Moreau-Yosida regularization of sweeping process with nonregular sets

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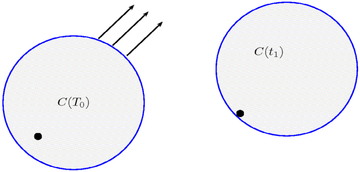
Introduction

Consider a large ring that contains a smaller ball inside, and the ring will start to move at time $t = T_0$.

Depending on the motion of the ring, the ball will just stay where it is (in case it is not hit by the ring), or otherwise it is swept towards the interior of the ring.

In this latter case the velocity of the ball has to point inwards to the ring in order not to leave.



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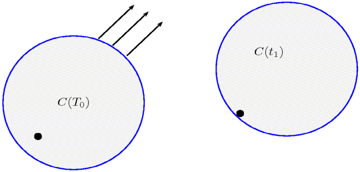

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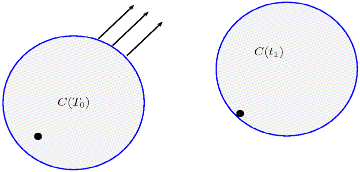

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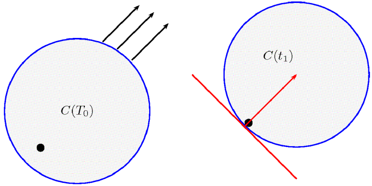

Introduction

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$$\begin{cases} -\dot{x}(t) \in N\left(C(t);x(t)\right) \ \text{ a.e. } t \in [T_0, T]; \\ x(T_0) = x_0 \in C(T_0), \end{cases} \tag{1.1}$$

where

- $x(t)$ is the position of the ball at time t.
- $C(t)$ is the moving set (the ring and its interior).
- $N\left(C(t);x(t)\right)$ is some appropriate outward normal cone of $C(t)$ at $x(t) \in C(t)$.

In the general setting, the set $C(t)$ is allowed to change its shape while moving.

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The sweeping process

The differential inclusion:

$$\begin{cases} -\dot{x}(t) \in N\left(C(t); x(t)\right) \ \text{a.e. } t \in [T_0, T]; \\ x(T_0) = x_0 \in C(T_0), \end{cases} \tag{1.2}$$

is called *Sweeping process* and it appears naturally in many problems from elastoplasticity, contact dynamics, non-regular electrical circuits, granular media, crowd motion, etc.

Tangent and Normal cones

Definition

Let $S \subset H$ be a closed set. For every $x \in S$, we define the Clarke tangent cone of S at x as:

$$T(S;x) = \liminf_{S \ni y \to x, t \downarrow 0} \frac{S-y}{t}.$$

This cone is closed and convex and its negative polar $N(S;x)$ is the Clarke normal cone to S at $x \in S$, that is,

$$N(S;x) = \{v \in H \colon \langle v, h \rangle \leq 0 \, \forall h \in T(S;x)\}.$$

As usual we set $N(S;x) = \emptyset$ if $x \notin S$.
Also, $N(S;x) = \{0\}$ if $x \in \operatorname{int} S$.

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Clarke subdifferential

The Clarke subdifferential of a lower semicontinuous function $f\colon H \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\partial f(x) = \{v \in H\colon (v,-1) \in N\,(\operatorname{epi} f, (x, f(x)))\},$$

where $\operatorname{epi} f = \{(y,r) \in H \times \mathbb{R}\colon f(y) \leq r\}$ is the epigraph of f.

Some properties of the Clarke subdifferential

Let $f : H \to \mathbb{R}$ be Lipschitz near x.

1. If f is L-Lipschitz near x then $\partial f(x) \subset L\mathbb{B}$.
2. If f admits a Gâteaux derivative $f'_G(x)$ at x, then $f'_G(x) \in \partial f(x)$.
3. If f is continuously differentiable at x, then $\partial f(x) = \{f'(x)\}$.

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Normal cone and distance function

Let $S \subset H$ be a closed set. We define the indicator function of S as:

$$I_S(x) := \begin{cases} 0 & \text{if } x \in S; \\ +\infty & \text{if } x \notin S, \end{cases}$$

and the distance function:

$$d_S(x) := \inf_{y \in S} \|x - y\|.$$

We denote $\mathrm{Proj}_S(x)$ the set (possibly empty):

$$\mathrm{Proj}_S(x) = \{s \in S \colon \|x - s\| = d_S(x)\}.$$

Normal cone and distance function

- If $x \in S$.

$$\partial I_S(x) = N(S;x) \quad \forall x \in S.$$

- If $x \in S$.

$$N(S;x) = \overline{\bigcup_{\lambda \geq 0} \lambda \partial d_S(x)}.$$

In particular, for every $x \in S$ $\partial d_S(x) \subset N(S;x)$.

- If S is ball-compact.

$$\frac{1}{2}\partial d_S^2(x) = x - \overline{\text{co}}\,\text{Proj}_S(x) \quad \forall x \in H.$$

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The sweeping process

Definition

We say that $x\colon [T_0,T]\to H$ is a solution of the sweeping process if it absolutely continuous and satisfies:

$$\begin{cases} -\dot{x}(t)\in N\left(C(t);x(t)\right) & \text{a.e. } t\in[T_0,T];\\ x(T_0)=x_0\in C(T_0), \end{cases}$$

where $N\left(C(t);x(t)\right)$ denotes the Clarke normal cone to $C(t)$ at $x(t)\in C(t)$.

The Hausdorff distance

Let $C, D \subset H$ closed sets. The Hausdorff distance between C and D is defined by:

$$\begin{aligned} \operatorname{Haus}(C,D) &= \max\{\sup_{x\in C} d(x,D), \sup_{x\in D} d(x,C)\} \\ &= \sup_{x\in H} |d_C(x) - d_D(x)| \\ &= \inf\{\eta \geq 0 \colon C \subset D + \eta\mathbb{B}, D \subset C + \eta\mathbb{B}\}. \end{aligned}$$

What about the variation of the moving set?

Let $C\colon [T_0, T] \rightrightarrows H$ be a set-valued map. We say that C is κ-Lipschitz if

$$\operatorname{Haus}(C(t), C(s)) \leq \kappa|t-s| \quad \text{for all } t, s \in [T_0, T]. \tag{3.1}$$

In particular, if $C(t) = S + v(t)$, where v is κ-Lipschitz, (3.1) holds.

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What about the shape of the moving set?

Definition

Let $\alpha \in]0,1]$. $S \subset H$ is positively α-far if there exists $\rho > 0$ such that if $x \in U_\rho(S)$

$$\text{if } \zeta \in \partial d_S(x) \text{ then } \|\zeta\| \geq \alpha,$$

where $U_\rho(S) := \{x \in H \colon 0 < d(x,S) < \rho\}$ is the ρ-tube around S.

If S is ball-compact then S is positively α-far if there exists $\rho > 0$ such that

$$0 < \alpha \leq \frac{d(x, \overline{\text{co}}\,\text{Proj}_S(x))}{d_S(x)} \quad \forall x \in U_\rho(S).$$

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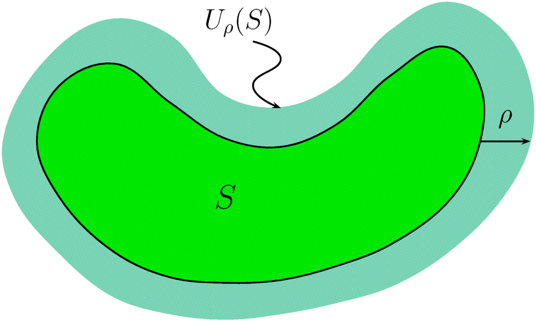

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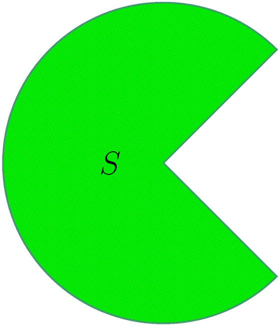

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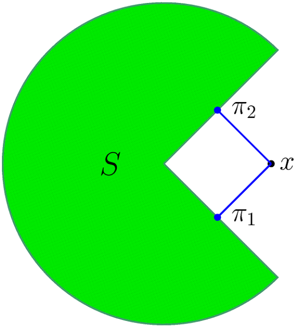

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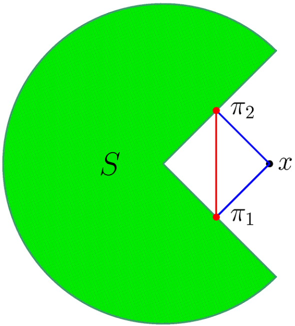

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Positively α-far sets

- If S is convex then S is 1-far (with $\rho = +\infty$).
- If S is ρ-uniformly prox-regular then S is 1-far (with the same ρ).
- The class of positively α-far set is very general and includes several classes of sets: convex sets, paraconvex sets, uniformly prox-regular sets, uniformly subsmooth sets, etc.

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Aim

Our aim is to prove the existence of solutions to the sweeping process:

$$\begin{cases} -\dot{x}(t) \in \partial I_{C(t)}(x(t)) & \text{a.e. } t \in [T_0, T]; \\ x(T_0) = x_0 \in C(T_0), \end{cases}$$

when $C\colon [T_0, T] \rightrightarrows H$ is κ-Lipschitz continuous with nonempty, closed and positively α-far values.

Moreau-Yosida envelope

Given $f\colon H \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous function bounded from below. For $\lambda > 0$, the Moreau-Yosida envelope of f is defined as:

$$f_\lambda(x) = \inf_{y\in H}\left(f(y) + \frac{1}{2\lambda}\|x-y\|^2\right).$$

The main property of the Moreau-Yosida envelope is that:

1. f_λ is locally-Lipschitz.
2. $f_\lambda(x) \nearrow f(x)$ as $\lambda \searrow 0$ for all $x \in H$.
3. If f is convex then f_λ is $C^{1,1}$.

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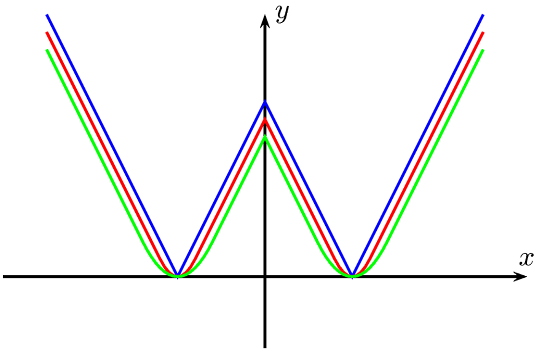
Moreau-Yosida regularization

- $h = 0.1$
- $h = 0.2$



Properties of Moreau-Yosida envelope

Let $S \subset H$ be a closed set. Then,

$$(I_S)_\lambda(x) = \frac{1}{2\lambda} d_S^2(x) \quad \forall x \in H.$$

Also, if S is ball-compact.

$$\partial(I_S)_\lambda(x) = \frac{1}{\lambda}(x - \overline{\text{co}}\ \text{Proj}_S(x)) \quad \forall x \in H.$$

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Therefore, we will approach the sweeping process through the following penalized differential inclusion:

$$\begin{cases} -\dot{x}_\lambda(t) \in \dfrac{1}{2\lambda}\partial d^2_{C(t)}(x(t)) & \text{a.e. } t \in [T_0, T]; \\ x_\lambda(T_0) = x_0 \in C(T_0). \end{cases} \tag{$\mathcal{P}$}$$

The existence of $(\mathcal{P})$ can be obtained through a theorem of existence for differential inclusions with compact values due to Bothe [1]. This was never noticed before!

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Existence result

Theorem (Jourani-Vilches, 2016)

Assume that:

1. $\mathrm{Haus}(C(t), C(s)) \leq \kappa|t - s|$ *for all* $t, s \in [T_0, T]$.
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Then, there exists at least one solution of

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1. $\mathrm{Haus}(C(t), C(s)) \leq \kappa|t-s|$ *for all* $t, s \in [T_0, T]$.
2. $C(t)$ *is positively* α*-far for all* $t \in [T_0, T]$.
3. $C(t)$ *is ball-compact for all* $t \in [T_0, T]$.

Then, there exists at least one solution of

$$\begin{cases} -\dot{x}(t) \in N\left(C(t); x(t)\right) & a.e.\ t \in [T_0, T]; \\ x(T_0) = x_0 \in C(T_0). \end{cases}$$

Sketch of proof

The proof can be divided in several steps.

(i) Step 1:

$$d_{C(t)}(x_\lambda(t)) \leq \frac{\kappa}{\alpha^2}\lambda \quad \forall t \in [T_0, T].$$

(ii) Step 2: $x_\lambda : [T_0, T] \to H$ is $\frac{\kappa}{\alpha^2}$-Lipschitz continuous.

$$\|\dot{x}_\lambda(t)\| \leq \frac{\kappa}{\alpha^2} \quad \text{a.e. } t \in [T_0, T].$$

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(iii) Step 3: There exists a subsequence $(\lambda_k)_k$ of $(\lambda)_{\lambda>0}$ and $x \in \mathrm{AC}\left([T_0,T];H\right)$ such that

$$\begin{aligned} x_{\lambda_k}(t) &\rightharpoonup x(t) && \forall t \in [T_0,T]; \\ x_{\lambda_k} &\rightharpoonup x && \text{in } L^1_w\left([T_0,T];H\right); \\ \dot{x}_{\lambda_k} &\rightharpoonup \dot{x} && \text{in } L^1_w\left([T_0,T];H\right). \end{aligned}$$

(iv) Step 4: $x_{\lambda_k}(t) \to x(t)$ and $x(t) \in C(t)$ for all $t \in [T_0,T]$.

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Sketch of proof

(v) Step 5: For a.e. $t \in [T_0, T]$

$$-\dot{x}_{\lambda_k}(t) \subseteq \frac{\kappa}{\alpha^2}\overline{\text{co}}\left\{\partial d_{C(t)}(x_{\lambda_k}(t)) \cup \{0\}\right\}.$$

(vi) Step 6: Pass to the limit:

$$-\dot{x}(t) \in \frac{\kappa}{\alpha^2}\partial d_{C(t)}(x(t)) \quad \text{a.e. } t \in [T_0, T].$$

This completes the proof because $\partial d_{C(t)}(x(t)) \subset N\left(C(t); x(t)\right)$.

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Further results

The same proof applies, mutatis mutandis, to the state-depedent sweeping process in the sense of measure differential inclusion.

Theorem (Jourani-Vilches, 2016)

Assume that

(i) *There exists* $v \in \mathrm{CBV}([T_0, T]; \mathbb{R})$ *and* $L \in [0, 1[$ *such that for all* $s, t \in [T_0, T]$ *and all* $x, y \in H$

$$\mathrm{Haus}(C(t,x), C(s,x)) \leq |v(t) - v(s)| + L\|x - y\|.$$

(ii) *The family* $\{C(t,v) \colon (t,v) \in [T_0, T] \times H\}$ *is equi-uniformly subsmooth.*

Further results

Theorem (Jourani-Vilches, 2016)

(iii) *There exists* $k \in L^1(T_0, T)$ *such that for every* $t \in [T_0, T]$, *every* $r > 0$ *and every bounded set* $A \subset H$

$$\gamma(C(t,A) \cap r\mathbb{B}) \leq k(t)\gamma(A),$$

where $\gamma = \alpha$ *or* $\gamma = \beta$ *is either the Kuratowski or the Hausdorff measure of non-compactness and* $k(t) < 1$ *for all* $t \in [T_0, T]$.

Then, there exists at least one solution $x \in \mathrm{CBV}\left([T_0, T]; H\right)$ *of*

$$\begin{cases} -dx \in N\left(C(t,x(t)); x(t)\right); \\ x(T_0) = x_0 \in C(T_0, x_0). \end{cases}$$

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Moreau-Yosida regularization of sweeping process with nonregular sets

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Thanks!