On a generalized perturbed sweeping process with nonregular sets

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Summary

1. Introduction and motivation
2. Position of the problem
3. Basic assumptions
4. An existence result for the GPSP
5. Some consequences
6. Uniqueness
7. References
1 Introduction and motivation

2 Position of the problem

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Introduction and motivation

Consider a large ring that contains a smaller ball inside, and the ring will start to move at time $t = T_0$. Depending on the motion of the ring, the ball will just stay where it is (in case it is not hit by the ring), or otherwise it is swept towards the interior of the ring. In this latter case the velocity of the ball has to point inwards to the ring in order not to leave.
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Mathematically,

\[
\begin{cases}
-\dot{v}(t) \in N(C(t); v(t)) \text{ a.e. } t \in [T_0, T]; \\
v(T_0) = v_0 \in C(T_0),
\end{cases}
\]  

(1.1)

where

- \(v(t)\) is the position of the ball at time \(t\).
- \(C(t)\) is the moving set (the ring and its interior).
- \(N(C(t); v(t))\) is some appropriate outward normal cone of \(C(t)\) at \(v(t) \in C(t)\).

In the general setting, the set \(C(t)\) is allowed to change its shape while moving.
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In the general setting, the set \(C(t)\) is allowed to change its shape while moving.
Here we consider the Clarke normal cone: For $x \in S$

$$N(S; x) = \{\zeta \in H : \langle \zeta, v \rangle \leq 0 \forall v \in T_S(x)\},$$

where $T_S(x)$ is the Clarke tangent cone:

$$v \in T_S(x) \iff \forall x_i \to x, \forall t_i \to 0, \exists v_i \to v \text{ such that } x_i + t_i v_i \in S \forall i.$$  

Also, we set $N(S, x) = \emptyset$ if $x \not\in S$.  

Moreau’s perturbed sweeping process

\[
\begin{cases}
-\dot{v}(t) \in N(C(t); v(t)) + F(t, v(t)) \quad \text{a.e. } t \in [T_0, T]; \\
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\end{cases}
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where

- \( C : [T_0, T] \Rightarrow H \) is a set-valued map with nonempty closed values.
- \( N(S, \cdot) \) is the Clarke normal cone to \( S \).
- \( F : [T_0, T] \times H \Rightarrow H \) is a set-valued map with nonempty closed convex values satisfying some standard conditions.
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It appears, for example, in:

- Granular materials (J.J. Moreau)
- Electrical circuits (V. Acary - B. Brogliato)
- Crowd motion (F. Bernicot - J. Venel)
- Hysteresis in elasto-plastic models (P. Krejčí)
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Moreau’s perturbed sweeping process: Existence theory

Main contributions:

- J.J. Moreau (1971 [28], 1972 [29], 1977 [30], 1999 [31])
  - $C(t)$ convex and $F \equiv 0$.

- C. Castaing - T.D. Ha - M. Valadier (1993 [14])
  - $C(t)$ convex and complement of a convex and $F$ usc.

- M. Kunze - Monteiro-Marques (1996 [25], 2000 [27])
  - $C(t)$ convex and $F \equiv 0$.

- G. Colombo - V. Goncharov (1999 [16])
  - $C(t)$ closed and $F \equiv 0$.

- H. Benabdellah (2000 [6])
  - $C(t)$ closed and $F \equiv 0$. 
Moreau’s perturbed sweeping process: Existence theory

Main contributions:

- M. Bounkhel - L. Thibault (2005 [12])
  - $C(t)$ prox-regular and $F$ usc.

- J. Edmond - L. Thibault (2005 [17], 2006 [18])
  - $C(t)$ prox-regular and $F$ usc.

  - $C(t)$ $\alpha$-far and $F$ mixed usc.

- Thibault (2003 [33], 2008 [34], 2016 [35])
  - $C(t)$ convex and prox-regular.

- A. Jourani - E. Vilches (2016 [24])
  - $C(t)$ $\alpha$-far and $F$ usc.
State-dependent perturbed sweeping process

\[
\begin{cases}
-\dot{v}(t) \in N(C(t, v(t)); v(t)) + F(t, v(t)) \quad \text{a.e. } t \in [T_0, T]; \\
v(T_0) = v_0 \in C(T_0, v_0),
\end{cases}
\]

where

1. \(C: [T_0, T] \times H \Rightarrow H\) is a set-valued map with nonempty closed values.
2. \(N(S, \cdot)\) is the Clarke normal cone to \(S\).
3. \(F: [T_0, T] \times H \Rightarrow H\) is a set-valued map with nonempty closed convex values satisfying some standard conditions.
State-dependent perturbed sweeping process: Existence theory

Main contributions:

- M. Kunze - M. Monteiro-Marques (1998 [26])
  - $C(t, x)$ convex and $F \equiv 0$.
- N. Chemetov - M. Monteiro-Marques (2007 [15])
  - $C(t, x)$ prox-regular and $F$ continuous.
- M. Bounkhel - C. Castaing (2012 [11])
  - $C(t, x)$ convex and $F \equiv 0$.
- T. Haddad (2013 [19])
  - $C(t, x)$ convex and $F$ usc.
Main contributions:

  - $C(t, x)$ convex and $F$ mixed usc.

- J. Noel - L. Thibault (2014 [32])
  - $C(t, x)$ subsmooth and $F$ usc.

  - $C(t, x)$ prox-regular and $F$ mixed usc.

- A. Jourani - E. Vilches (2016 [22])
  - $C(t, x)$ subsmooth and $F \equiv 0$. 
Second-order perturbed sweeping process

\[
\begin{cases}
-\ddot{u}(t) \in N(C(t, u(t), \dot{u}(t)); \dot{u}(t)) + F(t, u(t), \dot{u}(t)) & \text{a.e. } t \in [T_0, T]; \\
u(T_0) = u_0, \dot{u}(T_0) = v_0 \in C(T_0, u_0, v_0),
\end{cases}
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where,

1. \( C : [T_0, T] \times H \times H \rightrightarrows H \) is a set-valued map with nonempty closed values.
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3. \( F : [T_0, T] \times H \times H \rightrightarrows H \) is a set-valued map with nonempty closed convex values satisfying some standard conditions.
Main contributions:

- C. Castaing (1988 [13])
  - $C(t, u, v) = C(u)$ convex and $F \equiv 0$.

- M. Bounkhel et al (2003 [8], 2004 [10], 2010 [9])
  - $C(t, u, v) = C(u)$ prox-regular and $F(t, u, v) = F(t, v)$ usc.

  - $C(t, u, v) = C(t)$ or $C(u)$ prox-regular and $F$ usc.

- F. Bernicot - J. Venel (2012 [7])
  - $C(t, u, v) = C(t)$ prox-regular and $F(t, u, v) = F(t, u)$ Lipschitz.

- S. Adly - B. Le (2016 [1])
  - $C(t, u, v) = C(t, u)$ prox-regular and $F$ usc.
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Generalized perturbed sweeping process (GPSP)

The generalized perturbed sweeping process (GPSP):

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\begin{align*}
-\dot{u}(t) &= Bv(t) \\
-\dot{v}(t) &\in N(C(t, u(t), v(t)); v(t)) + F(t, u(t), v(t)) + Au(t) \\
u(T_0) &= u_0, v(T_0) = v_0 \in C(T_0, u_0, v_0),
\end{align*}
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where,

- $H$ is a separable Hilbert space.
- $A : H \to H$ and $B : H \to H$ are two bounded linear operators.
- $C : [T_0, T] \times H \times H \Rightarrow H$ is a set-valued map with nonempty and closed values.
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where,

- *H* is a separable Hilbert space.
- *A* : *H* → *H* and *B* : *H* → *H* are two bounded linear operators.
- *C* : [T_0, T] × *H* × *H* → *H* is a set-valued map with nonempty and closed values.
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why consider the GPSP?

- If \( C(t, u, v) = C(t), F(t, u, v) = F(t, v), A = 0 \) and \( B = 0 \) we recover the Moreau’s perturbed sweeping process.
- If \( C(t, u, v) = C(t, v), F(t, u, v) = F(t, v), A = 0 \) and \( B = 0 \) we recover the state-dependent perturbed sweeping process.
- If \( A = 0 \) and \( B = -I \) we recover the second-order perturbed sweeping process.
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v(t) &\in C(t, u(t), v(t)) & \forall t \in [T_0, T]; \\
u(T_0) &= u_0, v(T_0) = v_0 & \in C(T_0, u_0, v_0),
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Basic assumptions

\((\mathcal{H}^F)\): \(F: [T_0, T] \times H \times H \rightrightarrows H\) has nonempty closed and convex values.

- For each \((u, v) \in H \times H\), \(F(\cdot, u, v)\) is measurable.
- For a.e. \(t \in [T_0, T]\), \(F(t, \cdot, \cdot)\) is upper semicontinuous from \(H \times H\) into \(H_w\).
- There exist \(c, d \in L^1(T_0, T)\) such that
  \[d(0, F(t, u, v)) := \inf\{\|w\| : w \in F(t, u, v)\} \leq c(t)\|u, v\| + d(t),\]
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- For each \((u, v) \in H \times H\), \(F(\cdot, u, v)\) is measurable.
- For a.e. \(t \in [T_0, T]\), \(F(t, \cdot, \cdot)\) is upper semicontinuous from \(H \times H\) into \(H_w\).
- There exist \(c, d \in L^1(T_0, T)\) such that

\[
d(0, F(t, u, v)) := \inf\{\|w\| : w \in F(t, u, v)\} \leq c(t)\|(u, v)\| + d(t),
\]

for a.e. \(t \in [T_0, T]\) and all \((u, v) \in H \times H\).
Basic assumptions (continued)

\((\mathcal{H}^C)\) \( C: [T_0, T] \times H \times H \rightarrow H \) has nonempty closed values.

- There exist \( \zeta \in \text{AC}([T_0, T]; \mathbb{R}) \), \( L_1 \geq 0 \) and \( L_2 \in [0, 1] \) such that for all \( s, t \in [T_0, T] \) and all \( x, y, u, v \in H \)

\[
\text{Hauss}(C(t, x, u), C(s, y, v)) \leq |\zeta(t) - \zeta(s)| + L_1\|x - y\| + L_2\|u - v\|.
\]

- For every \( t \in [T_0, T] \), every \( r > 0 \) and every pair of bounded sets \( A, B \subset H \), the set \( C(t, A, B) \cap r\mathbb{B} \) is relatively compact.
Basic assumptions (continued)

\((\mathcal{H}^C)\): \([0, T] \times H \times H \Rightarrow H\) has nonempty closed values.
- There exist \(\zeta \in AC([0, T]; \mathbb{R}), L_1 \geq 0\) and \(L_2 \in [0, 1]\) such that for all \(s, t \in [0, T]\) and all \(x, y, u, v \in H\)

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Basic assumptions (continued)

\((\mathcal{H}^C)\) \(C : [T_0, T] \times H \times H \Rightarrow H\) has nonempty closed values.

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\]

- For every \(t \in [T_0, T]\), every \(r > 0\) and every pair of bounded sets \(A, B \subset H\), the set \(C(t, A, B) \cap r\mathbb{B}\) is relatively compact.
Uniformly subsmooth sets

Definition

* S is *uniformly subsmooth*, if for every $\varepsilon > 0$ there exists $\delta > 0$, such that

$$ \langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\varepsilon \| x_1 - x_2 \|,$$

holds for all $x_1, x_2 \in S$ satisfying $\| x_1 - x_2 \| < \delta$ and all $x_i^* \in N(S; x_i) \cap \mathbb{B}$ for $i = 1, 2$. 
equi-uniformly subsmooth sets

**Definition**

If \( E \neq \emptyset \) the family \( (S(t))_{t \in E} \) is *equi-uniformly subsmooth*, if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \), such that for all \( t \in E \)

\[
\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\varepsilon \| x_1 - x_2 \|,
\]

holds for all \( x_1, x_2 \in S(t) \) satisfying \( \| x_1 - x_2 \| < \delta \) and all \( x_i^* \in N(S(t); x_i) \cap \mathbb{B} \) for \( i = 1, 2 \).
Positively $\alpha$-far sets

**Definition**

Let $\alpha \in ]0, 1]$. A set $S \subset H$ is *positively $\alpha$-far* if there exists $\rho > 0$ such that if $x \in U_\rho(S)$ then the following implication holds:

$$\zeta \in \partial d_S(x) \quad \text{then} \quad \|\zeta\| \geq \alpha, \tag{3.1}$$

where $U_\rho(S) := \{x \in H : 0 < d(x, S) < \rho\}$ is the $\rho$-tube around $S$.

Moreover, if $E \neq \emptyset$, we say that the family $(S(t))_{t \in E}$ is *positively $\alpha$-far* if every $S(t)$ satisfies (3.1) with the same $\alpha$ and the same $\rho > 0$. 
Definition

Let $\alpha \in ]0, 1]$. A set $S \subset H$ is positively $\alpha$-far if there exists $\rho > 0$ such that if $x \in U_\rho(S)$ then the following implication holds:

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Moreover, if $E \neq \emptyset$, we say that the family $(S(t))_{t \in E}$ is positively $\alpha$-far if every $S(t)$ satisfies (3.1) with the same $\alpha$ and the same $\rho > 0$. 
Relation between some classes

- If $S$ is convex then $S$ is 1-far (with $\rho = +\infty$).
- If $S$ is $\rho$-uniformly prox-regular then $S$ is 1-far (with the same $\rho$).
- If $S$ is uniformly subsmooth then $S$ is $\sqrt{1 - \varepsilon}$-far for all $\varepsilon \in ]0, 1[$.
Relation between some classes

- If $S$ is convex then $S$ is $1$-far (with $\rho = +\infty$).
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Relation between some classes

- If $S$ is convex then $S$ is 1-far (with $\rho = +\infty$).
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- If $S$ is uniformly subsmooth then $S$ is $\sqrt{1 - \varepsilon}$-far for all $\varepsilon \in ]0, 1[$.
Positively $\alpha$-far sets: An example

$S$ is $\frac{\sqrt{2}}{2}$-far but not subsmooth.

**Figure:** $S = \{(x, y) \in \mathbb{R}^2 : |y| \geq x\} \cap B$
Proposition

Assume that the following assumptions holds true:

- $H^C$ holds.
- The family $\{C(t, u, v)\}_{(t, u, v) \in [T_0, T] \times H \times H}$ is equi-uniformly subsmooth.

Then, for all $t \in [T_0, T]$ the set-valued map $(u, v) \mapsto \partial d(\cdot, C(t, u, v))(v)$ is upper semicontinuous from $H \times H$ into $H_w$. 
Subsmooth sets and sweeping process

**Proposition**

Assume that the following assumptions holds true:

- $\mathcal{H}^C$ holds.
- The family $\{C(t, u, v)\}\{(t,u,v)\in[T_0,T]\times H \times H\}$ is equi-uniformly subsmooth.

Then, for all $t \in [T_0, T]$ the set-valued map $(u, v) \mapsto \partial d(\cdot, C(t, u, v))(v)$ is upper semicontinuous from $H \times H$ into $H_w$. 
Subsmooth sets and sweeping process

Proposition

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Reduction of sweeping process

To prove existence of the GPSP, we use the reduction technique, i.e.,

\[
\begin{aligned}
-\dot{u}(t) &= Bv(t) & \text{a.e. } t \in [T_0, T]; \\
-\dot{v}(t) &\in N(C(t, u(t), v(t)); v(t)) + F(t, u(t), v(t)) + Au(t) & \text{a.e. } t \in [T_0, T]; \\
u(T_0) &= u_0, v(T_0) = v_0 \in C(T_0, u_0, v_0).
\end{aligned}
\]
Reduction of sweeping process

To prove existence of the GPSP, we use the reduction technique, i.e.,

\[
\begin{align*}
-\dot{u}(t) &= Bv(t) \quad \text{a.e. } t \in [T_0, T]; \\
-\dot{v}(t) &\in m(t, u(t), v(t))\partial d_{C(t,u(t),v(t))}(v(t)) \\
&\quad + \tilde{F}(t, u(t), v(t)) + Au(t) \quad \text{a.e. } t \in [T_0, T]; \\
\end{align*}
\]

\[u(T_0) = u_0, \quad v(T_0) = v_0 \in C(T_0, u_0, v_0),\]

where \(m(t, u, v)\) is a positive function and

\[\tilde{F}(t, u, v) = F(t, u, v) \cap (c(t)\|u, v\| + d(t)) \mathbb{B}.\]
Reduction of sweeping process

By using the inclusion:

$$\partial d_S(x) \subseteq N(S; x) \cap B \quad x \in S.$$  

If we can prove that

$$\nu(t) \in C(t, u(t), \nu(t)) \text{ for all } t \in [T_0, T].$$

Then, any solution of $P_{\text{Red}}$ is a solution of GPSP.
1. Introduction and motivation

2. Position of the problem

3. Basic assumptions

4. An existence result for the GPSP

5. Some consequences

6. Uniqueness

7. References
First main result

Theorem (Jourani-Vilches, 2016 [23])

Assume that the following assumptions hold true:

1. \((\mathcal{H}^F)\) and \((\mathcal{H}^C)\) hold.

2. The family \((C(t,u,v))\) for \((t,u,v) \in [T_0,T] \times H \times H\) is equi-uniformly subsmooth.

Then, there exists at least one solution of the GPSP:

\[
\begin{align*}
-\dot{u}(t) &= Bv(t) \quad a.e. \ t \in [T_0,T]; \\
-\dot{v}(t) &\in N(C(t,u(t),v(t));v(t)) + F(t,u(t),v(t)) + Au(t) \quad a.e. \ t \in [T_0,T]; \\
u(T_0) &= u_0, \ v(T_0) = v_0 \in C(T_0,u_0,v_0),
\end{align*}
\]
First main result

Theorem (Jourani-Vilches, 2016 [23])

Assume that the following assumptions hold true:

1. \((\mathcal{H}^F)\) and \((\mathcal{H}^C)\) hold.
2. the family \((C(t, u, v))_{(t,u,v)\in[T_0,T] \times H \times H}\) is equi-uniformly subsmooth.

Then, there exists at least one solution of the GPSP:

\[
\begin{cases}
-\dot{u}(t) = Bv(t) & a.e. \ t \in [T_0, T]; \\
-\dot{v}(t) \in N(C(t, u(t), v(t)); v(t)) + F(t, u(t), v(t)) + Au(t) & a.e. \ t \in [T_0, T]; \\
u(T_0) = u_0, v(T_0) = v_0 \in C(T_0, u_0, v_0),
\end{cases}
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Then, there exists at least one solution of the GPSP:

\[
\begin{align*}
-\dot{u}(t) &= Bv(t) & a.e. \ t \in [T_0, T]; \\
-\dot{v}(t) &\in N\left(C(t, u(t), v(t)); v(t)\right) + F(t, u(t), v(t)) + Au(t) & a.e. \ t \in [T_0, T]; \\
u(T_0) &= u_0, \ v(T_0) = v_0 \in C(T_0, u_0, v_0),
\end{align*}
\]
Second main result

Theorem (Jourani-Vilches, 2016 [23])

Assume that the following assumptions hold true:

1. \((\mathcal{H}^F)\) and \((\mathcal{H}^C)\) hold.
2. The family \((C(t))\)\(\{t \in [T_0, T]\}\) is positively \(\alpha\)-far.

Then, there exists at least one solution of the GPSP:

\[
\begin{cases}
-\dot{u}(t) = Bv(t) & \text{a.e. } t \in [T_0, T]; \\
-\dot{v}(t) \in N(C(t); v(t)) + F(t, u(t), v(t)) + Au(t) & \text{a.e. } t \in [T_0, T]; \\
u(T_0) = u_0, v(T_0) = v_0 \in C(T_0),
\end{cases}
\]
1. Introduction and motivation
2. Position of the problem
3. Basic assumptions
4. An existence result for the GPSP
5. Some consequences
6. Uniqueness
7. References
Corollary

Assume that the following assumptions hold true:

- $(H^F)$ and $(H^C)$ hold.
- The family $(C(t))_{t \in [T_0, T]}$ is uniformly positively $\alpha$-far.

Then, there exists at least one solution of

$$\begin{cases} 
-\dot{v}(t) \in N(C(t); v(t)) + F(t, v(t)) & \text{a.e. } t \in [T_0, T]; \\
 v(T_0) = v_0 \in C(T_0). 
\end{cases}$$
Corollary

Assume that the following assumptions hold true:

- $(\mathcal{H}^F)$ and $(\mathcal{H}^C)$ hold.
- The family $(C(t))_{t \in [T_0, T]}$ is uniformly positively $\alpha$-far.

Then, there exists at least one solution of

\[
\begin{cases}
-\dot{v}(t) \in N(C(t); v(t)) + F(t, v(t)) & \text{a.e. } t \in [T_0, T]; \\
v(T_0) = v_0 \in C(T_0).
\end{cases}
\]
Corollary

Assume that the following assumptions hold true:

- \((\mathcal{H}^F)\) and \((\mathcal{H}^C)\) hold.
- The family \((C(t))_{t \in [T_0, T]}\) is uniformly positively \(\alpha\)-far.

Then, there exists at least one solution of

\[
\begin{cases}
-\dot{v}(t) \in N(C(t); v(t)) + F(t, v(t)) & \text{a.e. } t \in [T_0, T]; \\
v(T_0) = v_0 \in C(T_0).
\end{cases}
\]
State-dependent sweeping process

Corollary

Assume that the following assumptions hold true:

- $(\mathcal{H}^F)$ and $(\mathcal{H}^C)$ hold.
- The family $\{C(t, v): (t, v) \in [T_0, T] \times H\}$ is equi-uniformly subsmooth.

Then, there exists at least one solution of

$$
\begin{cases}
-\dot{v}(t) \in N(C(t, v(t)); v(t)) + F(t, v(t)) & \text{a.e. } t \in [T_0, T]; \\
\phantom{-}v(T_0) = v_0 \in C(T_0, v_0).
\end{cases}
$$
Corollary

Assume that the following assumptions hold true:

- $(\mathcal{H}^F)$ and $(\mathcal{H}^C)$ hold.
- The family $\{C(t,v) : (t,v) \in [T_0,T] \times H\}$ is equi-uniformly subsmooth.

Then, there exists at least one solution of

\[
\begin{cases}
-\dot{v}(t) \in N(C(t,v(t));v(t)) + F(t,v(t)) & \text{a.e. } t \in [T_0,T]; \\
v(T_0) = v_0 \in C(T_0,v_0).
\end{cases}
\]
Corollary

Assume that the following assumptions hold true:

- $(\mathcal{H}_F)$ and $(\mathcal{H}_C)$ hold.
- The family $\{C(t, v) : (t, v) \in [T_0, T] \times H\}$ is equi-uniformly subsmooth.

Then, there exists at least one solution of

$$
\begin{cases}
-\dot{v}(t) \in N(C(t, v(t)); v(t)) + F(t, v(t)) & a.e. \ t \in [T_0, T]; \\
v(T_0) = v_0 \in C(T_0, v_0).
\end{cases}
$$
Corollary

Assume that the following assumptions hold true:

- $(\mathcal{H}^F)$ and $(\mathcal{H}^C)$ hold.
- The family $\{C(t, u, v) : (t, u, v) \in [T_0, T] \times H \times H\}$ is equi-uniformly subsMOOTH.

Then, there exists at least one solution of

\[
\begin{cases}
-\ddot{u}(t) \in N(C(t, u(t), \dot{u}(t)); \dot{u}(t)) + F(t, u(t), \dot{u}(t)) & \text{a.e. } t \in [T_0, T]; \\
u(T_0) = u_0, \dot{u}(T_0) = v_0 \in C(T_0, u_0, v_0).
\end{cases}
\]
Corollary

Assume that the following assumptions hold true:

1. $(\mathcal{H}^F)$ and $(\mathcal{H}^C)$ hold.
2. The family $\{C(t,u,v) : (t,u,v) \in [T_0,T] \times H \times H\}$ is equi-uniformly subsmooth.

Then, there exists at least one solution of

$$\begin{cases} -\ddot{u}(t) \in N(C(t,u(t),\dot{u}(t));\dot{u}(t)) + F(t,u(t),\dot{u}(t)) & a.e. \ t \in [T_0,T]; \\ u(T_0) = u_0, \dot{u}(T_0) = v_0 \in C(T_0,u_0,v_0). \end{cases}$$
Second-order sweeping process

Corollary

Assume that the following assumptions hold true:

- \((\mathcal{H}^F)\) and \((\mathcal{H}^C)\) hold.
- The family \(\{C(t,u,v): (t,u,v) \in [T_0,T] \times H \times H\}\) is equi-uniformly subsmooth.

Then, there exists at least one solution of

\[
\begin{cases}
-\ddot{u}(t) \in N(C(t,u(t),\dot{u}(t));\dot{u}(t)) + F(t,u(t),\dot{u}(t)) & \text{a.e. } t \in [T_0,T]; \\
u(T_0) = u_0, \dot{u}(T_0) = v_0 \in C(T_0,u_0,v_0).
\end{cases}
\]
1 Introduction and motivation
2 Position of the problem
3 Basic assumptions
4 An existence result for the GPSP
5 Some consequences
6 Uniqueness
7 References
Uniqueness of Moreau’s sweeping process

Let us consider the Moreau’s sweeping process:

\[
\begin{cases}
-\dot{v}(t) \in N(C(t); v(t)) & \text{a.e. } t \in [T_0, T]; \\
v(T_0) = v_0 \in C(T_0),
\end{cases}
\]

It is known that if $C(t)$ is convex for all $t \in [T_0, T]$ then uniqueness hold.
Uniqueness of Moreau’s sweeping process

Consider \( v_1(t) = (-t/2, t/2) \) and \( v_2(t) = (-t/2, -t/2) \) defined over \([0, 1]\). Then \( v_1 \) and \( v_2 \) are solutions of

\[
\begin{cases}
-\dot{v}(t) \in N(C(t); v(t)) & \text{a.e. } t \in [0, 1]; \\
v(0) = (0, 0) \in C(0),
\end{cases}
\]

where \( C(t) = S - (t, 0) \) for \( t \in [0, 1] \).

Figure: \( S = \{(x, y) \in \mathbb{R}^2 : |y| \geq x\} \cap \mathbb{B} \)
Unbounded second-order state-dependent Moreau’s sweeping processes in Hilbert spaces.  

A second order differential inclusion with proximal normal cone in Banach spaces.  

Mixed semicontinuous perturbation of a second order nonconvex sweeping process.  

Existence of solutions for second-order perturbed nonconvex sweeping process.  

Mixed semicontinuous perturbation of nonconvex state-dependent sweeping process.  

Existence of solutions to the nonconvex sweeping process.  

Existence of solutions for second-order differential inclusions involving proximal normal cones.  

General existence results for second order nonconvex sweeping process with unbounded perturbations.  
First and second order convex sweeping processes in reflexive smooth Banach spaces.

Existence results on the second-order nonconvex sweeping processes with perturbations.

State dependent sweeping process in $p$-uniformly smooth and $q$-uniformly convex Banach spaces.

Nonconvex sweeping process and prox-regularity in Hilbert space.

Quelques problèmes d’évolution du second ordre. Exposé 5.

Evolution equations governed by sweeping process.
_Set-Valued Anal., 1:109–139, 1993._

Nonconvex quasi-variational differential inclusions.

The sweeping process without convexity.
_Set-Valued Anal., 7:357–374, 1999._
Relaxation of an optimal control problem involving a perturbed sweeping process.

Bv solutions of nonconvex sweeping process differential inclusion with perturbation.

State-dependent sweeping process with perturbation.

Reduction of sweeping process to unconstrained differential inclusion.

Reduction of state dependent sweeping process to unconstrained differential inclusion.

Moreau-Yosida regularization of state-dependent sweeping processes with nonregular sets.
Submitted, 2016.

Partial galerkin method and generalized perturbed sweeping process with nonregular sets.
Submitted, 2016.

Positively $\alpha$-far sets and existence results for generalized perturbed sweeping processes.
Yosida-Moreau regularization of sweeping processes with unbounded variation.

On parabolic quasi-variational inequalities and state-dependent sweeping processes.

An introduction to Moreau’s sweeping process.

Rafle par un convexe variable I. Exposé 15.

Rafle par un convexe variable II. Exposé 3.

Evolution problem associated with a moving convex set in a Hilbert space.

Numerical aspects of the sweeping process.

Nonconvex sweeping process with a moving set depending on the state.
[33] L. Thibault.
Sweeping process with regular and nonregular sets.

[34] L. Thibault.
Regularization of nonconvex sweeping process in Hilbert space.

Moreau sweeping process with bounded truncated retraction.
On a generalized perturbed sweeping process with nonregular sets

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Thanks!