq S^-(t) \leq x\} \geq I\!P\{1 - \phi(t) \leq S^{ex}(t) \leq x\}$ for all $x \geq 0$. The remaining inequality is easily seen to follow also from (12). □

6 Examples

We now compute the limiting distribution of the transient search-cost for some examples of random and non random partitions. Their stationary regimes were analyzed in [2]. In each example, w_i are i.i.d random variables following a specified law, or deterministic quantities, and we use in each case the results of the previous section accordingly.
The first three are examples of random partitions yielding simple expressions for $f_{S(t)}$.

1) Let $w_i \sim$ Bernoulli(p), then

$$f_{S(t)}(x) = \frac{1}{p}\mathbf{1}_{[0,p(1-e^{-t}))}(x) + \frac{e^{-t}}{1-p+pe^{-t}}\mathbf{1}_{[p(1-e^{-t}),1]}(x)\ .$$

2) Let $w_i \sim$ Gamma$(1, \alpha)$, then

$$f_{S(t)}(x) = \left(1 + \frac{1}{\alpha}\right)(1-x)^{1/\alpha}\mathbf{1}_{[0,u(t))}(x) + (1+t)^{-1}\mathbf{1}_{[u(t),1]}(x)\ ,$$

with $u(t) = 1 - (1+t)^{-\alpha}$.

3) If $w_i \sim$ Geometric(p), then

$$f_{S(t)}(x) = \frac{2(1-x)-p}{1-p}\mathbf{1}_{[0,u(t))}(x) + \frac{pe^{-t}}{1-(1-p)e^{-t}}\mathbf{1}_{[u(t),1]}(x)\ ,$$

where $u(t) = \frac{(1-p)(1-e^{-t})}{p+(1-p)(1-e^{-t})}$.

Let now $\gamma \in (-1, 0)$ and define

$$\begin{aligned} f_\gamma(x) &= -\frac{1}{\gamma}x^{1/\gamma-1}\mathbf{1}_{[1,\infty)}(x),\ \text{(Pareto density)} \\ \phi(s) &= -\frac{1}{\gamma}\int_1^\infty e^{-xs}x^{1/\gamma-1}dx, \\ g_t(y) &= -\frac{1}{\gamma}\int_1^y e^{-xt}x^{1/\gamma-1}dx. \end{aligned}$$

4.i) If $w_i = i^\gamma$, the we have

$$f_{S(t)}(x) = -\frac{1}{\gamma+1}\frac{\phi''(\phi^{-1}(1-x))}{\phi'(\phi^{-1}(1-x))}\mathbf{1}_{[0,1-\phi(t))}(x) + \frac{1}{\gamma+1}g_t^{-1}(1-x)\mathbf{1}_{[1-\phi(t),1]}(x).$$

4.ii) If $w_i \sim f_\gamma$, then

$$f_{S(t)}(x) = -\frac{1}{\gamma+1}\frac{\phi''(\phi^{-1}(1-x))}{\phi'(\phi^{-1}(1-x))}\mathbf{1}_{[0,1-\phi(t))}(x) + \frac{1}{1+\gamma}\frac{|\phi'(t)|}{\phi(t)}\mathbf{1}_{[1-\phi(t),1]}(x) ,$$

Let $\gamma > 0$ and set now

$$\begin{aligned}\phi(s) &= \frac{1}{\gamma}\int_0^1 e^{-xs}x^{1/\gamma-1}dx, \\ g_t(y) &= \frac{1}{\gamma}\int_y^1 e^{-xt}x^{1/\gamma-1}dx.\end{aligned}$$

5.i) If $w_i = i^\gamma$, we have

$$f_{S(t)}(x) = -\frac{1}{\gamma+1}\frac{\phi''(\phi^{-1}(1-x))}{\phi'(\phi^{-1}(1-x))}\mathbf{1}_{[0,1-\phi(t))}(x) + \frac{1}{\gamma+1}(1-g_t)^{-1}(x)\mathbf{1}_{[1-\phi(t),1]}(x) ,$$

5.ii) If $w_i \sim \mathrm{Beta}(1, 1/\gamma)$, then

$$f_{S(t)}(x) = -\frac{1}{\gamma+1}\frac{\phi''(\phi^{-1}(1-x))}{\phi'(\phi^{-1}(1-x))}\mathbf{1}_{[0,1-\phi(t))}(x) + \frac{1}{1+\gamma}\frac{|\phi'(t)|}{\phi(t)}\mathbf{1}_{[1-\phi(t),1]}(x) .$$

Finally,

6) if $w_i = 1$ or equivalently, $w_i \sim \delta_1$, we get $f_{S(t)}(x) = 1$, using any of **(LLN)ex**, **(LLN)**$^+$ or **(LLN)**$^-$. That is, the limiting search cost is uniform for all $t \geq 0$.

The stationary distributions associated with examples 4.i) and 5.i) were studied in Fill [9]. The fact that examples 4.i) and 4.ii) share the same stationary distribution, so as for 5.i) and 5.ii), was remarked by Barrera *et al.* in [2]. Nevertheless, an explanation to that fact was not furnished.
Our results show that indeed, such coincidence is expectable for every symmetric functional of random or deterministic partitions giving raise to the same deterministic probability measure P in its large numbers limit. In the move-to-front rule search cost, that was (asymptotically) the case for the stationary search-cost and the "equilibrium" part of the transient search-cost, but not for its "out of equilibrium" part. However, under the assumption of exchangeability of the partition, limits of expectations of any functional can be studied, replacing it by a "symmetrized version" as in the proof of Theorem 4.2.

References

[1] J. Barrera, T. Huillet, and C. Paroissin(2005). Size-biased permutation of Dirichlet partitions and search-cost distribution. *Probab. Engrg. Inform. Sci.* **19** no. 1 83–97.

[2] J. Barrera, T. Huillet, and C. Paroissin(2006). Limiting search cost distribution for move-to-front rule with random request probabilities. *Oper. Res. Lett.* **35** no. 5 557-563.

[3] J. Barrera and C. Paroissin(2004). On the distribution of the search cost for the move-to-front rule with random weights. *J. Appl. Probab.* **41** no. 1 250–262.

[4] C.M. Bender and S.A. Orszag(1999). *Advanced mathematical methods for scientists and engineers: Asymptotic methods and perturbation theory.* Springer, New-York.

[5] J. Bodell (1997)*Cost of searching - probablistic analysis of the self organizing Move-to-Front and Move-to-Root sorting rules*, Ph.D. thesis, Mathematics Department, Royal Institute of Technology, Sweden.

[6] F. Bolley, A. Guillin and C. Villani *Quantitative Concentration Inequalities for Empirical Measures on Non-compact Space* to appear in *Probab. Theory Related Fields.*

[7] R.P. Dobrow and J.A. Fill(1995) Rates of convergence for the move-to-root Markov chain for binary search trees, *Ann. Appl. Probab.* **5** no. 1 20–36.

[8] W. Feller(1966) *An introduction to Probability Theory and its applications.* Vol. II, John Wiley and Sons.

[9] J.A. Fill(1996). Limits and rates of convergence for the distribution of search cost under the move-to-front rule. *Theoret. Comput. Sci.* **164** no. 1-2 185–206.

[10] J.A. Fill(1996). An exact formula for the move-to-front rule for the self-organised list. *J. Appl. Probab.* **9** no. 1 113–160.

[11] J.A. Fill and L. Holst(1996) On the distribution of search cost for the move-to-front rule. *Random Structures Algorithms* **8**, no. 3, 179–186.

[12] P. Flajolet, D. Gardy, and L. Thimonier(1992) Birthday paradox, coupon collectors, caching algorithms and self-organizing search. *Discrete Appl. Math.* **39** , no. 3, 207–229.

[13] P. Joyce, S. Tavaré(1992) A convergence theorem for symmetric functionals of random partitions. *J. Appl. Probab.* **29** no. 2 280–290.

[14] J.F.C. Kingman(1993) *Poisson processes*, Oxford Studies in Probability, vol. 3, The Clarendon Press Oxford University Press, New York, . Oxford Science Publications.

[15] A.S. SZNITMAN(1991) *Topics in propagation of chaos*, Ecole d'été de probabilités de Saint-Flour XIX-1989, L.N. in Math. 1464, Springer.

[16] C. VILLANI(2003) *Topics in optimal transportation.* Graduate Studies in Mathematics, 58. American Mathematical Society, Providence, RI.