Some variational problems in the KPZ universality class

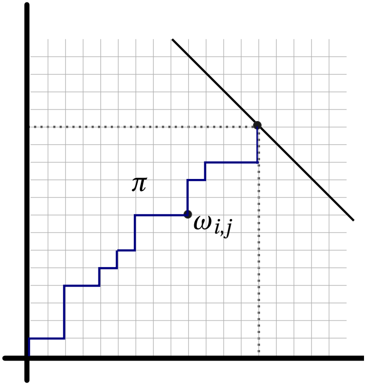
Daniel Remenik

Universidad de Chile

July 28, 2014

Stochastic Processes and Their Applications

Last passage percolation and Airy$_2$



i.i.d. geometric weights $\omega_{i,j}$, $i, j \in \mathbb{Z}^+$.

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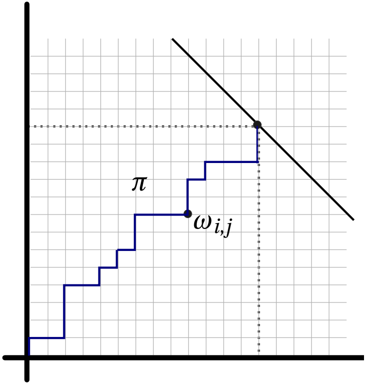
Last passage percolation and Airy_2



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[Johansson '00]

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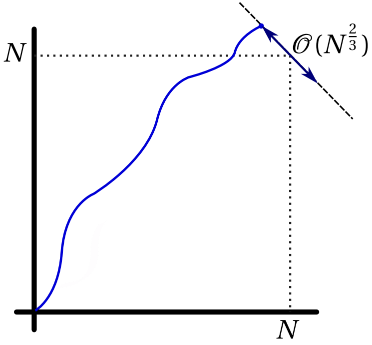

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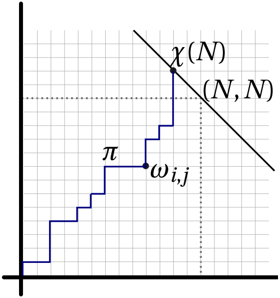
Spatial fluctuations: Let

$$H_N(u) = \frac{G(N + c_3 N^{2/3} u, N - c_3 N^{2/3} u) - c_1 N}{c_2 N^{\frac{1}{3}}}.$$

Then $H_N(u) \xrightarrow[N\to\infty]{} \mathcal{A}_2(u) - u^2$ with $\mathcal{A}_2$ the *Airy_2 process*.

[Prähofer-Spohn '01, Johansson '03]

Point-to-line last passage percolation



Now choose among all paths π of length $2N$

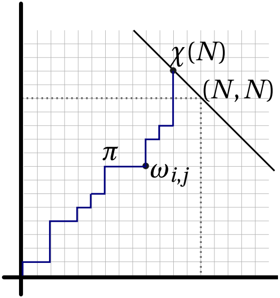
$\chi(N)$: endpoint of the maximizing path

$$G^{\text{line}}(N) = \max_{|u|\le N} G^{\text{pt}}(N-u, N+u).$$

In particular

$$\frac{G^{\text{line}}(N) - c_1 N}{c_2 N^{1/3}} = \max_{u \in c_3^{-1} N^{-2/3}\mathbb{Z}, |u| \le c_3^{-1} N^{1/3}} H_N(u)$$

Point-to-line last passage percolation



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$\approx \tilde{\zeta}_{\text{GOE}}$ [Baik-Rains '00]

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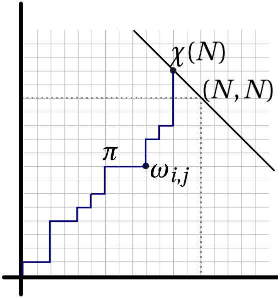

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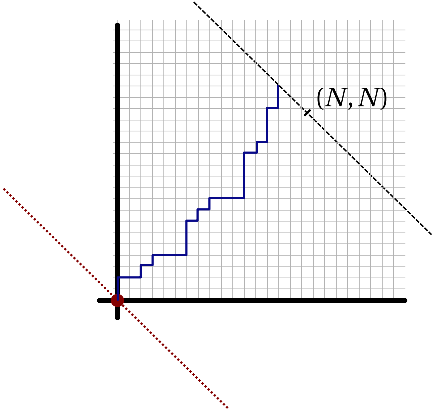
$$\approx \tilde{\zeta}_{\text{GOE}}$$ [Baik-Rains '00]

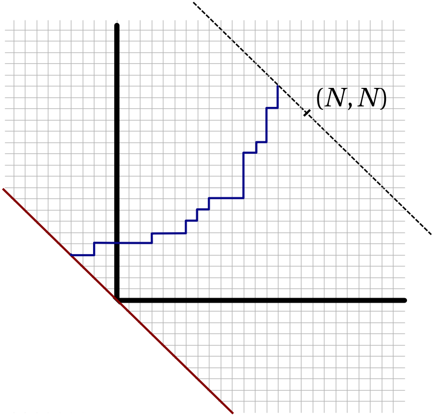
and therefore

$$\sup_{u \in \mathbb{R}} \left\{ \mathscr{A}_2(u) - u^2 \right\} \stackrel{d}{=} \tilde{\zeta}_{\text{GOE}}$$

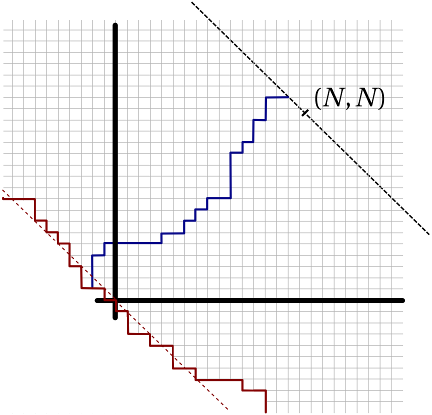
[Johansson '03]



Point-to-point, $g_N(x) = \mathbf{1}_{x=0}$



Point-to-line, $g_N \equiv 1$



General case, $g_N \xrightarrow[N\to\infty]{} g$ under diffusive scaling

One expects that the limiting fluctuations of the last passage time are distributed in this case as

$$\sup_{u\in\mathbb{R}} \left\{ \mathscr{A}_2(u) - u^2 + g(u) \right\}$$

Continuum statistics of the Airy$_2$ process

$$\mathbb{P}\Big(\mathscr{A}_2(t_0) \le x_0, \ldots, \mathscr{A}_2(t_n) \le x_n\Big) = \det\Big(I - P_{\mathbf{x}} K^{\text{ext}}_{\text{Ai}} P_{\mathbf{x}}\Big)_{L^2(\{t_0,\ldots,t_n\}\times\mathbb{R})}$$

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$$= \det(I - K_{\text{Ai}} + \bar{P}_{x_0} e^{(t_0-t_1)H} \bar{P}_{x_1} e^{(t_1-t_2)H} \cdots \bar{P}_{x_n} e^{(t_n-t_0)H} K_{\text{Ai}})_{L^2(\mathbb{R})}$$

[Prähofer and Spohn '02]

$H = -\partial_x^2 + x$ Airy Hamiltonian, K_{Ai} Airy kernel, $\bar{P}_a f(x) = \mathbf{1}_{x \le a} f(x)$

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Theorem (Corwin-Quastel-R ’11)

For $g \in H^1([\ell, r])$,

$$\mathbb{P}\Big(\mathcal{A}_2(t) \le g(t) + t^2 \text{ for } t \in [\ell, r]\Big) = \det\Big(I - K_{\mathrm{Ai}} + \Theta^g_{[\ell,r]} e^{(r-\ell)H} K_{\mathrm{Ai}}\Big),$$

where $\Theta^g_{[\ell,r]} f(x) = u(r,x)$ is the solution operator at time r of the b.v.p.

$$\begin{cases} \partial_t u + Hu = 0 & x < g(t) + t^2 \\ u(t,x) = 0 & x \ge g(t) + t^2 \end{cases} \quad \text{with } u(\ell, x) = f(x)$$

Continuum statistics of the Airy$_2$ process

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Application 1: direct proof of $\sup_{u\in\mathbb{R}} \{\mathcal{A}_2(u) - u^2\} \stackrel{d}{=} \tilde{\zeta}_{\mathrm{GOE}}$.

Continuum statistics of the Airy$_2$ process

$$\mathbb{P}\Big(\mathscr{A}_2(t_0) \le x_0, \dots, \mathscr{A}_2(t_n) \le x_n\Big) = \det\Big(I - P_{\mathbf{x}} K_{\mathrm{Ai}}^{\mathrm{ext}} P_{\mathbf{x}}\Big)_{L^2(\{t_0,\dots,t_n\}\times\mathbb{R})}$$

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Application 2: endpoint distribution, $\mathscr{T} = \lim_{N\to\infty} N^{-2/3} \chi(N)$.

[Moreno-Quastel-R '11]

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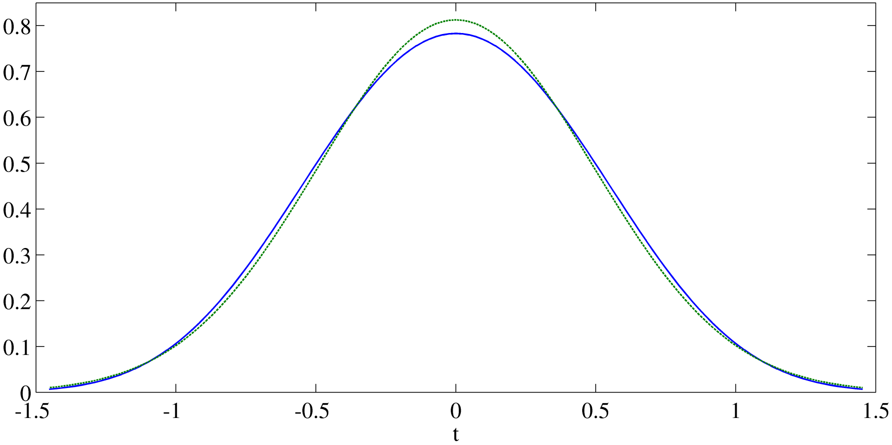
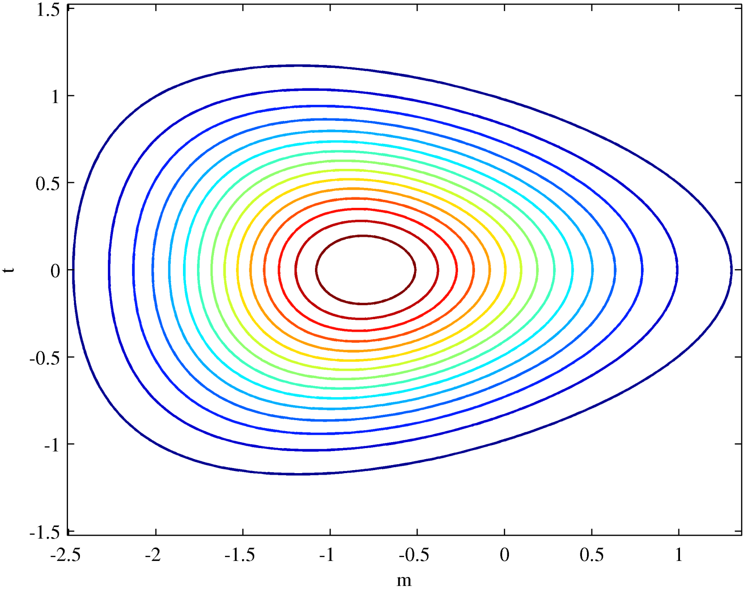
Application 3: numerical study of persistence probabilities

$$\mathbb{P}\Big(\mathscr{A}_2(t) \le m_2,\ t \in [0, L]\Big)$$

[Ferrari-Frings '13]



The polymer endpoint distribution compared to a Gaussian with the same variance. The tails decay like $e^{-\frac{4}{3}x^3}$.



Contour plot of the joint density of $\mathscr{M}$ and $\mathscr{T}$

Similar formulas hold for many other processes

General theorem proved in [Borodin-Corwin-R '13].

I will just focus on some examples.

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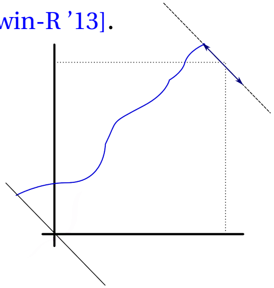

Airy$_1$ process [Quastel-R' 13]
governs the spatial fluctuations of last passage times with flat initial condition.

For $g \in H^1([\ell, r])$,

$$\mathbb{P}\big(\mathscr{A}_1(t) \le g(t) \text{ for } t \in [\ell, r]\big) = \det\big(I - B_0 + \Theta^g_{[\ell,r]} e^{(\ell - r)\Delta} B_0\big),$$

where $\Theta^g_{[\ell,r]} f(x) = u(r,x)$ is the solution operator at time r of the b.v.p.

$$\begin{cases} \partial_t u + \Delta u = 0 & x < g(t) \\ u(t,x) = 0 & x \ge g(t) \end{cases} \quad \text{with } u(\ell, x) = f(x)$$

with $B_0(x,y) = \mathrm{Ai}(x+y)$, $\quad e^{a\Delta} B_0(x,y) = e^{\frac{2}{3}a^3 + (x+y)a} \mathrm{Ai}(x+y+a^2)$ $\forall a \in \mathbb{R}$.

Applications:
- The paths of the Airy$_1$ process are Hölder $\frac{1}{2}-$.
- The Airy$_1$ process is "locally Brownian".
- Numerical study of Airy$_1$ persistance probabilities.

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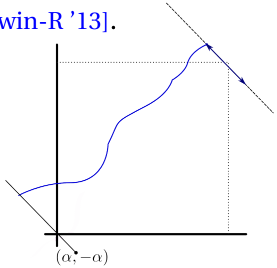

Airy$_{2\to1}$ process

governs the spatial fluctuations of last passage times with half-flat initial condition.

For $g \in H^1([\ell, r])$ (and $\alpha = 0$),

$$\mathbb{P}\big(\mathcal{A}_{2\to1}(t) \le g(t) + t^2 \mathbf{1}_{t\ge0} \text{ for } t \in [\ell, r]\big) = \det\big(I - K_{2\to1} + \Theta^g_{[\ell,r]} e^{(\ell-r)\Delta} K_{2\to1}\big),$$

where $\Theta^g_{[\ell,r]} f(x) = u(r, x)$ is the solution operator at time r of the b.v.p.

$$\begin{cases} \partial_t u + \Delta u = 0 & x < g(t) + t^2 \mathbf{1}_{t\ge0} \\ u(t,x) = 0 & x \ge g(t) + t^2 \mathbf{1}_{t\ge0} \end{cases} \quad \text{with } u(\ell, x) = f(x)$$

with $\quad K_{2\to1}(x_1, x_2) = K_{\mathrm{Ai}}(x_1, x_2) + \int_0^\infty d\lambda\, \mathrm{Ai}(x_1 - \lambda)\, \mathrm{Ai}(x_2 + \lambda).$

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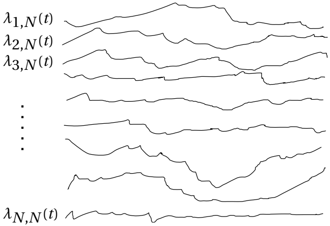

Stationary GUE Dyson B.M.

evolution of the largest eigenvalue of a GUE random matrix with each entry undergoing independent Ornstein-Uhlenbeck diffusions

For $g \in H^1([\ell, r])$,

$$\mathbb{P}\big(\lambda_{1,N}(t) \leq g(t) \text{ for } t \in [\ell, r]\big) = \det\big(I - K^N_{\text{Herm}} + \Theta^g_{[\ell,r]} e^{(r-\ell)D} K^N_{\text{Herm}}\big),$$

where $\Theta^g_{[\ell,r]} f(x) = u(r,x)$ is the solution operator at time r of the b.v.p.

$$\begin{cases} \partial_t u + Du = 0 & x < g(t) \\ u(t,x) = 0 & x \geq g(t) \end{cases} \quad \text{with } u(\ell, x) = f(x)$$

with $D = -\frac{1}{2}(\partial_x^2 - x^2 + 1)$ and $K^N_{\text{Herm}} = \sum_{k=0}^{N-1} \varphi_k(x)\varphi_k(y)$, where $\varphi_k(x) = e^{-x^2/2} p_k(x)$ and p_k is the k-th Hermite polynomial.

Back to the Airy$_2$ process

Question: Can we get a formula for $\mathbb{P}\big(\mathscr{A}_2(t) \le g(t) + t^2\ \forall\, t \in \mathbb{R}\big)$?

We have

$$\mathbb{P}\big(\mathscr{A}_2(t) \le g(t) + t^2 \text{ for } t \in [-L, L]\big) = \det\big(I - K_{\mathrm{Ai}} + e^{LH} K_{\mathrm{Ai}} \Theta^g_L e^{LH} K_{\mathrm{Ai}}\big),$$

Using Feynman-Kac to find Θ^g_L yields

$$\Theta^g_L(x_1, x_2) = e^{\frac{2}{3}L^3 - (x_1 + x_2)L} p(2L, x_1 - x_2) \\ \times \mathbb{P}_{-L, x_1 - L^2; L, x_2 - L^2}\big(B(t) \le g(t) \text{ on } [-L, L]\big).$$

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Using Feynman-Kac to find Θ_L^g yields

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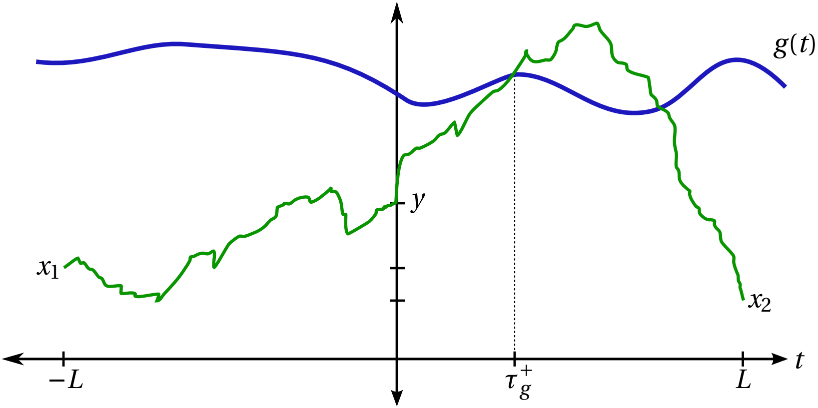
Then one proves that

$$e^{-\ell_1 H} K_{\mathrm{Ai}} \Theta_L^g e^{\ell_2 H} K_{\mathrm{Ai}}(x_1, x_2) = K_{\mathrm{Ai}} e^{\ell_1 \Delta} \widetilde{\Theta}_L^g e^{-\ell_2 \Delta} K_{\mathrm{Ai}}$$

(this makes sense because $K_{\mathrm{Ai}} = B_0 P_0 B_0$ and $e^{-s\Delta} B_0$ is ok)

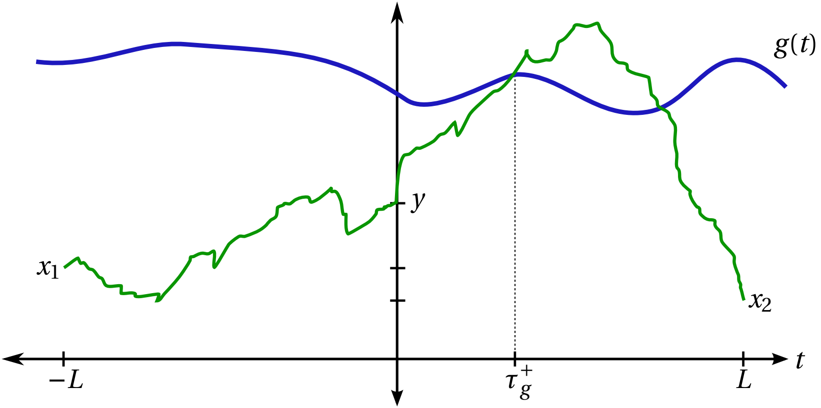
with

$$\widetilde{\Theta}_L^g(x_1, x_2) = p(2L, x_1 - x_2) \mathbb{P}_{-L, x_1; L, x_2}\big(B(t) \le g(t) \text{ on } [-L, L]\big) \\ = \mathbb{P}_{-L, x_1}\big(B(t) \le g(t) \text{ on } [-L, L],\; B(L) \in dx_2\big) / dx_2.$$



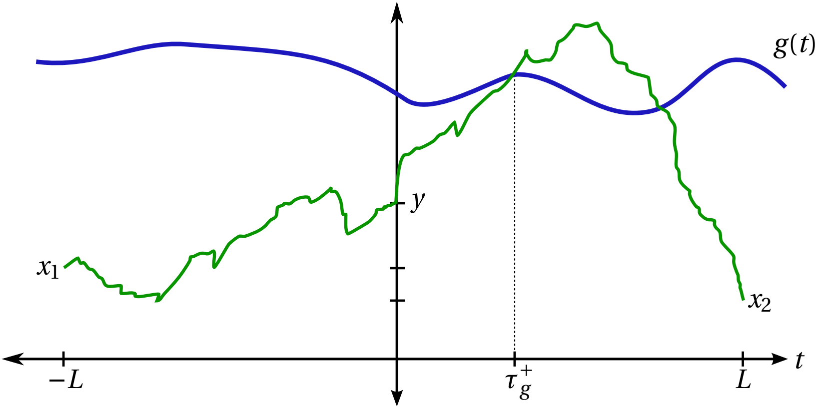
$$\widetilde{\Theta}_L^g(x_1,x_2) = \int_{-\infty}^{g(0)} \mathbb{P}_{-L,x_1;L,x_2}\big(B(0)\in dy\big)\mathbb{P}_{-L,x_1;0,y}\big(B(t)<g(t) \text{ on } [-L,0]\big)$$

$$\times\, \mathbb{P}_{0,y;L,x_2}\big(B(t)<g(t) \text{ on } [0,L]\big).$$



$$
\widetilde{\Theta}^{g}_{L}(x_1,x_2) = \int_{-\infty}^{g(0)} dy \Big(p(L, x_1 - y) - \int_0^L dt_1\, \mathbb{P}_y(\tau_g^- \in dt_1)\, p(L - t_1, x_1 - g(-t_1)) \Big)
$$

$$
\times \Big(p(L, x_2 - y) - \int_0^L dt_2\, \mathbb{P}_y(\tau_g^+ \in dt_2)\, p(L - t_2, x_2 - g(t_2)) \Big).
$$



$$e^{-L\Delta}\widetilde{\Theta}_L^g e^{-L\Delta}(x_1,x_2) = \int_{-\infty}^{g(0)} dy \Big(\delta_{x_1-y} - \int_0^L dt_1\, \mathbb{P}_y(\tau_g^- \in dt_1) e^{-t_1\Delta}(g(-t_1),x_1)\Big)$$

$$\times \Big(\delta_{x_2-y} - \int_0^L dt_2\, \mathbb{P}_y(\tau_g^+ \in dt_2) e^{-t_2\Delta}(g(t_2),x_2)\Big).$$

Theorem (Quastel-R '14)

For $g \in H^1_{\mathrm{loc}}(\mathbb{R})$ such that $g(t) \geq c - \gamma t^2$ for some $c \in \mathbb{R}$, $\gamma \in (0, 3/4)$,

$$K_{\mathrm{Ai}} - K_{\mathrm{Ai}} e^{-L\Delta} \widetilde{\Theta}^g_L e^{-L\Delta} K_{\mathrm{Ai}} \xrightarrow[L\to\infty]{} K_{\mathrm{Ai}} - K_{\mathrm{Ai}} \Omega^g_\infty K_{\mathrm{Ai}}$$

in trace class norm, where

$$\Omega^g_\infty(x_1, x_2) = \int_{-\infty}^{g(0)} dy \left[\delta_{x_1 - y} - \int_0^\infty dt_1 \, \mathbb{P}_y(\tau^-_g \in dt_1) e^{-t_1\Delta}(g(-t_1), x_1) \right] \times \left[\delta_{x_2 - y} - \int_0^\infty dt_2 \, \mathbb{P}_y(\tau^+_g \in dt_2) e^{-t_2\Delta}(g(t_2), x_2) \right].$$

As a consequence, for such g we have

$$\mathbb{P}\big(\mathcal{A}_2(t) \leq g(t) + t^2 \ \forall\, t \in \mathbb{R}\big) = \det\big(I - K_{\mathrm{Ai}} + K_{\mathrm{Ai}} \Omega^g_\infty K_{\mathrm{Ai}}\big).$$

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Example 1: If $g(0) = 0$, $g(t) = \infty$ for $t \neq 0$, then $\Omega^{g+m}_\infty = \bar{P}_m$ so that

$$\det\Big(I - K_{\mathrm{Ai}} + K_{\mathrm{Ai}} \Omega^{g+m}_\infty K_{\mathrm{Ai}}\Big) = \det(I - K_{\mathrm{Ai}} P_m K_{\mathrm{Ai}}) = F_{\mathrm{GUE}}(m).$$

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As a consequence, for such g we have

$$\mathbb{P}\Big(\mathscr{A}_2(t) \leq g(t) + t^2 + m \; \forall \, t \in \mathbb{R}\Big) = \det\Big(I - K_{\mathrm{Ai}} + K_{\mathrm{Ai}} \Omega^{g+m}_\infty K_{\mathrm{Ai}}\Big).$$

Example 2: If $g \equiv 0$, then $\Omega^{g+m}_\infty f(x) = f(x) - f(2m - x)$ so that

$$\det\Big(I - K_{\mathrm{Ai}} + K_{\mathrm{Ai}} \Omega^{g+m}_\infty K_{\mathrm{Ai}}\Big) = \det\big(I - K_{\mathrm{Ai}}(I - \varrho_m) K_{\mathrm{Ai}}\big) = F_{\mathrm{GOE}}(4^{1/3} m).$$

Example 3: square-root boundary,

$$g(t) = b_1\sqrt{-t}\mathbf{1}_{t<0} + b_2\sqrt{t}\mathbf{1}_{t\geq 0}.$$

Here $\mathbb{P}_y(\tau_g^+ \in dt)$ is explicit: for $g(t) = b\sqrt{t}$ $(t \geq 0)$, this measure has density

$$\rho_{y,b}(t) = \sum_{n=1}^{\infty} \frac{2^{\nu_n(b)+1}(2t)^{\nu_n(b)-1}(y^2/2)^{-\nu_n(b)}}{\partial_\nu \Psi(\nu_n(b), \frac{1}{2}, \frac{b^2}{4})},$$

with Ψ the Tricomi confluent hypergeometric function

$$\Psi(\tfrac{p}{2}, \tfrac{1}{2}, \tfrac{z^2}{2}) = \frac{\sqrt{\pi}}{\Gamma((1+p)/2)}\,{}_1F_1(\tfrac{p}{2}, \tfrac{1}{2}, \tfrac{z^2}{2}) - \frac{\sqrt{2\pi}}{\Gamma(p/2)}\,{}_1F_1(\tfrac{1+p}{2}, \tfrac{3}{2}, \tfrac{z^2}{2})$$

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With this we get

$$\mathbb{P}\Big(\mathcal{A}_2(t) \leq b_1\sqrt{-t}\,\mathbf{1}_{t<0} + b_2\sqrt{t}\,\mathbf{1}_{t\geq 0} + t^2 + m \;\forall\, t \in \mathbb{R}\Big) = \det(I - P_0(K_{\mathrm{Ai}} - \Gamma_m)P_0)$$

with

$$\begin{aligned}\Gamma_m(x_1, x_2) = &\int_{-\infty}^{0} dy \int_0^\infty dt_1 \int_0^\infty dt_2\, \rho_{y,b_1}(t_1)\rho_{y,b_2}(t_2) e^{-2t_1^3/3 - t(x_1 + b_1\sqrt{t_1})} \\ &\times \mathrm{Ai}(x_1 + b_1\sqrt{t_1} + t_1^2)\,\mathrm{Ai}(x_2 + b_2\sqrt{t_2} + t_2^2) \\ &- \int_{-\infty}^{0} dy \int_0^\infty dt\, \mathrm{Ai}(x_1 + y + m)\rho_{y,b_2}(t) e^{-2t^3/3 - t(x_2 + b_2\sqrt{t})}\,\mathrm{Ai}(x_2 + b_2\sqrt{t} + t^2) \\ &- \int_{-\infty}^{0} dy \int_0^\infty dt\, \rho_{y,b_1}(t) e^{-2t^3/3 - t(x_1 + b_1\sqrt{t})}\,\mathrm{Ai}(x_1 + b_1\sqrt{t} + t^2)\,\mathrm{Ai}(x_2 + y + m).\end{aligned}$$