Nonsmooth Lyapunov pairs for differential equations and perturbed sweeping processes

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Classical setting

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a locally Lipschitz function and consider the differential equation:

$$\dot{x}(t) = f(x(t)) \quad t \geq 0. \tag{1.1}$$

According to the classical Cauchy-Lipschitz theorem, for any initial condition $x(0) = x_0 \in \mathbb{R}^n$, the differential equation (1.1) has a unique solution $x(\cdot; x_0)$ defined on $[0, \tau)$ for some $\tau > 0$.
We further assume that all the solutions of (1.1) are defined on $[0, +\infty)$.

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Some stability notions

Definition (Stability and Attractiveness)

1. We say that the origin is stable for (1.1) if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x_0 \in B_\delta(0)$,
$$x(t, x_0) \in B_\varepsilon(0) \quad \text{for all } t \geq 0.$$

2. We say that the origin is attractive for (1.1) if it is stable and there exists $\delta > 0$ such that for any $x_0 \in B_\delta(0)$,
$$\lim_{t \to +\infty} x(t, x_0) = 0.$$

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The Lyapunov method

Definition

A continuous function $V\colon \mathbb{R}^n \to \mathbb{R}$ is said a a-Lyapunov function, $a \geq 0$, for (1.1) if for every solution of x of (1.1) and for all $t_1, t_2 \geq 0$ the following implication holds:

$$t_1 \leq t_2 \quad \Rightarrow \quad e^{at_2}V(x(t_2)) \leq e^{at_1}V(x(t_1)),$$

that is, the function $t \mapsto e^{at}V(x(t))$ is nonincreasing.

The Lyapunov method

Theorem

Assume that there exists a a-Lyapunov function for (1.1) such that

(a) $V(x) \geq 0$ *for all* $x \in \mathbb{R}^n$.
(b) $V(x) = 0$ *if and only if* $x = 0$.
(c) *The sublevel sets* $\{x \in \mathbb{R}^n : V(x) \leq \lambda\}$ *are bounded for all* $\lambda \in \mathbb{R}$.

1. *If* $a = 0$, *then the origin is stable.*
2. *If* $a > 0$, *then the origin is attractive.*

Proof.

The function $t \mapsto e^{at}V(x(t))$ is nonincreasing, that is, for every $x_0 \in \mathbb{R}^n$

$$e^{at}V(x(t;x_0)) \leq V(x_0) \quad \text{for all } t \geq 0.$$

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The Lyapunov method: The smooth case

Assume that V is a C^1 and let us consider $\varphi(t) := e^{at}V(x(t))$, where $x(\cdot)$ is a solution of (1.1). Then, for all $t \geq 0$

$$\dot{\varphi}(t) = \left(\langle \nabla V(x(t)), f(x(t))\rangle + aV(x(t))\right) e^{at}.$$

Therefore, if

$$\langle \nabla V(x), f(x)\rangle + aV(x) \leq 0 \quad \text{for all } x \in \mathbb{R}^n,$$

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Assume that V is C^1. Then V is a a-Lyapunov function for (1.1) if and only if

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Aim of this talk

The aim of this talk is to give explicit necessary and sufficient conditions for nonsmooth a-Lyapunov function for differential equations and sweeping processes.

Nonsmooth setting

From now on we consider extended-valued lower semicontinuous functions $V\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$.

Extended-valued lsc Lyapunov functions provides much flexibility in constructing Lyapunov functions.

Many operations become possible:

- Truncating a function
- Taking maximum or absolute value of functions.
- Using indicator functions.

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From smooth to nonsmooth analysis

A function $f\colon \mathbb{R}^n \to \mathbb{R}$ if Fréchet differentiable at x if there exists a linear operator $x^* \in \mathcal{L}(\mathbb{R}^n)$ such that

$$\lim_{\|h\|\to 0} \frac{f(x+h) - f(x) - \langle x^*, x\rangle}{\|h\|} = 0.$$

In this case, we denote $f'(x) := x^*$ the derivative of f at x.

The Fréchet subdifferential

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc function. We say that f is Fréchet subdifferentiable at x if there exists a linear operator $x^* \in \mathcal{L}(\mathbb{R}^n)$ such that

$$\liminf_{\|h\|\to 0} \frac{f(x+h) - f(x) - \langle x^*, x\rangle}{\|h\|} \geq 0.$$

We denote the set of all Fréchet-subderivatives x^* of f at x by $\partial_F f(x)$ and call this object the Fréchet subdifferential of f at x.
For convenience we define $\partial_F f(x) = \emptyset$ if $x \notin \operatorname{dom} f$.

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The Fréchet subdifferential: Examples

- $f(x) = |x|$, $\partial_F f(0) = [-1, 1]$.
- $f(x) = -|x|$, $\partial_F f(0) = \emptyset$.
- $f(x) = \sqrt{|x|}$, $\partial_F f(0) = (-\infty, \infty)$.
- $f(x) = \max\{x, 0\}$, $\partial_F f(0) = [0, 1]$.
- $f(x) := I_{[0,1]}(x)$, $\partial_F f(0) = (-\infty, 0]$.

Density theorem

Theorem (Density Theorem)

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a lsc function. Then, the set*

$$\{x \in \mathbb{R}^n : \partial_F f(x) \neq \emptyset\}$$

is dense in $\operatorname{dom} f$.

Distance function and normal cone

Let $S \subseteq \mathbb{R}^n$ be a closed set.

- The distance function to S is defined by $d_S(x) := \inf_{s\in S} \|x - s\|$.
- The closed ball around S is defined by $B_r(S) := \{x \in \mathbb{R}^n : d_S(x) \leq r\}$.
- The Fréchet normal cone of S is defined by

$$N^F(S;x) = \partial_F I_S(x) \quad x \in S.$$

- If $x \in S$, then

$$\partial_F d_S(x) = N^F(S;x) \cap \mathbb{B}.$$

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The Lyapunov method: Analogy with the smooth case

Let $\varphi(t) := e^{at}V(x(t))$.
When V is C^1, to show that φ is nonincreasing, we had used to main ingredients:

- First ingredient: Monotonicity, that is, if $\dot{\varphi} \leq 0$, then φ is nonincreasing.
- Second ingredient: Chain rule.

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First ingredient: Monotonicity

Proposition (Monotonicity lemma)

Let $\varphi\colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ *be a lsc function. Suppose that*

$$\partial_F \varphi(x) \subseteq (-\infty, 0] \quad \textit{for all } x \in \mathbb{R}.$$

Then φ *is nonincreasing.*

Second ingredient: Approximate Chain Rule

Theorem (Approximate Chain Rule)

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and let $F : \mathbb{R}^m \to \mathbb{R}^n$ be a locally Lipschitz mapping.
Suppose that $x^* \in \partial_F (f \circ F)(\bar{x})$.
Then, for any $\varepsilon > 0$, there exist $x \in B_\varepsilon(\bar{x})$, $y \in B_\varepsilon(F(\bar{x}))$, $y^* \in \partial_F f(y)$, $\|\lambda - y^*\| < \varepsilon$ and $z^* \in \partial_F \langle \lambda, F \rangle (x)$ such that $|f(y) - f(F(\bar{x}))| < \varepsilon$,

$$\max\left(\|\lambda\|, \|y^*\|, \|z^*\|\right) \|y - F(x)\| < \varepsilon$$

and $\|x^* - z^*\| < \varepsilon$.

The Lyapunov method: Nonsmooth case

Theorem (Zhu, 2003)

Let $V : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a lsc function. Suppose that*

$$\langle x^*, f(x)\rangle + aV(x) \leq 0 \quad \textit{for all } x^* \in \partial_F V(x)$$

Then V *is a a-Lyapunov function for (1.1).*

Proof.

Apply the Monotonicity lemma and the Approximate Chain Rule to the function $\varphi(t) := f \circ F(t)$, where $f(x, y) = yV(x)$ and $F(t) = (x(t; x_0), e^{at})$, where $x(t; x_0)$ is a solution of (1.1). $\square$

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Application to stability

Definition (Stable and Attractive sets)

Let S be a closed subset of $\mathbb{R}^n$.

1. We say that S is stable for (1.1) if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x_0 \in B_\delta(S)$,

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Let $S \subseteq \mathbb{R}^n$ be a compact set and let $V : \mathbb{R} \to [0, +\infty]$ be a lsc function such that

1. $S \subseteq \{x \in \mathbb{R}^n : V(x) = 0\}$,
2. *for any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $B_\delta(S) \subseteq \{x \in \mathbb{R}^n : V(x) < \varepsilon\}$,
3. *for any* $\varepsilon > 0$ *there exists* $\delta, \eta > 0$ *such that*

$$\{x \in \mathbb{R}^n : V(x) < \delta\} \cap B_{\varepsilon+\eta}(S) \subseteq B_\varepsilon(S).$$

(S is said critical set for V)
Suppose that there exists a constant $a \geq 0$, *for any* $x^* \in \partial_F V(x)$

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Example

Let us consider

$$\dot{x} = f(x) = -x(x+2)(x-1).$$

Define

$$V(x) := \begin{cases} (x+2)^2 & x \leq -1/2, \\ +\infty & x \in (-1/2, 1/2), \\ (x-1)^2 & x \geq 1/2. \end{cases}$$

Then, $S_1 = \{1\}$ and $S_2 = \{-2\}$ are critical sets of V.

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Example (continuation)

- If $x < -1/2$ and $x > 1/2$, then V is differentiable and

$$V'(x) \cdot f(x) = \begin{cases} -2x(x-1)V(x) & x < -1/2, \\ -2x(x+2)V(x) & x > 1/2 \end{cases} \leq -3/2V(x).$$

- If $x = -1/2$, then $\partial_F V(-1/2) = [2(-1/2+2), +\infty)$ and for all $p \in \partial_F V(-1/2)$

$$p \cdot f(-1/2) \leq -3/2V(-1/2).$$

- If $x = 1/2$, then $\partial_F V(1/2) = (-\infty, 2(1/2-1)]$ and for all $p \in \partial_F V(1/2)$

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Example (continuation)

Therefore, for all $x \in \mathbb{R}$ and all $p \in \partial_F V(x)$

$$p \cdot f(x) + \frac{3}{2} V(x) \leq 0.$$

Thus both S_1 and S_2 are attractive, i.e., 1 and -2 are asymptotic equilibrium points.

Perturbed sweeping processes

The sweeping process is the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in -N^F\left(C(t);x(t)\right) + f(x(t)) & \text{a.e. } t \geq 0, \\ x(t) \in C(t) & t \geq 0, \\ x(T_0) = x_0 \in C(0), \end{cases} \tag{3.1}$$

where $C(t) \subseteq \mathbb{R}^n$ are closed set for all $t \geq 0$.

Remark

If $C(t) \equiv \mathbb{R}^n$, then the sweeping process becomes (1.1).

Perturbed sweeping processes

The sweeping process is the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in -N^F\left(C(t); x(t)\right) + f(x(t)) & \text{a.e. } t \geq 0, \\ x(t) \in C(t) & t \geq 0, \\ x(T_0) = x_0 \in C(0), \end{cases} \tag{3.1}$$

where $C(t) \subseteq \mathbb{R}^n$ are closed set for all $t \geq 0$.

Remark

If $C(t) \equiv \mathbb{R}^n$, then the sweeping process becomes (1.1).

Perturbed sweeping processes governed by a fixed set

In this talk, for simplicity, we consider the case $C(t) \equiv S$ for $S \subseteq \mathbb{R}^n$ is a closed and convex set, that is,

$$\begin{cases} \dot{x}(t) \in -N^F(S; x(t)) + f(x(t)) & \text{a.e. } t \geq 0, \\ x(t) \in S & t \geq 0, \\ x(T_0) = x_0 \in S. \end{cases} \qquad (SP)$$

Perturbed sweeping processes governed by a fixed set

Theorem

For all $x_0 \in S$ *there exists a unique solution* $x(\cdot; x_0)$ *of (SP) defined over* $[0, +\infty)$.

Lyapunov pairs

Let $V: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc function and $W: \mathbb{R}^n \to \mathbb{R}$ be Lipschitz continuous. We say that (V, W) forms a Lyapunov pair for the sweeping process (SP) if for every $x_0 \in S$

$$V(x(t;x_0)) + \int_0^t W(x(s;x_0))ds \leq V(x_0) \quad \text{for all } t \geq 0.$$

Characterization of Lyapunov pairs

Theorem (V., 2017)

Assume that $\operatorname{dom} V \subseteq S$ *and* $W \geq 0$. *Then, the following conditions are equivalent:*

1. (V, W) *forms a Lyapunov pair for (SP).*
2. *For all* $x \in \mathbb{R}^n$ *and all* $x^* \in \partial_F V(x)$

$$\inf\{\langle x^*, v\rangle : v \in -\|f(x)\|\partial_F d_S(x) + f(x)\} \leq -W(x).$$

Characterization of Lyapunov pairs

Corollary (V., 2017)

Assume that $W \geq 0$. Then, the following conditions are equivalent:

1. *(V,W) forms a Lyapunov pair for $\dot{x} = f(x)$.*
2. *For all $x \in \mathbb{R}^n$ and all $x^* \in \partial_F V(x)$*

$$\langle x^*, f(x) \rangle \leq -W(x).$$

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Nonsmooth Lyapunov pairs for differential equations and perturbed sweeping processes

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