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Remarks on Lipschitz Solutions to Measurable Differential Inclusions and an Existence Result for some Nonconvex Variational Problems*

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In this paper we examine the problem of finding a Lipschitz function on an open domain with prescribed boundary values and whose gradient is required to satisfy some nonhomogeneous pointwise constraints a.e. in the domain. These constraints are supposed to be given by a measurable set-valued mapping with convex, uniformly compact and nonempty-interior values. We discuss existence and metric properties of maximal solutions of such a problem. We exploit some connections with weak solutions to discontinuous Hamilton-Jacobi equations, and we provide a variational principle that characterizes maximal solutions. We investigate the case where the original problem is supplemented with bilateral obstacle constraints on the function values. Finally, as an application of these results, we prove existence for a specific class of nonconvex problems from the calculus of variations, with and without obstacle constraints, under mild regularity hypotheses on the data.

1. Introduction

Throughout this paper, Ω is an open subset of $\mathbb{R}^N$ with $N \geq 1$, and $C : \Omega \rightrightarrows \mathbb{R}^N$ is a Borel measurable set-valued mapping which is supposed to satisfy the following conditions:

$$C(x) \text{ is closed and convex for a.e. } x \in \Omega. \tag{1a}$$

$$\text{For all compact set } K \subset \Omega,\ \exists c_1 > 0,\ \text{for a.e. } x \in K,\ c_1 B \subset C(x). \tag{1b}$$

$$\text{For all compact set } K \subset \Omega,\ \exists c_2 > 0,\ \text{for a.e. } x \in K,\ C(x) \subset c_2 B. \tag{1c}$$

Here, B stands for the closed unit ball of $\mathbb{R}^N$. Let us consider the following first-order Dirichlet differential inclusion problem: find $u \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{cases} \nabla u \in C(x) & \text{a.e. in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{2}$$

for $g \in Lip(\partial\Omega) := \{w \in C(\partial\Omega) \mid \exists L \geq 0, \forall x, y \in \partial\Omega,\ w(y) - w(x) \leq L \operatorname{dist}_\Omega(x,y)\}$. In fact, by considering a Lipschitz extension of g to $\overline{\Omega}$ [15], we may assume that $g \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega})$.

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Several problems of the calculus of variations are related to this type of differential inclusion. For instance, (2) may describe the feasible functions for a variational problem involving convex constraints on the gradient. In a different direction, the solutions of (2) may be the a.e. *subsolutions* of a Hamilton-Jacobi equation $H(x, \nabla u) = 0$ in Ω, whose Hamiltonian $H(x, \xi)$ is supposed to be convex and coercive with respect to ξ, and $C(x) = \{\xi \in \mathbb{R}^N \mid H(x, \xi) \leq 0\}$ is the 0-sublevel set of $H(x, \cdot)$. In the latter case, the a.e. solutions are related to the more restrictive differential inclusion:

$$\begin{cases} \nabla u \in \partial C(x) & \text{a.e. in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Other interesting cases that require to solve a problem as (3) are some attainment results for relaxed problems of the calculus of variations, where $C(x)$ is the closure of the *detachment* set $\mathcal{D}_f(x) = \{\xi \in \mathbb{R}^N \mid f(x, \xi) > f^{**}(x, \xi)\}$ between the integrand in the energy functional and its convex envelope.

Under appropriate continuity conditions, it is a well known result of the Hamilton-Jacobi equations theory that (viscosity) solutions to some classes of equations are characterized by a variational principle, which can be interpreted as a sort of *minimal Lipschitz extension* of the boundary data to the whole domain. This is accomplished by considering the *Finsler metric* structures induced by the support functions of 0-sublevel sets of the corresponding Hamiltonians. Recently, these metric methods have been adapted to prove the variational principle together with other interesting results for more general Hamilton-Jacobi equations under either semicontinuity [3] or just measurability [4] conditions, with or without convexity (in this connection, see also [24]).

This note is intended to show how to exploit the results of [3, 4] to provide some metric properties of the solutions of (2), compatibility conditions on the boundary data, and a variational principle for the solutions to (3). In so doing, we compare some aspects of the aforementioned works and we establish some new links between them. To this end, some preliminaries are given in Section 2. The metric properties are discussed in Section 3. The connection with measurable Hamilton-Jacobi equations is reviewed in Section 4. Additionally, in Section 5, we investigate the case where the original problem is supplemented with bilateral obstacle constraints on the function values. As an application of these results, in Section 6 we prove existence for a specific class of nonconvex problems from the calculus of variations, with and without obstacle constraints, under mild regularity hypotheses on the data.

Finally, let us mention that the existence issue has been extensively treated in quite general cases, including the vectorial case, by different methods as the convex integration theory of Gromov and Baire's category method. The book [9] provides an explanation of these approaches. See also [25, 20].

2. Preliminaries on the induced metric structure

Given $x \in \Omega$, let us denote by $\sigma_C(x, \cdot) : \mathbb{R}^N \to [0, \infty)$ the *support function* of $C(x)$, that is,

$$\sigma_C(x, \xi^*) := \sigma_{C(x)}(\xi^*) = \sup_{\xi \in C(x)} \langle \xi, \xi^* \rangle, \quad \xi^* \in \mathbb{R}^N. \tag{4}$$

By (1), for a.e. $x \in \Omega$, $\sigma_C(x,\cdot)$ is a continuous *gauge* which is strictly positive except at the origin, namely,

1. $\sigma_C(x,\cdot) : \mathbb{R}^N \to [0,\infty)$ is continuous and convex.
2. $\forall\lambda > 0$, $\forall\xi^* \in \mathbb{R}^N$, $\sigma_C(x,\lambda\xi^*) = \lambda\sigma_C(x,\xi^*)$.
3. $\sigma_C(x,\xi^*) = 0$ iff $\xi^* = 0$.

The gauge $\sigma_C(x,\cdot)$ is not a norm unless $C(x)$ is symmetric in the sense that $C(x) = -C(x)$. In the general case, the triangle inequality holds: $\forall\xi_1^*,\xi_2^* \in \mathbb{R}^N$, $\sigma_C(x,\xi_1^*+\xi_2^*) \le \sigma_C(x,\xi_1^*) + \sigma_C(x,\xi_2^*)$. Furthermore, it follows from (1b)-(1c) that

$$\begin{aligned}&\text{For all compact set } K \subset \Omega,\ \exists c_2 \ge c_1 > 0,\\ &\text{for a.e. } x \in K,\ \forall\xi^* \in \mathbb{R}^N,\ c_1|\xi^*| \le \sigma_C(x,\xi^*) \le c_2|\xi^*|,\end{aligned} \tag{5}$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^N$. By convexity, $\sigma_C(x,\xi^*)$ is Lipschitz continuous with respect to ξ^* for a.e. $x \in \Omega$, and Borel measurable in x for each fixed ξ^* (for the last assertion, the reader is referred to [5] or [23, Example 14.51]). In particular, $\sigma_C(\cdot,\cdot)$ is a Carathéodory function, hence Borel measurable as a function on $\Omega \times \mathbb{R}^N$.

The function σ_C defines a measurable *Finsler metric* on Ω, a generalization of a Riemannian metric that induces an intrinsic distance on Ω. Under measurability conditions, the introduction of a proper metric structure has been investigated in [11, 12, 13] for the more general case of Lipschitz manifolds. Interesting refinements which will be useful in our case in the context of Hamilton-Jacobi equations theory can be found in [3, 4]. For any $x,y \in \Omega$, we first denote by $\Gamma(\Omega;x,y)$ the set of all the Lipschitz continuous curves $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = y$, we set $\Gamma(\Omega) := \cup_{x,y\in\Omega}\Gamma(\Omega;x,y)$, and for every curve $\gamma \in \Gamma(\Omega)$ we define

$$\mathbb{L}(\gamma \mid \sigma_C) = \begin{cases}\int_0^1 \sigma_C(\gamma(t),\dot\gamma(t))dt & \text{if the integral is well defined,}\\ +\infty & \text{otherwise.}\end{cases}$$

Next, for every $n \ge 1$ we denote by $\mathcal{L}_n$ the corresponding n-dimensional Lebesgue measure. Following [14], we say that a curve $\gamma \in \Gamma(\Omega)$ is *transversal* to a Borel subset E of Ω if $\mathcal{L}_1(\gamma^{-1}(E)) = \mathcal{L}_1(\{t \in [0,1] \mid \gamma(t) \in E\}) = 0$. We write $\gamma \pitchfork E$ when γ is transversal to E. Then, it is possible to prove that the length functional $\mathbb{L}(\cdot \mid \sigma_C) : \Gamma(\Omega) \to [0,\infty]$ induces the finite-valued (possibly nonsymmetric) intrinsic distance $\mathrm{dist}_\Omega(\cdot,\cdot \mid \sigma_C) : \Omega \times \Omega \to [0,\infty)$ given by

$$\mathrm{dist}_\Omega(x,y \mid \sigma_C) = \sup_{\mathcal{L}_N(E)=0} \inf_\gamma \{\mathbb{L}(\gamma \mid \sigma_C) \mid \gamma \in \Gamma(\Omega;x,y),\ \gamma \pitchfork E\}. \tag{6}$$

We extend $\mathrm{dist}_\Omega(\cdot,\cdot \mid \sigma_C)$ to $\overline{\Omega} \times \overline{\Omega}$ by taking appropriate lower limits. In particular, if $x \in \partial\Omega$ and $y \in \partial\Omega$, then

$$\mathrm{dist}_\Omega(x,y \mid \sigma_C) = \liminf_{(\zeta,\eta)\to(x,y)} \mathrm{dist}_\Omega(\zeta,\eta \mid \sigma_C),$$

where $(\zeta,\eta) \in \Omega \times \Omega$. In general, the function $\mathrm{dist}_\Omega(\cdot,\cdot \mid \sigma_C)$ so defined is just a pseudodistance on $\overline{\Omega}$ because it may fail to satisfy the triangle inequality. Actually, without additional conditions on the geometry of Ω, we may only have that $\mathrm{dist}_\Omega(x,y \mid \sigma_C) \le$

$\mathrm{dist}_\Omega(x,z \mid \sigma_C) + \mathrm{dist}_\Omega(z,y \mid \sigma_C)$ for every $x, y \in \overline{\Omega}$ and $z \in \Omega$. By (5) we have that $\mathrm{dist}_\Omega(\cdot,\cdot \mid \sigma_C)$ is locally equivalent to the usual geodesic distance on Ω. More precisely, we have that for all compact set $K \subset \Omega$, $\forall x, y \in K$, $c_1 \mathrm{dist}_\Omega(x,y) \leq \mathrm{dist}_\Omega(x,y \mid \sigma_C) \leq c_2 \mathrm{dist}_\Omega(x,y)$, where $\mathrm{dist}_\Omega(x,y) := \inf\{\int_0^1 |\dot{\gamma}| \mid \gamma \in \Gamma(\Omega; x, y)\}$. As a consequence, we get that $c_1|x-y| \leq \mathrm{dist}_\Omega(x,y \mid \sigma_C) \leq c_2|x-y|$ locally in Ω.

Remark 2.1. It is possible to simplify the expression (6) for the intrinsic distance under appropriate semicontinuity assumptions. In fact, suppose that $C : \Omega \rightrightarrows \mathbb{R}^N$ is upper semicontinuous (u.s.c.). Under local boundedness of images, which is valid here due to (1c), the latter is equivalent to $\forall \bar{x} \in \Omega$, $\limsup_{x \to \bar{x}} C(x) \subset C(\bar{x})$ (cf. [23, Theo. 5.19]), which, in turn, amounts to the upper semicontinuity of $\sigma_C(\cdot, \xi^*)$ for any $\xi^* \in \mathbb{R}^N$ (cf. [23, Ex. 5.6(c) and Cor. 11.35(a)]). Then, by either [12, Theorem 3.3] or [3, Proposition 3.4], we have that $\mathrm{dist}_\Omega(x,y \mid \sigma_C) = \inf\{\mathbb{L}(\gamma \mid \sigma_C) \mid \gamma \in \Gamma(\Omega; x, y)\}$.

Next, we introduce a dual metric on Ω via *polar* operations (for a detailed exposition of polarity the reader is referred to [22, Part III]). We begin by recalling that $C(x)$, being a closed convex neighborhood of the origin, can be expressed as

$$C(x) = \{\xi \in \mathbb{R}^N \mid \sigma_C^\circ(x,\xi) \leq 1\}, \tag{7}$$

where $\sigma_C^\circ(x,\xi) = \sigma_{C(x)}^\circ(\xi)$ is the polar gauge of $\sigma_C(x,\cdot) = \sigma_{C(x)}$ and in this case is given by

$$\sigma_C^\circ(x,\xi) = \sup_{\xi^* \neq 0} \frac{\langle \xi, \xi^* \rangle}{\sigma_C(x,\xi^*)} = \max_{|\xi^*|=1} \frac{\langle \xi, \xi^* \rangle}{\sigma_C(x,\xi^*)}. \tag{8}$$

It is worth pointing out that

$$\operatorname{int} C(x) = \{\xi \in \mathbb{R}^N \mid \sigma_C^\circ(x,\xi) < 1\}, \tag{9}$$

$$\partial C(x) = \{\xi \in \mathbb{R}^N \mid \sigma_C^\circ(x,\xi) = 1\}. \tag{10}$$

Furthermore, it is well known that $\forall \xi \in \mathbb{R}^N$, $\sigma_C^\circ(x,\xi) = \sigma_{C(x)^\circ}(\xi)$, where $C(x)^\circ$ is the polar of $C(x)$, that is, the convex set given by

$$C(x)^\circ = \{\xi^* \in \mathbb{R}^N \mid \forall \xi \in C(x), \langle \xi, \xi^* \rangle \leq 1\} = \{\xi^* \in \mathbb{R}^N \mid \sigma_C(x,\xi^*) \leq 1\}.$$

Let $C^\circ : \Omega \rightrightarrows \mathbb{R}^N$ be the set-valued mapping defined by $C^\circ(x) := C(x)^\circ$. Therefore, we can write

$$\sigma_C^\circ(x,\xi) = \sigma_{C^\circ}(x,\xi). \tag{11}$$

Both C° and σ_{C° are Borel measurable. On the other hand, it follows from (1b)-(1c) that for all compact set $K \subset \Omega$ and for a.e. $x \in K$, $c_2^{-1}B \subset C^\circ(x) \subset c_1^{-1}B$, hence $\forall \xi \in \mathbb{R}^N$, $c_2^{-1}|\xi| \leq \sigma_{C^\circ}(x,\xi) \leq c_1^{-1}|\xi|$. By virtue of (8) we have the following Cauchy-Schwartz type inequality

$$\forall x \in \Omega,\ \forall \xi, \xi^* \in \mathbb{R}^N, \quad \langle \xi, \xi^* \rangle \leq \sigma_{C^\circ}(x,\xi)\sigma_C(x,\xi^*). \tag{12}$$

3. Differential inclusion problems under Dirichlet boundary conditions

Suppose (1) and consider the Dirichlet differential inclusion problem (2). It follows from (7) and (11) that for every Lipschitz function u we have

$$\nabla u(x) \in C(x) \text{ for a.e. } x \in \Omega \iff \sigma_{C^\circ}(x, \nabla u(x)) \leq 1 \text{ for a.e. } x \in \Omega. \tag{13}$$

The boundary data is supposed to satisfy the following *compatibility condition*:

$$L_{\partial\Omega}(g \mid \sigma_C) \leq 1, \tag{14}$$

where

$$L_{\partial\Omega}(g \mid \sigma_C) := \sup_{x,y\in\partial\Omega} \frac{g(y) - g(x)}{\mathrm{dist}_\Omega(x, y \mid \sigma_C)}. \tag{15}$$

The necessity of (14) is a direct consequence of (13) together with

Proposition 3.1. *Under* (1), *if* $u \in W^{1,\infty}(\Omega)$ *then*

$$\forall x, y \in \Omega,\ u(y) - u(x) \leq \|\sigma_{C^\circ}(\cdot, \nabla u)\|_\infty \mathrm{dist}_\Omega(x, y \mid \sigma_C), \tag{16}$$

where $\|\sigma_{C^\circ}(\cdot, \nabla u)\|_\infty = \textit{ess-sup}\{\sigma_{C^\circ}(x, \nabla u(x)) \mid x \in \Omega\}$. *If in addition* $u \in C(\overline{\Omega})$ *then the latter holds for all* $x, y \in \overline{\Omega}$. *In any case, we have*

$$\|\sigma_{C^\circ}(\cdot, \nabla u)\|_\infty = \sup_{x\neq y} \frac{u(y) - u(x)}{\mathrm{dist}_\Omega(x, y \mid \sigma_C)}. \tag{17}$$

If $C(x) = B$ for all $x \in \Omega$, then (16) amounts to $|u(x) - u(y)| \leq \|\nabla u\|_\infty \mathrm{dist}_\Omega(x, y)$, which is a well known inequality in Sobolev spaces theory (cf. [2, Chap. IX, Rem. 8]). This result has been proved in the context of minimal Lipschitz extension problems in the homogeneous case [9, Thm. 2.17] and in the continuous nonhomogeneous case [6, Props. 2.8 and 2.9]. Here we give a proof for the general measurable case.

Proof of Proposition 3.1. We denote by A_u the set of all the points in Ω where u is not differentiable. By Rademacher's theorem, $\mathcal{L}_N(A_u) = 0$. Let $\gamma \in \Gamma(\Omega; x, y)$ be such that $\gamma \pitchfork A_u$. Then $u(\gamma(\cdot))$ is differentiable almost everywhere in $[0, 1]$ and moreover $u(\gamma(1)) - u(\gamma(0)) = \int_0^1 \langle \nabla u(\gamma(t)), \dot\gamma(t)\rangle dt$. By (12), we deduce that $u(y) - u(x) \leq \int_0^1 \sigma_{C^\circ}(\gamma(t), \nabla u(\gamma(t)))\sigma_C(\gamma(t), \dot\gamma(t))dt \leq \|\sigma_{C^\circ}(\nabla u)\|_\infty \mathbb{L}(\gamma \mid \sigma_C)$. and (16) follows.

In the case where $x \in \partial\Omega$ or $y \in \partial\Omega$, we first fix $\varepsilon > 0$. By continuity, there exists $r_0 > 0$ such that for any $\zeta \in \Omega \cap B_{r_0}(x)$ and $\eta \in \Omega \cap B_{r_0}(y)$ we have $u(y) - u(x) \leq \varepsilon + u(\eta) - u(\zeta)$. From our previous analysis it follows that $u(y) - u(x) \leq \varepsilon + \|\sigma_{C^\circ}(\cdot, \nabla u)\|_\infty \mathrm{dist}_\Omega(\zeta, \eta \mid \sigma_C)$. Letting $(\zeta, \eta) \to (x, y)$ and since $\varepsilon > 0$ is arbitrary, we get the desired inequality.

Next, set $M := \sup_{x\neq y} \frac{u(y)-u(x)}{\mathrm{dist}_\Omega(x,y|\sigma_C)}$. By (16), we have that $M \leq \|\sigma_{C^\circ}(\cdot, \nabla u)\|_\infty$. For every $x \in A_u$ and $\xi^* \in \mathbb{R}^N$ we have that $\langle \nabla u(x), \xi^*\rangle = \lim_{h\to 0^+} \frac{u(x)-u(x-h\xi^*)}{h} \leq M \limsup_{h\to 0^+} \frac{\mathrm{dist}_\Omega(x-h\xi^*,x|\sigma_C)}{h}$. On the other hand, by [4, Prop. 4.2] (see also [3, Prop. 2.11] in the upper semicontinuous case), for every $\xi \in \mathbb{R}^N$, $\limsup_{h\to 0^+} \frac{\mathrm{dist}_\Omega(x-h\xi^*,x|\sigma_C)}{h} \leq \sigma_C(x, \xi^*)$. Thus, for every $x \in A_u$, we have that $\sigma_{C^\circ}(x, \nabla u(x)) = \sup_{\xi^*\neq 0} \frac{\langle\nabla u(x),\xi^*\rangle}{\sigma_C(x,\xi^*)} \leq M$. Since $\mathcal{L}_N(A_u) = 0$, we get $\|\sigma_{C^\circ}(\cdot, \nabla u)\|_\infty \leq M$, which completes the proof. □

Under (14), it follows from (13) and Proposition 3.1 that any solution u to (2) satisfies $L_{\partial\Omega}(g \mid \sigma_C) \leq \|\sigma_{C^\circ}(\cdot, \nabla u)\|_\infty \leq 1$. Given $\lambda > 0$, consider the following Lipschitz *McShane* type extensions of g to $\overline{\Omega}$ (see [1] and the references therein):

$$u^\lambda(x) := \inf_{y\in\partial\Omega} \{g(y) + \lambda \operatorname{dist}_\Omega(y, x \mid \sigma_C)\}, \, x \in \overline{\Omega}, \tag{18}$$

$$u_\lambda(x) := \sup_{y\in\partial\Omega} \{g(y) - \lambda \operatorname{dist}_\Omega(x, y \mid \sigma_C)\}, \, x \in \overline{\Omega}. \tag{19}$$

It is simple to see that u_λ, $u^\lambda \in W^{1,\infty}(\Omega)$ and moreover both functions satisfy

$$\forall x, y \in \Omega, \, u(y) - u(x) \le \lambda \operatorname{dist}_\Omega(x, y \mid \sigma_C), \tag{20}$$

which amounts to $\|\sigma_{C^\circ}(\cdot, \nabla u)\|_\infty \le \lambda$. Notice that if u satifies (20) then

$$u_\lambda \le u \le u^\lambda \quad \text{on } \Omega. \tag{21}$$

If $\lambda \ge L_{\partial\Omega}(g \mid \sigma_C)$ then it is clear that $u^\lambda = u_\lambda = g$ on $\partial\Omega$. By arguing, for instance, exactly as in the first part of the proof of [15, Thm. 1.8] where the specific case $\sigma_C \equiv |\cdot|$ is treated, we can easily verify that $u^\lambda, u_\lambda \in C(\overline{\Omega})$. Therefore, if

$$L_{\partial\Omega}(g \mid \sigma_C) \le \lambda \le 1. \tag{22}$$

then u_λ and u^λ solve (2), and are respectively minimal and maximal in the class of all the solutions u of (2) satisfying (20). It turns out that (14) is necessary and sufficient for the solvability of (2).

Take $\lambda_0 := L_{\partial\Omega}(g \mid \sigma_C)$. The corresponding functions u_{λ_0} and u^{λ_0} satisfy $\|\sigma_{C^\circ}(\cdot, \nabla u)\|_\infty = \lambda_0$. Consequently, both u_{λ_0} and u^{λ_0} solve the following *nonhomogeneous minimal Lipschitz extension* problem:

$$\min\{\|\sigma_{C^\circ}(\cdot, \nabla u)\|_\infty \mid u \in g + W_0^{1,\infty}(\Omega)\},$$

whose optimal value is indeed given by λ_0. Notice that any solution u of the latter satisfies $u_{\lambda_0} \le u \le u^{\lambda_0}$ in Ω. This motivates the introduction of the *uniqueness set* $\mathcal{U}(g) := \{x \in \Omega \mid u_{\lambda_0}(x) = u^{\lambda_0}(x)\}$. It is easy to see that $x \in \Omega$ belongs to $\mathcal{U}_\lambda(g)$ iff there exist boundary points $y_1, y_2 \in \partial\Omega$ such that $|g(y_1) - g(y_2)| = \lambda_0 \operatorname{dist}_\Omega(y_1, y_2 \mid \sigma_C)$ and $\operatorname{dist}_\Omega(y_1, x \mid \sigma_C) + \operatorname{dist}_\Omega(x, y_2 \mid \sigma_C) = \operatorname{dist}_\Omega(y_1, y_2 \mid \sigma_C)$. The first condition forces y_1, y_2 to realize the supremum in the definition of λ_0, while the second one means that $\mathcal{U}(g)$ consists of $\operatorname{dist}_\Omega(\cdot, \cdot \mid \sigma_C)$*-segments* joining those boundary points if they exist.

4. Maximal solutions and Hamilton-Jacobi equations

From now on, we suppose that the compatibility condition (14) holds. In the previous section we have seen that for every λ satisfying (22) the corresponding McShane extension (18) provides a solution to (2). The maximal of those solutions is obtained by taking $\lambda = 1$, that is,

$$\bar{u}(x) := \inf_{y\in\partial\Omega} \{g(y) + \operatorname{dist}_\Omega(y, x \mid \sigma_C)\}, \quad x \in \overline{\Omega}. \tag{23}$$

Motivated by Bellman's approach to optimal control problems, this type of formula is well-known in connection with Hamilton-Jacobi equations, providing a sort of extended Hopf-Lax variational principle. The aim of this section is to discuss such a connection, from the continuous to the general measurable case.

Let $H_C : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be the Hamiltonian defined by

$$H_C(x,\xi) := \sigma_{C^\circ}(x,\xi) - 1. \tag{24}$$

This function is convex continuous with respect to ξ for a.e. $x \in \Omega$, and we have the following:

$$C(x) = \{\xi \in \mathbb{R}^N \mid H_C(x,\xi) \leq 0\}. \tag{25a}$$

$$\partial C(x) = \{\xi \in \mathbb{R}^N \mid H_C(x,\xi) = 0\}. \tag{25b}$$

When H_C is continuous, which in our setting amounts to the continuity of $C : \Omega \rightrightarrows \mathbb{R}^N$, general results of the Hamilton-Jacobi theory (see, for instance, [17, Chap. 5]) together with Remark 2.1, ensure that $\bar{u}$ given by (23) is the unique *viscosity solution* of the following Hamilton-Jacobi equation with boundary conditions:

$$\begin{cases} H_C(x,\nabla u) = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{26}$$

Before recalling the standard definition of viscosity solutions, let us introduce the notion of tangent functions.

Definition 4.1. Given a l.s.c. (resp. u.s.c.) function u, a C^1 function ψ is called *subtangent* (resp. *supertangent*) to u at x_0 if x_0 is a local minimizer (resp. maximizer) of $(u - \psi)$.

Consider the Hamilton-Jacobi equation

$$H_C(x,\nabla u) = 0 \quad \text{in } \Omega. \tag{27}$$

Definition 4.2 (Crandall-Lions). Suppose that H_C is continuous. We say that a l.s.c. (u.s.c.) function u is a *viscosity super* (resp. *sub*) *solution* of (27) if for any $x_0 \in \Omega$ and ψ C^1-subtangent (resp. supertangent) to u at x_0 we have

$$H_C(x_0, \nabla\psi(x_0)) \geq 0 \quad (\text{resp.} \leq 0).$$

A continuous function that is a viscosity super and sub solution of (27) is called a *viscosity solution.* A function $u \in C(\overline{\Omega})$ is a *viscosity solution* (resp. *subsolution*) of the Dirichlet problem (26) if it is a viscosity solution (resp. subsolution) of (27) and satisfies $u = g$ (resp. $u \leq g$) on $\partial\Omega$.

Under the continuity assumption on H_C, because $\bar{u}$ is a Lipschitz viscosity solution of (27), it follows that $\bar{u}$ is an *a.e. solution* of (26) in the sense that $H_C(x, \nabla u(x)) = 0$ for a.e. $x \in \Omega$ and satisfies $u = g$ on $\partial\Omega$. As a consequence, by (25b), it follows that $\bar{u}$ solves (3).

Next, suppose that $C : \Omega \rightrightarrows \mathbb{R}^N$ is only u.s.c. so that $H_C(\cdot,\xi)$ is l.s.c. for fixed ξ. Then the classical notion of viscosity solution is not appropriate. In the specific case of eikonal equations of the type $F(\nabla u) = n(x)$ with l.s.c. right-hand side, this drawback was overcome in [21] by the introduction of the so called *Monge solutions*:

Definition 4.3 (Newcomb-Su). A function $u \in C(\Omega)$ is said to be a *Monge solution* (resp. *subsolution, supersolution*) of (27) if for every $x_0 \in \Omega$ we have that

$$\liminf_{x \to x_0} \frac{u(x) - u(x_0) + \mathrm{dist}_\Omega(x, x_0 \mid \sigma_C)}{|x - x_0|} = 0 \quad (\text{resp. } \geq, \leq). \tag{28}$$

A function $u \in C(\overline{\Omega})$ is said to be a *Monge solution* (resp. *subsolution*) of the Dirichlet problem (26) if it is a Monge solution (resp. subsolution) of (27) and satisfies $u = g$ (resp. $u \leq g$) on $\partial\Omega$.

Under the l.s.c. hypothesis on $n(x)$, it is proved in [21] that a comparison principle holds for Monge subsolutions and supersolution of $F(\nabla u) = n(x)$ as well as existence/uniqueness results for Dirichlet boundary conditions. All these results were substantially extended in [3] to more general Hamilton-Jacobi equations with discontinuities. Furthermore, as in [3, Theo. 5.4], it turns out that $\bar{u}$ is indeed the unique Monge solution of (26) and it is maximal in the class of all Monge subsolutions.

In the general case where $H_C(x, \xi)$ is only measurable in x for fixed $\xi \in \mathbb{R}^N$, and following the 0-sublevel set approach of [4], we begin by recalling a couple of definitions about some notions of limits for measurable set-valued mappings (see [14] for further details). For any subset E and $x \in \mathbb{R}^N$ the density $\Delta_x(E)$ of E at x is given by the formula

$$\Delta_x(E) = \lim_{r \to 0} \frac{\mathcal{L}_N(E \cap B(x,r))}{\mathcal{L}_N(B(x,r))}$$

and the *approximate limsup* of $C : \Omega \rightrightarrows \mathbb{R}^N$ at x_0 is defined as

$$\operatorname{ap}\limsup_{x \to x_0} C(x) = \bigcap \{K \text{convex, compact} \mid \Delta_{x_0}(\{x \mid C(x) \subset K\}) = 1\}. \tag{29}$$

Under our hypotheses, we have that

$$\operatorname{ap}\limsup_{x \to x_0} C(x) = C(x_0) \quad \text{for a.e. } x_0 \in \Omega.$$

See, for instance. [4, Prop. 4.1(i)].

Definition 4.4 (Camilli-Siconolfi). An u.s.c. function u is said to be a *CS-viscosity subsolution* of (27) provided that for any x_0 and any C^1-supertangent ψ to u at x_0, it results

$$\nabla\psi(x_0) \in \operatorname{ap}\limsup_{x \to x_0} C(x).$$

A l.s.c function v is said to be a CS viscosity supersolution of (27) if

$$\operatorname{ess}\limsup_{x \to x_0} d^*(\nabla\psi(x), C(x)) \geq 0 \tag{30}$$

for any x_0, and any Lipschitz continuous subtangent ψ to v at x_0, where $\operatorname{ess}\limsup$ denotes the essential limsup, defined for a measurable function g as $\operatorname{ess}\limsup_{x \to x_0} g(x) = \inf_{\varepsilon > 0}\{\operatorname{ess\,sup}_{B(x_0,\varepsilon)} g\}$, and $d^*(\xi, A)$ stands for the signed distance from the point ξ to the set A.

Definition 4.4 was introduced in [4], and it copes with the nonconvex case as well. Monge and CS-viscosity solutions are equivalent to their classical viscosity analogues when H_C is continuous (see, for instance, [3, Prop. 4.5]).

Remark 4.5. A comparison principle is proved in [4] under the additional hypothesis of the existence of a strict a.e. subsolution. Notice that under (1b), by taking $\lambda = c_1/2$ in (18) the corresponding function $u^{c_1/2}$ is a strict a.e. subsolution, hence the comparison principle for CS-viscosity solutions holds in our setting on any compact subset K of Ω, which will be sufficient in the following.

Concerning a.e. solutions in the CS-viscosity theory, let us mention that the comparison between CS-viscosity subsolutions and a.e. subsolutions was made in [4], and results in an equivalence as follows:

Proposition 4.6. *Under* (1a) *and* (1c), *the following assertions are equivalent for* $u \in C(\overline{\Omega})$*:*

(*i*) u *is an a.e. subsolution of* (26).
(*ii*) u *is a CS-viscosity subsolution of* (26).
(*iii*) $\forall x, y \in \overline{\Omega}$, $u(x) - u(y) \le \mathrm{dist}_\Omega(y, x \mid \sigma_C)$.

Now, we answer positively whether CS-viscosity supersolutions are a.e. supersolutions without any regularity assumption.

Proposition 4.7. *Suppose* (1), *and let* $u \in W^{1,\infty}(\Omega)$ *be a CS-viscosity supersolution of* (27), *then:*

(*i*) u *is a Monge supersolution of* (27).
(*ii*) u *is also an a.e. supersolution of* (27).

Proof. (*i*) We must prove that u satisfies

$$\liminf_{x \to x_0} \frac{u(x) - u(x_0) + \mathrm{dist}_\Omega(x, x_0 \mid \sigma_C)}{|x - x_0|} \le 0. \tag{31}$$

Inspired by [21, Prop. 2.5], suppose on the contrary that there exists $x_0 \in \Omega$, and two positive constants r, δ such that $u(x) - u(x_0) + \mathrm{dist}_\Omega(x, x_0 \mid \sigma_C) \ge \delta |x - x_0|$, $\forall x \in \overline{B_r(x_0)}$. Without loss of generality, suppose that $u(x_0) = 0$. Let $\varphi(x) = -\mathrm{dist}_{B_r(x_0)}(x, x_0 \mid \sigma_C) + \delta r$. Notice that φ satisfies the compatibility condition (14) with respect to $\mathrm{dist}_{B_r(x_0)}(\cdot, \cdot \mid \sigma_C)$ and $u \ge \varphi$ on $\partial B_r(x_0)$. Next, recall that $w(x) = \min_{y \in \partial B_r(x_0)} \{\varphi(y) + \mathrm{dist}_{B_r(x_0)}(y, x \mid \sigma_C)\}$ is a CS-viscosity solution (the unique indeed) of

$$\begin{cases} H_C(x, \nabla w) = 0 & \text{in } B_r(x_0), \\ w = \varphi & \text{on } \partial B_r(x_0). \end{cases}$$

By virtue of the comparison principle, $u \ge w$ in $\overline{B_r(x_0)}$. But $w(x_0) = \delta r > 0 = u(x_0)$, which is a contradiction.

(*ii*) The proof consists mainly in noticing that the upper semicontinuity required in [3, Prop. 4.6 (ii)] to prove the same assertion for Monge supersolutions is used only to appeal to [3, Prop. 2.11], which has an analogous in [4, Prop. 4.2] but without the semicontinuity requirement. We omit the details. □

It was shown in [4] that the McShane extension $\bar{u}$ is the unique CS-viscosity solution of (26). Summarizing we have the following:

Proposition 4.8. *Under* (1) *and* (14), *the McShane Lipschitz extension $\bar{u}$ is the unique CS-viscosity solution of the Dirichlet problem* (26). *Moreover, $\bar{u}$ solves* (3) *and $u \leq \bar{u}$ for every solution u of* (2).

Remark 4.9. In [10] it is proposed to define a viscosity solution for a homogeneous differential inclusion as a viscosity solution of $d(\nabla u, K) = 0$ for a suitable set K which is supposed to be compact but not necessarily convex. It is proved in [10] that the function $\widetilde{u}(x) = \inf_{y\in\partial\Omega} \sigma_K(x-y)$, $x \in \overline{\Omega}$, is a viscosity solution satisfying a null boundary condition when Ω is convex. In the scalar case, we think that the Definition 4.4 is well suited to cover the nonhomogeneous, nonregular and with more general boundary condition case.

5. Solutions under bilateral obstacle constraints

In this section, we assume that the Dirichlet inclusion problem (2) is supplemented with a pointwise bilateral constraint on u, namely

$$\begin{cases} (u, \nabla u) \in K(x) & \text{a.e. in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{32}$$

where $K : \Omega \rightrightarrows \mathbb{R}^{N+1}$ is given by

$$K(x) = [a(x), b(x)] \times C(x)$$

for a set-valued map $C : \Omega \rightrightarrows \mathbb{R}^N$ satisfying (1), and some measurable functions $a, b : \Omega \to \mathbb{R}$ such that $\sup a < \inf b$ for all $x \in \Omega$. We may assume without lost of generality that

$$-\infty < \inf a \leq \sup a < 0 < \inf b \leq \sup b < +\infty. \tag{33}$$

In fact, we have that any solution $u \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega})$ of (2) satisfies $|u(x)| \leq \|g\|_{\infty,\partial\Omega} + \operatorname{diam}_\Omega(\Omega \mid \sigma_C)$, $x \in \Omega$, where $\operatorname{diam}_\Omega(\Omega \mid \sigma_C) := \sup_{x,y\in\Omega} \operatorname{dist}_\Omega(x, y \mid \sigma_C)$. Thus, with no loss of generality, we may assume that both functions a and b are bounded. On the other hand, if $\sup a \in [0, \infty)$ then we set $a_\delta(x) := a(x) - \sup a - \delta$ and $b_\delta(x) := b(x) - \sup a - \delta$, for any $\delta \in (0, \inf b - \sup a)$ so that $\sup a_\delta = -\delta < 0 < \inf b_\delta$. Then we solve the inclusion $(u_\delta, \nabla u_\delta) \in K_\delta(x)$ a.e. in Ω with the Dirichlet condition $u_\delta = g_\delta$ on $\partial\Omega$, where $K_\delta(x) = [a_\delta(x), b_\delta(x)] \times C(x)$ and $g_\delta(x) := g(x) - \sup a - \delta$. By taking $u(x) = u_\delta(x) + \sup a + \delta$, we recover a solution of the original problem (32).

We now show that it is possible to formulate (32) as a problem of the type (2) in one higher dimension and with unbounded domain given by $\mathbb{R} \times \Omega$. First, given a function $v \in C(\overline{\Omega})$, we define $z_v \in C(\mathbb{R} \times \overline{\Omega})$ by

$$z_v(y) = e^\tau v(x) \quad \text{for } y = (\tau, x) \in \mathbb{R} \times \overline{\Omega}. \tag{34}$$

Notice that $u = g$ on $\partial\Omega$ iff $z_u = z_g$ on $\mathbb{R} \times \partial\Omega$. On the other hand, $v \in W^{1,\infty}(\Omega)$ iff $z_v \in W^{1,\infty}(]\alpha, \beta[\times\Omega)$ for some $\alpha < \beta$, and moreover we have that

$$\nabla_y z_v(y) = \left(\frac{\partial z_v(y)}{\partial \tau}, \nabla_x z_v(y)\right) = e^\tau \left(v(x), \nabla v(x)\right), \; y = (\tau, x).$$

Thus, given a solution u of (32), the corresponding function z_u solves the differential inclusion problem

$$\begin{cases} \nabla_y z \in \widehat{C}(y) & \text{a.e. in } \widehat{\Omega} := \mathbb{R} \times \Omega, \\ z = z_g & \text{on } \partial\widehat{\Omega} = \mathbb{R} \times \partial\Omega, \end{cases} \tag{35}$$

where $\widehat{C}(y) = e^{\tau} K(x)$, $y = (\tau, x) \in \widehat{\Omega}$. This type of transformation has been used in [16] to deal with some Hamilton-Jacobi equations of the type $F(x, u, \nabla u) = 1$ in Ω. When $F = F(x, s, \xi)$ is positively homogeneous of degree 1 with respect to (s, ξ), it is shown in [16] how (34) yields naturally to the equation $\widehat{F}(y, \nabla_y z) = 1$ in $\widehat{\Omega}$ for $\widehat{F}((\tau, x), (s, \xi)) = e^{-\tau} F(x, s, \xi)$. In our case, following the approach discussed in the previous section to find maximal solutions of convex differential inclusions, the original problem (32) is related to the Hamilton-Jacobi equation

$$\begin{cases} H_K(x, u, \nabla u) = 0 & \text{a.e. in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{36}$$

where the Hamiltonian is given by

$$H_K(x, s, \xi) = \sigma_{K^\circ}(x, s, \xi) - 1, \quad x \in \Omega,\ s \in \mathbb{R},\ \xi \in \mathbb{R}^N.$$

On the other hand, the transformed problem (35) yields to the boundary value problem

$$\begin{cases} H_{\widehat{C}}(y, \nabla_y z) = 0 & \text{a.e. in } \widehat{\Omega}, \\ z = z_g & \text{on } \partial\widehat{\Omega}, \end{cases} \tag{37}$$

where $H_{\widehat{C}}(y, p) = \sigma_{\widehat{C}^\circ}(y, p) - 1 = e^{-\tau} \sigma_{K^\circ}(x, s, \xi) - 1$, $y = (\tau, x) \in \widehat{\Omega}$, $p = (s, \xi) \in \mathbb{R}^{N+1}$.

By arguing as in [16], and using the results of [4] in the measurable case, it is possible to verify that the McShane Lipschitz extension

$$\bar{z}(y) = \inf_{\eta \in \partial\widehat{\Omega}} \left\{ z_g(\eta) + \operatorname{dist}_{\widehat{\Omega}}(\eta, y \mid \sigma_{\widehat{C}}) \right\}, \quad y \in \mathbb{R} \times \overline{\Omega}, \tag{38}$$

is indeed a CS-viscosity solution of (37), even though $\widehat{\Omega}$ is unbounded, provided that the following compatibility condition holds:

$$L_{\partial\widehat{\Omega}}(z_g \mid \sigma_{\widehat{C}}) \leq 1. \tag{39}$$

Here, z_g is given by (34) and L is defined as in (15). See [17] for similar results in the continuous case.

Note that (39) is always stronger than (14). In fact, by considering curves with constant first component in the definition of $\operatorname{dist}_{\widehat{\Omega}}(\eta, \zeta \mid \sigma_{\widehat{C}})$ for $\eta = (\tau, x)$ and $\zeta = (\tau, y)$ with $\tau \in \mathbb{R}$ and $x, y \in \partial\Omega$, one can readily see that $\operatorname{dist}_{\widehat{\Omega}}(\eta, \zeta \mid \sigma_{\widehat{C}}) \leq e^{\tau} \operatorname{dist}_{\Omega}(x, y \mid \sigma_C)$, hence

$$\begin{aligned} L_{\partial\widehat{\Omega}}(z_g \mid \sigma_{\widehat{C}}) &= \sup_{\eta, \zeta \in \partial\widehat{C}} \frac{z_g(\eta) - z_g(\zeta)}{\operatorname{dist}_{\widehat{\Omega}}(\eta, \zeta \mid \sigma_{\widehat{C}})} \\ &\geq \sup_{\substack{x, y \in \partial C \\ \tau \in \mathbb{R}}} \frac{e^{\tau}[g(x) - g(y)]}{\operatorname{dist}_{\widehat{\Omega}}((\tau, x), (\tau, y) \mid \sigma_{\widehat{C}})} \\ &\geq L_{\partial\Omega}(g \mid \sigma_C) \end{aligned}$$

In general, the first inequality is a strict one.

By Proposition 4.8, $\bar{z}$ is also an a.e. solution of (37). In particular, $\bar{z}$ is an a.e. maximal solution to (35), that is, $\nabla_y \bar{z}(y) \in \partial\widehat{C}(y)$ for a.e. $y \in \widehat{\Omega}$. Another property of $\bar{z}$ is the following:

$$\bar{z}(\tau, x) = e^{\tau}\bar{z}(0, x).$$

This relation follows from $\sigma_{\widehat{C}}(y,p) = e^{\tau}\sigma_K(x,p)$, and it has been already used in [16]. Therefore, taking

$$\bar{u}(x) := \bar{z}(0, x),$$

we recover a solution to (32). In fact, we have that

$$\begin{aligned}&(\bar{u}(x), \nabla\bar{u}(x)) \in \partial K(x)\\ &= [a(x), b(x)] \times \partial C(x) \cup \{a(x)\} \times C(x) \cup \{b(x)\} \times C(x) \quad \text{for a.e. } x \in \Omega.\end{aligned} \tag{40}$$

Finally, notice that the additional hypothesis

$$a, b \in W^{1,\infty}(\Omega) \quad \text{and} \quad \nabla a(x), \nabla b(x) \in \overline{(\mathbb{R}^N \setminus C(x))}, \quad \text{for a.e. } x \in \Omega, \tag{41}$$

forces the stronger condition $(\bar{u}(x), \nabla\bar{u}(x)) \in [a(x), b(x)] \times \partial C(x)$ a.e. in Ω, hence $\bar{u}$ solves

$$\begin{cases} \nabla u \in \partial C(x) & \text{a.e. } x \in \Omega,\\ a \le u \le b & \text{in } \Omega,\\ u = g & \text{on } \partial\Omega. \end{cases} \tag{42}$$

The previous discussion is summarized in the following:

Proposition 5.1. *Suppose that* (39) *holds. Consider the function* $\bar{z}$ *defined in* (38), *and define* $\bar{u}(x) := \bar{z}(0, x)$. *Then* $\bar{u}$ *is a Lipschitz solution of* (32). *Moreover,* $\bar{u}$ *satisfies* (40). *Furthermore, under the additional hypothesis* (41), *the solution* $\bar{u}$ *satisfies* (42).

6. Application to a nonconvex problem of the calculus of variations

We will apply the previous results to prove existence of solutions to some nonconvex problems of the calculus of variations.

Let Ω be a bounded open subset of $\mathbb{R}^N$. We will assume in addition that the boundary $\partial\Omega$ of Ω is *deformable Lipschitz.* We refer the reader to [7, Def. 2.1] for the precise definition of the deformable Lipschitz boundary property. Let us only mention here that it permits to define an outward unit normal field and to establish an extended Gauss-Green formula (see (2.2) on page 96 of [7]), which we will use in the proof of our existence result. Examples of such a regular domain are the smooth (C^2) domains, the star-shaped domains and all the domains having the cone property.

Given a Carathodory function $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$, let us denote by $f^{**}(x,\xi)$ the lower convex envelope of $f(x,\xi)$ with respect to ξ, that is, the greatest convex function below $f(x,\cdot)$. Following the approach of [18, 19], we define the *detachment* set by

$$\mathcal{D}_f(x) = \{\xi \in \mathbb{R}^N \mid f(x,\xi) > f^{**}(x,\xi)\},$$

and consider the following conditions:

$$\text{The set valued map } C(x) := \overline{\mathcal{D}_f(x)},\ x \in \Omega, \text{ satisfies (1).} \tag{43a}$$

$$\begin{aligned}&\exists m \in L^\infty(\Omega;\mathbb{R}^N),\ q \in L^1(\Omega) : f^{**}(x,\xi)\\ &= \langle m(x),\xi\rangle + q(x),\ \forall \xi \in \mathcal{D}_f(x), \text{ a.e. in } \Omega.\end{aligned} \tag{43b}$$

$$\operatorname{div}(m) = 0 \text{ in the sense of distributions.} \tag{43c}$$

Theorem 6.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with deformable Lipschitz boundary. Let $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a Carathodory function satisfying* (43). *If $g \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega})$ satisfies the compatibility condition* (14) *then the function $\bar{u}$ given by* (23) *solves the following problem:*

$$\min\left\{\int_\Omega f(x,\nabla u)dx \mid u \in g + W_0^{1,\infty}(\Omega)\right\}. \tag{44}$$

Let $a, b : \Omega \to \mathbb{R}$ satisfy (33) *and* (41), *and assume in addition that g satisfies* (39), *then the function $\bar{u} = \bar{z}(0,\cdot)$ with $\bar{z}$ given by* (38) *solves the following bilateral obstacle problem:*

$$\min\left\{\int_\Omega f(x,\nabla u)dx \mid a \le u \le b,\ u \in g + W_0^{1,\infty}(\Omega)\right\}. \tag{45}$$

Proof. Let $\bar{u} \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega})$ be the function given by (23) with $C(x) = \overline{\mathcal{D}_f(x)}$. By Proposition 4.8, $\bar{u}$ solves the following differential inclusion under Dirichlet boundary conditions:

$$\begin{cases}\nabla u \in \partial\mathcal{D}_f(x), & \text{a.e. } x \in \Omega,\\ u = g & \text{on } \partial\Omega,\end{cases}$$

Next we proceed to show the optimality of $\bar{u}$ for (44). First, notice that for a.e. $x \in \Omega$, $f^{**}(x,\xi)$ satisfies the affine representation given by (43b) for any $\xi \in \partial\mathcal{D}_f(x) \subset \{\zeta \in \mathbb{R}^N \mid f(x,\zeta) = f^{**}(x,\zeta)\}$. Thus

$$\int_\Omega f(x,\nabla\bar{u})dx = \int_\Omega f^{**}(x,\nabla\bar{u})dx = \int_\Omega \langle m(x),\nabla\bar{u}(x)\rangle dx + \int_\Omega q(x)dx.$$

By the extended Gauss-Green formula [7, Theo. 2.2 and 3.1] together with (43c), we have that $\int_\Omega \langle m(x),\nabla\bar{u}(x)\rangle dx = \int_{\partial\Omega} \bar{u}\langle m,\nu\rangle d\mathscr{H}^{N-1}$, where ν is the outward unit normal to $\partial\Omega$ and $\mathscr{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure on $\mathbb{R}^N$. Hence

$$\int_\Omega f(x,\nabla\bar{u})dx = \int_{\partial\Omega} g\langle m,\nu\rangle d\mathscr{H}^{N-1} + \int_\Omega q(x)dx.$$

On the other hand, by convexity, $f^{**}(x,\xi) \ge \langle m(x),\xi\rangle + q(x)$ for all $\xi \in \mathbb{R}^N$, a.e. in Ω. Let $v \in g + W_0^{1,\infty}(\Omega)$ be another Lipschitz function, then

$$\int_\Omega f(x,\nabla v)dx \ge \int_\Omega f^{**}(x,\nabla v)dx \ge \int_\Omega \langle m(x),\nabla v(x)\rangle dx + \int_\Omega q(x)dx.$$

Using once more the extended Gauss-Green formula, we get

$$\int_\Omega f(x,\nabla v) \ge \int_{\partial\Omega} g\langle m,\nu\rangle d\mathscr{H}^{N-1} + \int_\Omega q(x)dx = \int_\Omega f(x,\nabla\bar{u})dx,$$

which proves the result for (44). The case of (45) is similar. □

Remark 6.2. Concerning problem (44), the previous result generalizes [19, Theo. 1.4], hence [18, Theo. 1.3], where the requirement on $f^{**}(x,\cdot)$ to be affine on $\mathcal{D}_f(x)$ is supplemented with some continuity properties on the integrand. We avoid such a regularity hypothesis by exploiting some recent existence results for measurable Hamilton-Jacobi equations and the connection with differential inclusions. Additionally, our result covers a wider class of domains by requiring less regularity on the boundary, and it provides a new existence result for bilateral obstacle problems.

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