An existence result for perturbed sweeping processes with nonregular sets

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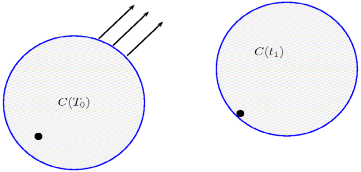
Summary

Introduction and motivation

Consider a large ring that contains a smaller ball inside, and the ring will start to move at time $t = T_0$.
Depending on the motion of the ring, the ball will just stay where it is (in case it is not hit by the ring), or otherwise it is swept towards the interior of the ring.
In this latter case the velocity of the ball has to point inwards to the ring in order not to leave.



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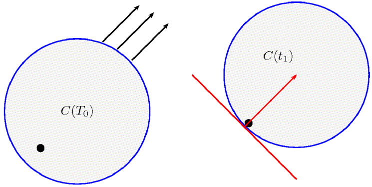

Introduction and motivation

Mathematically,

$$\begin{cases} -\dot{x}(t) \in N\left(C(t); x(t)\right) & \text{a.e. } t \in [T_0, T]; \\ x(T_0) = x_0 \in C(T_0), \end{cases} \tag{1.1}$$

where

- $x(t)$ is the position of the ball at time t.
- $C(t)$ is the moving set (the ring and its interior).
- $N\left(C(t); x(t)\right)$ is some appropriate outward normal cone of $C(t)$ at $x(t) \in C(t)$.

In the general setting, the set $C(t)$ is allowed to change its shape while moving.

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The perturbed sweeping process

$$\begin{cases} -\dot{x}(t) \in N\left(C(t); x(t)\right) + F(t, x(t)) & \text{a.e. } t \in [T_0, T]; \\ x(T_0) = x_0 \in C(T_0), \end{cases}$$

where

- $C\colon [T_0, T] \rightrightarrows H$ is a set-valued map with nonempty closed values.
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appears, for example, in the study of:

- Granular materials (J.J. Moreau)
- Electrical circuits (V. Acary - B. Brogliato)
- Crowd motion (F. Bernicot - J. Venel)
- Hysteresis in elasto-plastic models (P. Krejči)

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The Perturbed Sweeping Process: Existence theory

Main contributions:

- J.J. Moreau (1971 [15], 1972 [16], 1977 [17], 1999 [18])
  ▷ $C(t)$ convex and $F \equiv 0$.
- C. Castaing - T.D. Ha - M. Valadier (1993 [5])
  ▷ $C(t)$ convex and complement of a convex and F usc.
- M. Kunze - Monteiro-Marques (1996 [13], 2000 [14])
  ▷ $C(t)$ convex and $F \equiv 0$.
- H. Benabdellah (1999 [2])
  ▷$C(t)$ closed and $F \equiv 0$.
- G. Colombo - V. Goncharov (1999 [6])
  ▷ $C(t)$ closed and $F \equiv 0$.

The Perturbed Sweeping Process: Existence theory

Main contributions:

- M. Bounkhel - L. Thibault (2005 [4])
  ▷ $C(t)$ prox-regular and F usc.
- J. Edmond - L. Thibault (2005 [7], 2006 [8])
  ▷ $C(t)$ prox-regular and F usc.
- T. Haddad - A. Jourani - L. Thibault (2008 [9])
  ▷ $C(t)$ α-far, F mixed usc and $\dim H < +\infty$.
- Thibault (2003 [19], 2008 [20], 2016 [21])
  ▷ $C(t)$ convex and prox-regular.
- M. Bounkhel (2012 [3])
  ▷ $C(t)$ prox-regular F usc.

The Clarke normal cone

We will consider the Clarke normal cone: For $x \in S$

$$N\left(S;x\right) = \{\zeta \in H\colon\ \langle \zeta, v\rangle \leq 0\, \forall v \in T_S(x)\},$$

where $T_S(x)$ is the Clarke tangent cone:

$$v \in T_S(x) \Leftrightarrow \forall x_i \to x, \forall t_i \to 0^+, \exists v_i \to v \text{ such that } x_i + t_i v_i \in S\, \forall i.$$

Also, we set $N(S,x) = \emptyset$ if $x \notin S$.

Perturbed sweeping processes (PSP)

The perturbed sweeping process (PSP) is the following differential inclusion:

$$\begin{cases} -\dot{x}(t) \in N\left(C(t); x(t)\right) + F(t, x(t)) & \text{a.e. } t \in [T_0, T]; \\ x(T_0) = x_0 \in C(T_0), \end{cases}$$

where,

- H is a separable Hilbert space.
- $C\colon [T_0, T] \rightrightarrows H$ has nonempty and closed values.
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Basic assumptions

$(\mathcal{H}^F)$: $F\colon [T_0, T] \times H \rightrightarrows H$ has nonempty closed and convex values.

- For each $x \in H$, $F(\cdot, x)$ is measurable.
- For a.e. $t \in [T_0, T]$, $F(t, \cdot)$ is upper semicontinuous from H into H_w,
- There exist $c, d \in L^1(T_0, T)$ such that

$$d(0, F(t,x)) := \inf\{\|w\| : w \in F(t,x)\} \leq c(t)\|x\| + d(t),$$

for all $x \in H$ and a.e. $t \in [T_0, T]$.

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Basic assumptions (continued)

$(\mathcal{H}^C)$ $C\colon [T_0,T] \rightrightarrows H$ has nonempty closed values.

- There exist $\zeta \in \mathrm{AC}([T_0,T];\mathbb{R})$ such that for all $s,t \in [T_0,T]$

$$\mathrm{Hauss}(C(t),C(s)) \leq |\zeta(t)-\zeta(s)|.$$

- For every $t \in [T_0,T]$, every $r>0$ the set $C(t)\cap r\mathbb{B}$ is compact.

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Convex sets

Proposition

Let $S \subset H$ be a closed set. Then, S is convex if and only if

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq 0,$$

holds for all $x_1, x_2 \in S$ and all $x_i^* \in N(S; x_i) \cap \mathbb{B}$ for $i = 1, 2$.

ρ-uniformly prox-regular sets

Definition

Let $S \subset H$ be a closed set. We say that S is *ρ-uniformly prox-regular* if

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\frac{1}{\rho}\|x_1 - x_2\|^2,$$

holds for all $x_1, x_2 \in S$ and all $x_i^* \in N(S; x_i) \cap \mathbb{B}$ for $i = 1, 2$.

ρ-uniformly prox-regular sets

Proposition

Let $S \subset H$ be a closed set. Then S is ρ-uniformly prox-regular if and only if proj_S *is well defined and locally Lipschitz continuous on* $U_\rho(S) \cup S$.

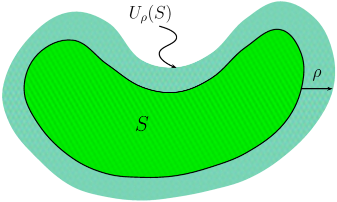

Figure: $U_\rho(S) := \{x \in H : 0 < d(x, S) < \rho\}$.

Proposition

Let $S \subset H$ be a closed set.

1. *Assume that S is ρ-uniformly prox-regular. Then,*

$$\partial d_S(x) = \frac{x - \mathrm{proj}_S(x)}{d_S(x)} \quad \forall x \in U_\rho(S).$$

   In particular $\|\partial d_S(x)\| = 1$ for all $x \in U_\rho(S)$.

2. *Assume that S is ball-compact. Then, for all $x \notin S$*

$$\partial d_S(x) = \frac{x - \overline{\mathrm{co}}\, \mathrm{Proj}_S(x)}{d_S(x)}.$$

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Positively α-far sets

Definition

Let $\alpha \in]0,1]$. A set $S \subset H$ is positively α-far if there exists $\rho > 0$ such that if $x \in U_\rho(S)$ then the following implication holds:

$$\zeta \in \partial d_S(x) \quad \text{then} \quad \|\zeta\| \geq \alpha, \tag{3.1}$$

where $U_\rho(S) := \{x \in H \colon 0 < d(x,S) < \rho\}$ is the ρ-tube around S.

Moreover, if $E \neq \emptyset$, we say that the family $(S(t))_{t\in E}$ is positively α-far if every $S(t)$ satisfies (3.1) with the same α and the same $\rho > 0$.

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Positively α-far sets

Proposition

Let $S \subset H$ be a ball-compact set and $\alpha \in]0, 1]$.

1. *Assume that*

$$\langle x - \pi_1, x - \pi_2 \rangle \geq \alpha^2 d_S^2(x) \quad \forall x \in U_\rho(S),$$

for all $\pi_1, \pi_2 \in \mathrm{Proj}_S(x)$. Then S is positively α-far.

2. *If S is positively α-far then*

$$\langle x - \pi_1, x - \pi_2 \rangle \geq (2\alpha^2 - 1) d_S^2(x) \quad \forall x \in U_\rho(S),$$

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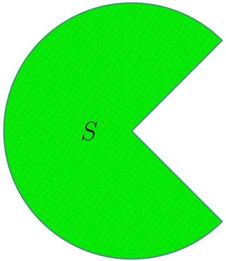
Positively α-far sets: An example

The set $S = \{(x,y) \in \mathbb{R}^2 \colon |y| \geq x\} \cap \mathbb{B}$ is positively $\frac{\sqrt{2}}{2}$-far but not uniformly prox-regular.



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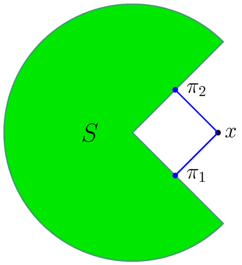

$\mathrm{Proj}_S(x) = \{\pi_1, \pi_2\}$

Existence for the perturbed sweeping process

$$\begin{cases} -\dot{x}(t) \in N\left(C(t); x(t)\right) + F(t, x(t)) & \text{a.e. } t \in [T_0, T]; \\ x(T_0) = x_0 \in C(T_0). \end{cases}$$

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Reduction of the perturbed sweeping process

To prove existence of the PSP, we use the reduction technique.

$$\begin{cases} -\dot{x}(t) \in N\left(C(t); x(t)\right) + F(t, x(t)) & \text{a.e. } t \in [T_0, T]; \\ x(T_0) = x_0 \in C(T_0). \end{cases}$$

Properties of the distance function

Proposition

Let $S \subset H$ be a closed set. Then,

- *For all $x \in S$*

$$\partial d_S(x) \subseteq N(S;x) \cap \mathbb{B}.$$

- *The set-valued map $x \rightrightarrows \partial d_S(x)$ is upper semicontinuous from H into H_w.*

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Reduction of sweeping process

To prove existence of the PSP, we consider the Reduced Problem:

$$\begin{cases} -\dot{x}(t) \in m(t,x)\partial d_{C(t)}\left(x(t)\right) + \tilde{F}(t,x(t)) & \text{a.e. } t \in [T_0,T]; \\ x(T_0) = x_0 \in C(T_0), \end{cases} \qquad (\mathcal{P}_{\text{Red}})$$

where

$$m(t,x) := \frac{1}{\alpha^2}\left(|\dot{\zeta}(t)| + c(t)\|x\| + d(t)\right),$$

and

$$\tilde{F}(t,x) := F(t,x) \cap \left(c(t)\|x\| + d(t)\right)\mathbb{B}.$$

Reduction of the Perturbed Sweeping Process

Proposition

The problema $(\mathcal{P}_{\text{Red}})$ has at least one solution which is also a solution of PSP.

Main result

Theorem (Jourani-Vilches, 2016 [12])

Assume that the following hold true:

1. $(\mathcal{H}^F)$ *and* $(\mathcal{H}^C)$ *hold.*
2. *The family* $(C(t))_{t\in[T_0,T]}$ *is uniformly positively* α*-far.*

Then, there exists at least one solution of the PSP:

$$\begin{cases} -\dot{x}(t) \in N\left(C(t); x(t)\right) + F(t, x(t)) & a.e.\ t \in [T_0, T]; \\ x(T_0) = x_0 \in C(T_0), \end{cases}$$

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Uniqueness of the Sweeping Process

Let us consider the Sweeping Process:

$$\begin{cases} -\dot{x}(t) \in N\left(C(t); x(t)\right) & \text{a.e. } t \in [T_0, T]; \\ x(T_0) = x_0 \in C(T_0). \end{cases} \tag{5.1}$$

Proposition

Assume that $C(t)$ is ρ-uniformly prox-regular for all $t \in [T_0, T]$ then (5.1) has at most one solution.

Uniqueness of the Sweeping Process

Consider $x_1(t) = (-t/2, t/2)$ and $x_2(t) = (-t/2, -t/2)$ defined over $[0, 1]$. Then x_1 and x_2 are solutions of

$$\begin{cases} -\dot{x}(t) \in N\left(C(t); x(t)\right) & \text{a.e. } t \in [0,1]; \\ \ x(0) = (0,0) \in C(0), \end{cases}$$

where $C(t) = S - (t, 0)$ for $t \in [0, 1]$.

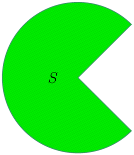

Figure: $S = \{(x,y) \in \mathbb{R}^2 \colon |y| \geq x\} \cap \mathbb{B}$

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Thank you!

An existence result for perturbed sweeping processes with nonregular sets

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