Régularisation de Moreau-Yosida du processus de rafle avec des ensembles non réguliers

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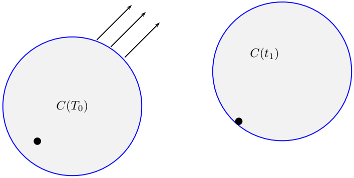
Summary

Introduction

Consider a large ring that contains a smaller ball inside, and the ring will start to move at time $t = T_0$.
Depending on the motion of the ring, the ball will just stay where it is (in case it is not hit by the ring), or otherwise it is swept towards the interior of the ring.
In this latter case the velocity of the ball has to point inwards to the ring in order not to leave.



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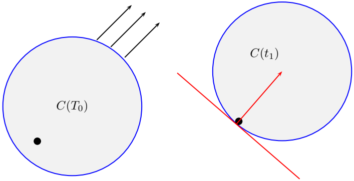

Introduction

Mathematically,

$$\begin{cases} -\dot{x}(t) \in N\left(C(t); x(t)\right) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in C(T_0), \end{cases} \tag{1.1}$$

where

- $x(t)$ is the position of the ball at time t.
- $C(t)$ is the moving set (the ring and its interior).
- $N\left(C(t); x(t)\right)$ is some appropriate outward normal cone of $C(t)$ at $x(t) \in C(t)$.

In the general setting, the set $C(t)$ is allowed to change its shape while is moving.

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The sweeping process

The differential inclusion:

$$\begin{cases} -\dot{x}(t) \in N\left(C(t); x(t)\right) \;\; \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in C(T_0), \end{cases} \tag{1.2}$$

is called *Sweeping process* (Processus de rafle) and it appears naturally in many problems from elastoplasticity, contact dynamics, non-regular electrical circuits, granular media, crowd motion, hysteresis, etc.

The sweeping process

Definition

Let H be a separable Hilbert space. We say that $x\colon [T_0, T] \to H$ is a solution of the sweeping process if it absolutely continuous and satisfies:

$$\begin{cases} -\dot{x}(t) \in N\left(C(t); x(t)\right) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in C(T_0), \end{cases}$$

where $N\left(C(t); x(t)\right)$ denotes the Clarke normal cone to $C(t)$ at $x(t) \in C(t)$.

What about the variation of the moving set?

Let $C, D \subseteq H$ be closed sets. The Hausdorff distance between C and D is defined by:

$$\begin{aligned}\operatorname{Haus}(C, D) &= \max\{\sup_{x\in C} d(x, D), \sup_{x\in D} d(x, C)\}\\ &= \sup_{x\in H} |d_C(x) - d_D(x)|\end{aligned}$$

What about the variation of the moving set?

Definition

Let $C\colon [T_0, T] \rightrightarrows H$ be a set-valued map. We say that C is κ-Lipschitz if

$$\mathrm{Haus}(C(t), C(s)) \leq \kappa|t-s| \quad \text{for all } t, s \in [T_0, T]. \tag{1.3}$$

In particular, if $C(t) = S + v(t)$, where v is κ-Lipschitz, (1.3) holds.
For all $t \in [T_0, T]$ and $x \in H$

$$|d_{C(t)}(x) - d_{C(s)}(x)| \leq \kappa|t-s|.$$

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Lemma

Let $x\colon [T_0, T] \to H$ be an absolutely continuous function and assume that $C\colon [T_0, T] \rightrightarrows H$ is κ-Lipschitz with nonempty and closed values. If $\varphi(t) := d_{C(t)}(x(t))$ for $t \in [T_0, T]$, then

1. *φ is absolutely continuous.*
2. *For a.e. $t \in [T_0, T]$*

$$\dot{\varphi}(t) \leq \kappa + \max_{v \in \partial d_{C(t)}(x(t))} \langle v, \dot{x}(t) \rangle$$

3. *For a.e. $t \in [T_0, T]$ where $x(t) \notin C(t)$*

$$\dot{\varphi}(t) \leq \kappa + \min_{v \in \partial d_{C(t)}(x(t))} \langle v, \dot{x}(t) \rangle$$

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Aim of this talk

Our aim is to prove the existence of solutions to the sweeping process:

$$\begin{cases} -\dot{x}(t) \in N\left(C(t); x(t)\right) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in C(T_0), \end{cases}$$

where $C\colon [T_0, T] \rightrightarrows H$ is κ-Lipschitz continuous with nonempty, closed convex or nonconvex values.

What about the shape of the moving set?

1. Convex case: $C(t)$ is convex for all $t \in [T_0, T]$.
2. Nonconvex case I: $C(t)$ is ρ-uniformly prox-regular for all $t \in [T_0, T]$.
3. Nonconvex case II: $C(t)$ is positively α-far for all $t \in [T_0, T]$.

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Moreau-Yosida envelope

Given $f : H \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous function, bounded from below. For $\lambda > 0$, the Moreau-Yosida envelope of f is defined as:

$$f_\lambda(x) = \inf_{y \in H} \left(f(y) + \frac{1}{2\lambda} \|x - y\|^2 \right).$$

The main properties of the Moreau-Yosida envelope are:

1. f_λ is locally-Lipschitz.
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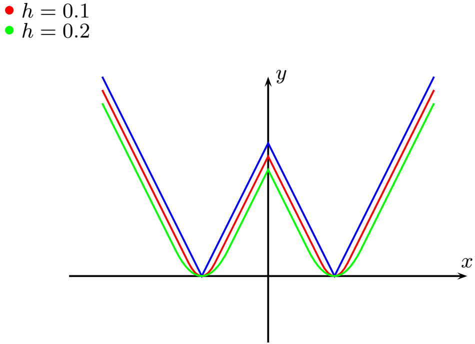

Figure: Moreau-Yosida envelope of $f(x) = \min\{|x-1|, |x+1|\}$.

Moreau-Yosida regularization: Convex case

If f is convex, then f_λ is $C^{1,1}$ with

$$\nabla f_\lambda(x) = \frac{1}{\lambda}\left(x - J_\lambda(x)\right),$$

where J_λ is the unique solution of

$$f_\lambda(x) = f(J_\lambda(x)) + \frac{1}{2\lambda}\|x - J_\lambda(x)\|^2.$$

Properties of Moreau-Yosida envelope: Convex case

Let $S \subseteq H$ be a closed and convex set, then

$$\begin{aligned}(I_S)_\lambda(x) &= \inf_{y\in H}\left(I_S(y) + \frac{1}{\lambda}\|x-y\|^2\right)\\ &= \frac{1}{\lambda}\inf_{y\in S}\|x-y\|^2\\ &= \frac{1}{2\lambda}d_S^2(x).\end{aligned}$$

$$\frac{1}{2\lambda}\nabla d_S^2(x) = \frac{x-\mathrm{proj}_S(x)}{\lambda}.$$

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Moreau-Yosida regularization: Convex case

Therefore, we will approach the sweeping process through the following penalized differential equation:

$$\begin{cases} -\dot{x}_\lambda(t) = \dfrac{1}{2\lambda}\nabla d^2_{C(t)}(x_\lambda(t)) & \text{a.e. } t \in [T_0, T], \\ x_\lambda(T_0) = x_0 \in C(T_0). \end{cases} \qquad (\mathcal{P}_\lambda)$$

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Therefore, we will approach the sweeping process through the following penalized differential equation:

$$\begin{cases} -\dot{x}_\lambda(t) = \dfrac{1}{\lambda}\Big(x_\lambda(t) - \mathrm{proj}_{C(t)}(x_\lambda(t))\Big) \quad \text{a.e. } t \in [T_0, T]; \\ x_\lambda(T_0) = x_0 \in C(T_0). \end{cases} \qquad (\mathcal{P}_\lambda)$$

Lemma

If A, B are two closed and convex sets, then

$$\|\operatorname{proj}_A(x) - \operatorname{proj}_B(y)\|^2 \leq \|x - y\|^2 + 2\,[d(x, A) + d(y, B)]\,\operatorname{Haus}(A, B).$$

Therefore, if $C\colon [T_0, T] \rightrightarrows H$ has closed and convex values, then

$$\begin{aligned}\|\operatorname{proj}_{C(t)}(x) - \operatorname{proj}_{C(s)}(y)\|^2 &\leq \|x - y\|^2 \\ &\quad + 2\,[d(x, C(t)) + d(y, C(t))]\,\operatorname{Haus}(C(t), C(s)).\end{aligned}$$

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The existence of $(\mathcal{P}_\lambda)$ can be obtained through the classical Cauchy-Lipschitz Theorem.

Existence result

Theorem (Moreau, 1971)

Assume that:

1. $\mathrm{Haus}(C(t), C(s)) \leq \kappa|t-s|$ *for all* $t, s \in [T_0, T]$.
2. $C(t)$ *is convex for all* $t \in [T_0, T]$.

Then, there exists a unique solution of the sweeping process

$$\begin{cases} -\dot{x}(t) \in N^{Fen}\left(C(t); x(t)\right) & a.e.\ t \in [T_0, T]; \\ x(T_0) = x_0 \in C(T_0). \end{cases}$$

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Sketch of proof

The proof can be divided in several steps.

(i) Step 1: For all $\lambda > 0$

$$d_{C(t)}(x_\lambda(t)) \leq \kappa\lambda \quad \forall t \in [T_0, T].$$

(ii) Step 2: $x_\lambda : [T_0, T] \to H$ is κ-Lipschitz continuous with

$$\|\dot{x}_\lambda(t)\| \leq \kappa \quad \text{a.e. } t \in [T_0, T].$$

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Step 3: $(x_\lambda)_{\lambda>0}$ is a Cauchy sequence in $C\left([T_0,T];H\right)$, thus, converges uniformly to a function x. If $\lambda,\mu>0$, then

$$-\dot{x}_\lambda(t)\in N\left(C(t);\operatorname{proj}_{C(t)}(x_\lambda(t))\right)$$
$$-\dot{x}_\mu(t)\in N\left(C(t);\operatorname{proj}_{C(t)}(x_\mu(t))\right)$$

For a.e. $t\in[T_0,T]$

$$\begin{aligned}\frac{d}{dt}\|x_\lambda(t)-x_\mu(t)\|^2 &\leq 2\left\langle -\dot{x}_\lambda(t)+\dot{x}_\mu(t),\lambda\dot{x}_\lambda(t)-\mu\dot{x}_\mu(t)\right\rangle\\ &\leq 4\kappa^2(\lambda+\mu).\end{aligned}$$

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(iv) Step 4: $x(t) \in C(t)$ for all $t \in [T_0, T]$.

(v) Step 5: For a.e. $t \in [T_0, T]$

$$-\dot{x}_\lambda(t) \in \kappa \,\overline{\mathrm{co}} \left\{ \partial d_{C(t)}(x_\lambda(t)) \cup \{0\} \right\}.$$

(vi) Step 6: Pass to the limit:

$$-\dot{x}(t) \in \kappa \, \partial d_{C(t)}(x(t)) \qquad \text{a.e. } t \in [T_0, T].$$

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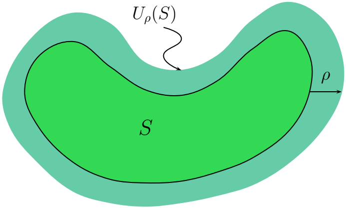
Uniformly prox-regular sets

Definition

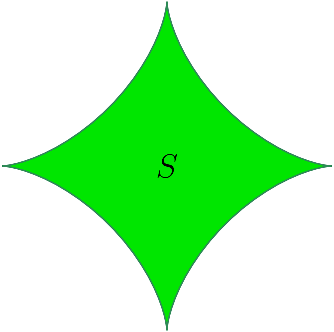
We say that $S \subseteq H$ is ρ-uniformly prox-regular if $x \mapsto \text{proj}_S(x)$ is single-valued and continuous in the set $S \cup U_\rho(S)$, where

$$U_\rho(S) := \{y \in H : 0 < d(y, S) < \rho\}.$$



Uniformly prox-regular sets

Example: An astroid is uniformly prox-regular



Theorem (Poliquin, Rockafellar, Thibault, 2000)

Let $S \subseteq H$ be a closed set. Then, the following statements are equivalent:

1. *S is ρ-uniformly prox-regular.*
2. *d_S^2 is $C^{1,1}$ with $\nabla d_S^2(x) = 2(x - \operatorname{proj}_S(x))$ on $U_\rho(S)$.*
3. *Whenever $x_i \in S$ and $v_i \in N(S; x_i) \cap \mathbb{B}$ one has*

$$\langle v_1 - v_2, x_1 - x_2 \rangle \geq -\frac{1}{\rho}\|x_1 - x_2\|^2.$$

Moreover, if S is weakly compact, then S is ρ-uniformly prox-regular if and only if $x \mapsto \operatorname{proj}_S(x)$ is single-valued on $U_\rho(S)$.

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Moreover, if S is weakly compact, then S is ρ-uniformly prox-regular if and only if $x \mapsto \mathrm{proj}_S(x)$ is single-valued on $U_\rho(S)$.

Moreau-Yosida regularization: Uniformly prox-regular case

As in the convex case, we approach the sweeping process through the following penalized differential equation:

$$\begin{cases} -\dot{x}_\lambda(t) = \dfrac{1}{\lambda}\left(x_\lambda(t) - \operatorname{proj}_{C(t)}(x_\lambda(t))\right) & \text{a.e. } t \in [T_0, T]; \\ x_\lambda(T_0) = x_0 \in C(T_0). \end{cases} \qquad (\mathcal{P}_\lambda)$$

The existence of $(\mathcal{P}_\lambda)$ can be obtained, again, through the classical Cauchy-Lipschitz Theorem.

Existence result: Uniformly prox-regular case

Theorem (Thibault, 2008)

Assume that:

1. $\mathrm{Haus}(C(t), C(s)) \leq \kappa |t-s|$ *for all* $t, s \in [T_0, T]$.
2. $C(t)$ *is* ρ*-uniformly prox-regular for all* $t \in [T_0, T]$.

Then, there exists a unique solution of the sweeping process

$$\begin{cases} -\dot{x}(t) \in N\left(C(t); x(t)\right) & a.e.\ t \in [T_0, T]; \\ x(T_0) = x_0 \in C(T_0). \end{cases}$$

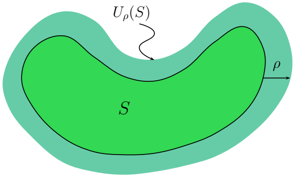
Positively α-far sets

Definition

Let $\alpha \in]0,1]$. $S \subseteq H$ is positively α-far if there exists $\rho > 0$ such that if $x \in U_\rho(S)$

$$\text{if } \zeta \in \partial d_S(x) \text{ then } \|\zeta\| \geq \alpha,$$

where $U_\rho(S) := \{x \in H \colon 0 < d(x,S) < \rho\}$ is the ρ-tube around S.



Positively α-far sets

If S is weakly compact, then

$$\frac{1}{2}\partial d_S^2(x) = x - \overline{\text{co}}\,\text{Proj}_S(x) \quad \forall x \in H.$$

If S is weakly compact, then S is positively α-far if there exists $\rho > 0$ such that

$$0 < \alpha \leq \frac{d(x, \overline{\text{co}}\,\text{Proj}_S(x))}{d_S(x)} \quad \forall x \in U_\rho(S).$$

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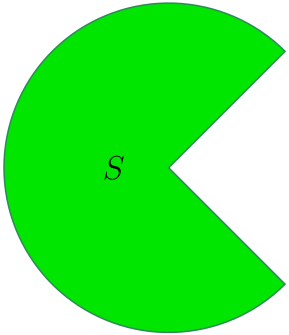

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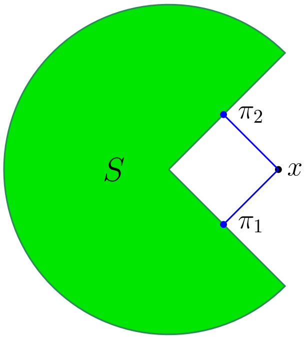

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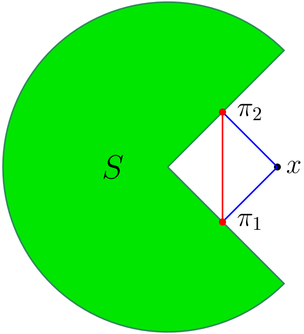

Positively α-far sets

- If S is convex, then S is positively 1-far (with $\rho = +\infty$).
- If S is ρ-uniformly prox-regular, then S is positively 1-far (with the same ρ).
- The class of positively α-far set is very general and includes several classes of sets: convex sets, paraconvex sets, uniformly prox-regular sets, uniformly subsmooth sets, etc.

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Moreau-Yosida regularization

Therefore, we will approach the sweeping process through the following penalized differential inclusion:

$$\begin{cases} -\dot{x}_\lambda(t) \in \dfrac{1}{2\lambda}\partial d^2_{C(t)}(x_\lambda(t)) & \text{a.e. } t \in [T_0, T]; \\ x_\lambda(T_0) = x_0 \in C(T_0). \end{cases} \tag{$\mathcal{P}_\lambda$}$$

The existence of $(\mathcal{P}_\lambda)$ can be obtained through some existence results for differential inclusions with compact values.

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The existence of $(\mathcal{P}_\lambda)$ can be obtained through some existence results for differential inclusions with compact values.

Existence result

Theorem (Jourani-V., 2016)

Assume that:

1. $\mathrm{Haus}(C(t), C(s)) \leq \kappa|t-s|$ *for all* $t, s \in [T_0, T]$.
2. $C(t)$ *is positively* α*-far for all* $t \in [T_0, T]$.
3. $C(t)$ *is ball compact for all* $t \in [T_0, T]$.

Then, there exists at least one solution of

$$\begin{cases} -\dot{x}(t) \in N\left(C(t); x(t)\right) & a.e.\ t \in [T_0, T]; \\ x(T_0) = x_0 \in C(T_0). \end{cases}$$

Sketch of proof

The proof can be divided in several steps.

(i) Step 1:

$$d_{C(t)}(x_\lambda(t)) \leq \frac{\kappa}{\alpha^2}\lambda \quad \forall t \in [T_0, T].$$

(ii) Step 2: $x_\lambda \colon [T_0, T] \to H$ is $\frac{\kappa}{\alpha^2}$-Lipschitz continuous with

$$\|\dot{x}_\lambda(t)\| \leq \frac{\kappa}{\alpha^2} \quad \text{a.e. } t \in [T_0, T].$$

Sketch of proof

(iii) Step 3: There exists a subsequence $(\lambda_k)_k$ of $(\lambda)_{\lambda>0}$ and $x \in \mathrm{AC}\left([T_0, T]; H\right)$ such that

$$\begin{aligned} x_{\lambda_k}(t) &\rightharpoonup x(t) && \forall t \in [T_0, T]; \\ x_{\lambda_k} &\rightharpoonup x && \text{in } L^1_w\left([T_0, T]; H\right); \\ \dot{x}_{\lambda_k} &\rightharpoonup \dot{x} && \text{in } L^1_w\left([T_0, T]; H\right). \end{aligned}$$

(iv) Step 4: $x_{\lambda_k}(t) \to x(t)$ and $x(t) \in C(t)$ for all $t \in [T_0, T]$.

Sketch of proof

(v) Step 5: For a.e. $t \in [T_0, T]$

$$-\dot{x}_{\lambda_k}(t) \subseteq \frac{\kappa}{\alpha^2}\overline{\text{co}}\left\{\partial d_{C(t)}(x_{\lambda_k}(t)) \cup \{0\}\right\}.$$

(vi) Step 6: Pass to the limit:

$$-\dot{x}(t) \in \frac{\kappa}{\alpha^2}\partial d_{C(t)}(x(t)) \quad \text{ a.e. } t \in [T_0, T].$$

This completes the proof because $\partial d_{C(t)}(x(t)) \subseteq N(C(t); x(t))$.

Further results I

Theorem (Jourani-V., 2016)

Assume that

(i) *There exists* $v \in \mathrm{CBV}([T_0, T]; \mathbb{R})$ *such that for all* $s, t \in [T_0, T]$ *and all* $x, y \in H$

$$\mathrm{Haus}(C(t), C(s)) \leq |v(t) - v(s)|.$$

(ii) $C(t)$ *if positively* α*-far for all* $t \in [T_0, T]$ *(with the same* $\rho > 0$*).*

(iii) $C(t)$ *is ball compact for all* $t \in [T_0, T]$.

Then, there exists at least one solution $x \in \mathrm{CBV}\left([T_0, T]; H\right)$ *of*

$$\begin{cases} -dx \in N\left(C(t); x(t)\right), \\ x(T_0) = x_0 \in C(T_0). \end{cases}$$

Further results II

The same proof applies, mutatis mutandis, to the state-depedent sweeping process in the sense of measure differential inclusion.

Theorem (Jourani-V., 2016)

Assume that

(i) *There exists* $v \in \mathrm{CBV}([T_0, T]; \mathbb{R})$ *and* $L \in [0, 1[$ *such that for all* $s, t \in [T_0, T]$ *and all* $x, y \in H$

$$\mathrm{Haus}(C(t,x), C(s,y)) \leq |v(t) - v(s)| + L\|x - y\|.$$

(ii) *The family* $\{C(t,v) \colon (t,v) \in [T_0, T] \times H\}$ *is equi-uniformly subsmooth.*

Further results II

Theorem (Jourani-V., 2016)

(iii) *There exists $k \in L^1(T_0, T)$ such that for every $t \in [T_0, T]$, every $r > 0$ and every bounded set $A \subseteq H$*

$$\gamma(C(t, A) \cap r\mathbb{B}) \leq k(t)\gamma(A),$$

where $\gamma = \alpha$ or $\gamma = \beta$ is either the Kuratowski or the Hausdorff measure of non-compactness and $k(t) < 1$ for all $t \in [T_0, T]$.

Then, there exists at least one solution $x \in \mathrm{CBV}([T_0, T]; H)$ of

$$\begin{cases} -dx \in N\left(C(t, x(t)); x(t)\right), \\ x(T_0) = x_0 \in C(T_0, x_0). \end{cases}$$

Further results III

Theorem (V., 2017)

Assume that

(i) *There exists $\kappa \geq 0$ and $L \in [0,1[$ such that for all $s,t \in [T_0,T]$ and all $x,y \in H$*

$$\mathrm{Haus}(C(t,x),C(s,x)) \leq \kappa|t-s| + L\|x-y\|.$$

(ii) *The family $\{C(t,v)\colon (t,v) \in [T_0,T] \times H\}$ is equi-uniformly subsmooth.*

(iii) *There exists $k \in L^1(T_0,T)$ such that for every $t \in [T_0,T]$, every $r > 0$ and every bounded set $A \subseteq H$*

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Further results III

Theorem (V., 2017)

(iv) *$f\colon [T_0, T] \times H \to H$ satisfies:*

- *For every $x \in H$, $f(\cdot, x)$ is measurable.*
- *There exists $\beta > 0$ such that, for all $t \in [T_0, T]$ and all $x, y \in H$*

$$\|f(t,x) - f(t,y)\| \leq \beta \|x - y\|.$$

- *There exists $M \geq 0$ such that, for all $t \in [T_0, T]$ and all $x \in H$ $\sup_{y \in \mathrm{Proj}_{C(t,x)}(x)} \|f(t,y)\| \leq M$.*

Then, there exists at least one solution $x \in \mathrm{Lip}\left([T_0, T]; H\right)$ of

$$\begin{cases} -\dot{x}(t) \in N\left(C(t, x(t)); x(t)\right) + f(t, x(t)), \\ x(T_0) = x_0 \in C(T_0, x_0). \end{cases}$$

Further results IV

Theorem (V., 2017)

Assume that

(i) *There exists* $\kappa \geq 0$ *and* $L \in [0,1[$ *such that for all* $s,t \in [T_0,T]$ *and all* $x,y \in H$

$$\mathrm{Haus}(C(t,x),C(s,x)) \leq \kappa|t-s| + L\|x-y\|.$$

(ii) *The family* $\{C(t,v)\colon (t,v) \in [T_0,T] \times H\}$ *is equi-uniformly subsmooth.*

(iii) *There exists* $k \in L^1(T_0,T)$ *such that for every* $t \in [T_0,T]$*, every* $r > 0$ *and every bounded set* $A \subseteq H$

$$\gamma(C(t,A) \cap r\mathbb{B}) \leq k(t)\gamma(A),$$

where $\gamma = \alpha$ *or* $\gamma = \beta$ *is either the Kuratowski or the Hausdorff measure of non-compactness and* $k(t) < 1$ *for all* $t \in [T_0,T]$.

Further result IV

Theorem (V., 2017)

(iv) *$\Phi\colon [T_0, T] \times H \to H$ satisfies:*

- *For every $x \in H$, $\Phi(\cdot, x)$ is convex.*
- *Φ is boundedly Lipschitz-continuous, i.e., for all $r > 0$, there exists $L_r > 0$ such that for all $t \in [T_0, T]$ and all $x, y \in r\mathbb{B}$*

$$|\Phi(t,x) - \Phi(t,y)| \leq L_r \|x - y\|.$$

- *There exists $m \geq 0$ such that $\Phi(t,0) \leq m$ for all $t \in [T_0, T]$.*

Then, there exists at least one solution $x \in \mathrm{Lip}\left([T_0, T]; H\right)$ of

$$\begin{cases} -\dot{x}(t) \in N\left(C(t, x(t)); x(t)\right) + \partial\Phi(t, x(t)), \\ x(T_0) = x_0 \in C(T_0, x_0). \end{cases}$$

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Régularisation de Moreau-Yosida du processus de rafle avec des ensembles non réguliers

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