Non-interesecting Browian bridges and the Laguerre Orthogonal Ensemble

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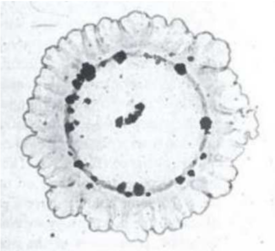
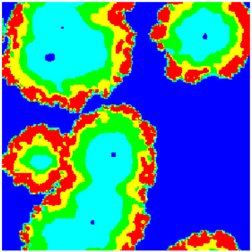
Joint work with Gia Bao Nguyen

The KPZ universality class

- Broad collection of models, including: interface growth models, directed polymers, interacting particle systems, reaction-diffusion models, randomly forced Hamilton-Jacobi equations.
- Main feature: $t^{1/3}$ scale of fluctuations, decorrelating at a $t^{2/3}$ spatial scale.
- Three special classes of initial data (scale invariance): curved, flat and stationary. Exact computations show that limiting fluctuations are related to random matrix theory (RMT).





KPZ $\longleftrightarrow$ RMT ? The *curved* case

- Very well-understood.

- Limiting fluct. described by the Tracy-Widom F_{GUE} distr.:

  Let A be a matrix from the Gaussian Unitary Ensemble:
  A is an Hermitian $N \times N$ matrix with

  $$A_{ij} = \mathcal{N}(0,1/4) + \mathrm{i}\,\mathcal{N}(0,1/4) \;\text{ for } i > j, \;\; A_{ii} = \mathcal{N}(0,1/2)$$

  and let $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ be its eigenvalues. Then

  $$F_{\text{GUE}}(r) = \lim_{N\to\infty} \mathbb{P}\big(\lambda_N \leq 4\sqrt{N} + 2N^{-1/6} r\big).$$

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- Simplest version of the curved/GUE connection (next slide):

  non-intersecting B.M. $\longleftrightarrow$ Dyson B.M.

- Other (deep) connections are available for many models:
  integrable probability (Macdonald processes, RSK, quantum integrable systems...).

Non-intersecting Brownian bridges

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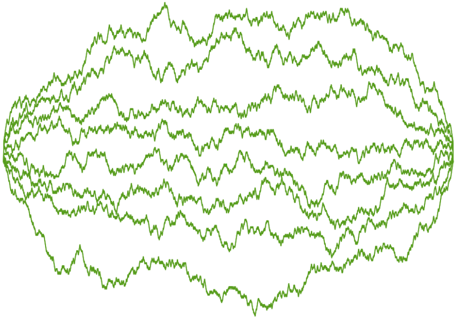

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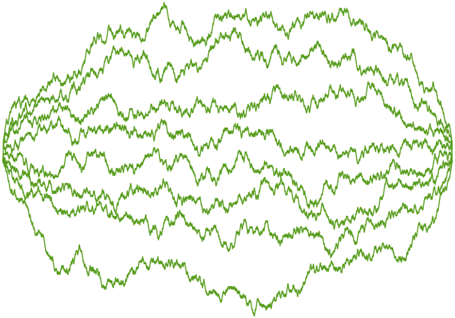

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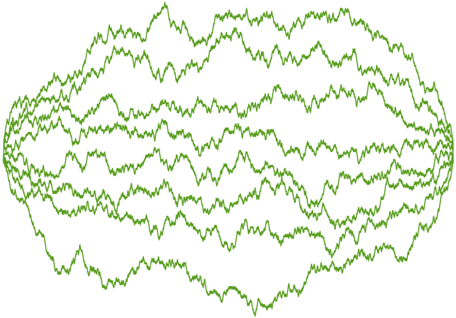

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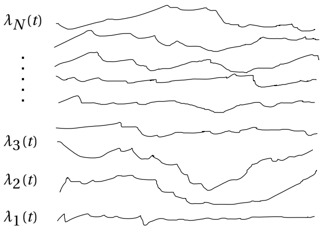
In particular, $\big(B_i(t)\big)_{i=1,\dots,N} \overset{(d)}{=} \Big(\sqrt{2t(1-t)}\,\lambda_i\Big)_{i=1,\dots,N}$.

So it is also an RMT model!

In fact, *more is true*:

If each entry of A undergoes an *Ornstein-Uhlenbeck diffusion* then $\lambda_1(t) < \lambda_2(t) < \cdots < \lambda_N(t)$, known as Dyson Brownian motion, defines a stationary process such that



$$\big(B_i(t)\big)_{i=1,\dots,N} \stackrel{(d)}{=} \left(\sqrt{2t(1-t)}\,\lambda_i\big(\tfrac{1}{2}\log(t/(1-t))\big)\right)_{i=1,\dots,N}.$$

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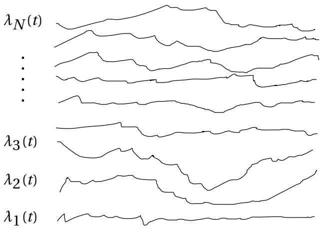

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Even more, one has $\sqrt{2}N^{1/6}\left(\lambda_N(N^{-1/3}t) - \sqrt{2N}\right) \xrightarrow[N\to\infty]{} \mathscr{A}_2(t)$,

which means $2N^{1/6}\left(B_N\big(\tfrac{1}{2}(1+N^{-1/3}t)\big) - \sqrt{N}\right) \xrightarrow[N\to\infty]{} \mathscr{A}_2(t) - t^2$.

$\mathscr{A}_2$ is the Airy$_2$ process, which describes the spatial fluctuations of models in the KPZ class with curved initial data.

KPZ $\longleftrightarrow$ RMT ? The *flat* case

- Limiting fluct. described by the Tracy-Widom F_{GOE} distr. associated to the Gaussian Orthogonal Ensemble, the real symmetric analogue of the GUE.
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- Most models are much more difficult to study in the flat case. Integrable methods seem to break down, except in the simplest situations.
- The flat/GOE connection is *essentially not understood at all*.
- In any case, it is clear that the flat/GOE connection is necessarily more tenuous than the curved/GUE case.

  For ex., it is known that the top line of the GOE Dyson B.M. *does not converge to the* Airy_1 *process.*

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Our goal: provide an explanation of the flat/GOE connection.

We use non-intersecting Brownian bridges, but focus on

$$\mathscr{M}_N = \max_{t\in[0,1]} B_N(t).$$

A slight detour (1): the Gaussian Orthogonal Ensemble

Consider a matrix A from the Gaussian Orthogonal Ensemble (GOE): A is an $N \times N$ (real) symmetric matrix with

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The eigenvalues concentrate on $[-2\sqrt{N}, 2\sqrt{N}]$, and the largest one satisfies

$$\lim_{N\to\infty} \mathbb{P}\big(\lambda_{\mathrm{GOE}}(N) \le 2\sqrt{N} + N^{-1/6} r\big) = F_{\mathrm{GOE}}(r)$$

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with

$$F_{\mathrm{GOE}}(r) = \det(I - P_0 B_r P_0)_{L^2(\mathbb{R})}.$$

where $P_r f(x) = f(x)\mathbf{1}_{x>r}$, B_r is the integral operator with kernel

$$B_r(x,y) = \mathrm{Ai}(x+y+r),$$

and the Fredholm determinant is defined as

$$\det(I-K)_{L^2(\mathbb{R})} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\mathbb{R}^n} \det\left[K(x_i,x_j)\right]_{i,j=1}^{n} d\vec{x}.$$

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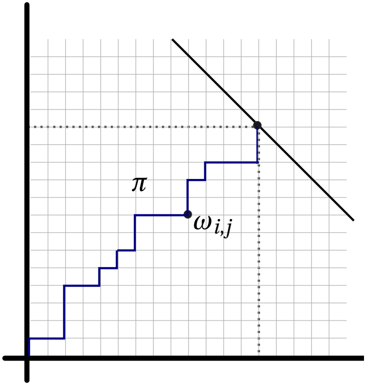
$$F_{\mathrm{GOE}}(r) = \det(I - P_0 B_r P_0)_{L^2(\mathbb{R})}.$$

For a GOE matrix the joint density of the eigenvalues $(\lambda_1, \ldots, \lambda_N)$ is

$$\frac{1}{Z_N} \prod_{i=1}^{N} e^{-\frac{1}{4}\lambda_i^2} \prod_{1\le i<j\le N} |\lambda_i - \lambda_j|.$$

The weights $e^{-\lambda^2/4}$ are those associated to the Hermite polynomials.

A slight detour (2): LPP and the Airy$_2$ process



i.i.d. geometric *waiting times*

$\omega_{i,j},\ i,j \in \mathbb{Z}^+$.

$$G^{\mathrm{pt}}(m,n) = \max_{\pi:(0,0)\to(m,n)} \sum_{i=0}^{m+n} w_{\pi_i}.$$

(point-to-point last passage time)

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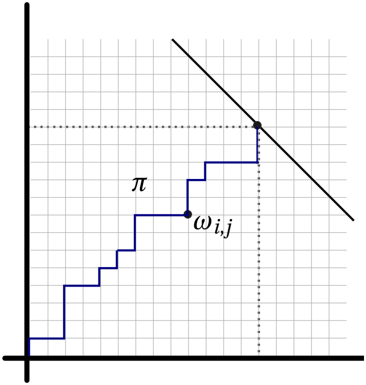

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Last passage time fluctuations:

$$\frac{G^{\mathrm{pt}}(N,N) - c_1 N}{c_2 N^{1/3}} \xrightarrow[N\to\infty]{} \zeta_{\mathrm{GUE}}.$$

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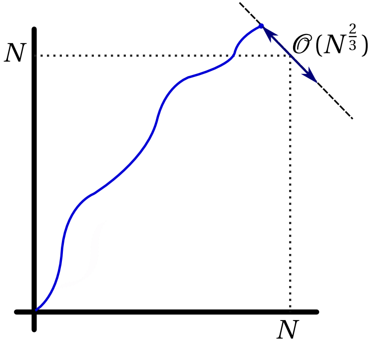

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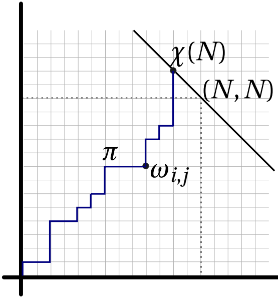
Spatial fluctuations: Let

$$H_N(u) = \frac{G(N + c_3 N^{2/3} u, N - c_3 N^{2/3} u) - c_1 N}{c_2 N^{\frac{1}{3}}}.$$

Then $H_N(u) \xrightarrow[N\to\infty]{} \mathcal{A}_2(u) - u^2$ with $\mathcal{A}_2$ the *Airy$_2$ process*.

[Prähofer-Spohn '01, Johansson '03]

Point-to-line last passage percolation



Now choose the path which maximizes the passage time among all paths π of length $2N$

$\chi(N)$: endpoint of the maximizing path

$$G^{\text{line}}(N) = \max_{|u|\le N} G^{\text{pt}}(N-u, N+u).$$

In particular

$$\frac{G^{\text{line}}(N) - c_1 N}{c_2 N^{1/3}} = \max_{u\in c_3^{-1}N^{-2/3}\mathbb{Z},\, |u|\le c_3^{-1}N^{1/3}} H_N(u)$$

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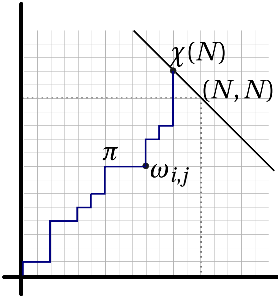

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[Baik-Rains '00, Borodin-Ferrari-Sasamoto '08]

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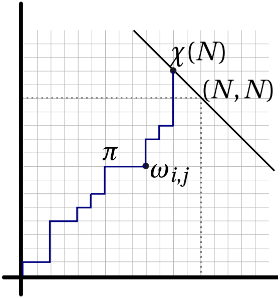

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and therefore

$$\sup_{u \in \mathbb{R}} \left\{ \mathscr{A}_2(u) - u^2 \right\} \overset{(d)}{=} 4^{1/3} \zeta_{\text{GOE}}$$

[Johansson '03]

Back to non-intersecting Brownian bridges

Recall that

$$2N^{1/6}\Big(B_N\big(\tfrac{1}{2}(1+N^{-1/3}t)\big)-\sqrt{N}\Big)\xrightarrow[N\to\infty]{}\mathscr{A}_2(t)-t^2 \qquad (\star)$$

(in the sense of finite-dimensional distributions).

This suggests that

$$\boxed{2N^{1/6}\left(\max_{t\in[0,1]}B_N(t)-\sqrt{N}\right)\xrightarrow[N\to\infty]{}4^{1/3}\zeta_{\mathrm{GOE}}}$$

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Proving this was the subject of intense research in the physics literature [Schehr, Majumdar, Rambeau, Comtet, Randon-Furling, Forrester '08-'12].

It was proved rigorously for Brownian br. on the half-line in [Liechty '12].

It actually follows from a stronger version of $(\star)$ in [Corwin-Hammond '14] (and also from our main theorem).

We may rewrite the result as

$$\max_{t\in[0,1]} B_N(t) = \sqrt{N} + 2^{-1/3}N^{-1/6}\zeta_{\mathrm{GOE}} + o(N^{-1/6}) \qquad \text{as } N \to \infty$$

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Obs: The case $N=1$ is easy. By the reflection principle,

$$\mathbb{P}\Big(\max_{t\in[0,1]} B_N(t) \le r\Big) = 1-e^{-r^2} = \mathbb{P}(\chi_2^2 \le 2r^2).$$

Observe that $\chi_2^2 \overset{(d)}{=} (Z_1\ Z_2)\cdot\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$ with Z_1, Z_2 independent $\mathcal{N}(0,1)$.

The Laguerre Orthogonal Ensemble

Let X be an $n \times N$ matrix with i.i.d. $\mathcal{N}(0,1)$ entries $\quad (n > N)$.

Then $M = X^{\top} X$ is said to be a matrix from the Laguerre Orthogonal Ensemble (LOE) (also called a Wishart matrix).

The joint density of the eigenvalues of M is now given by

$$\frac{1}{Z_N} \prod_{1 \le i < j \le N} |\lambda_i - \lambda_j| \prod_{i=1}^{N} \lambda_i^a e^{-\lambda_i/2},$$

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Now the eigenvalues concentrate on $[0, 4N]$. The fluctuations at the *soft edge* coincide with those of GOE [Johnstone '01]: if a is a constant, then

$$2^{-4/3} N^{-1/3} \big(\lambda_{\mathrm{LOE}}(N) - 4N\big) \xrightarrow[N \to \infty]{} \zeta_{\mathrm{GOE}}.$$

Main result

Take $a = 0$ (which means X is size $(N+1) \times N$) and let $F_{\text{LOE},N}$ be the distribution of the largest eigenvalue of $M = X^{\mathsf{T}} X$.

Theorem (Nguyen-R '15)

$$\mathbb{P}\left(\max_{t \in [0,1]} \sqrt{2} B_N(t) \leq r\right) = F_{\text{LOE},N}(2r^2).$$

In other words, $4 \max_{t \in [0,1]} B_N(t)^2$ *is distributed as the largest eigenvalue of an LOE matrix.*

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$\max_{t \in [0,1]} B_N(t) \overset{(d)}{\approx} \sqrt{N} + 2^{-1/3} N^{-1/6} \zeta_{\mathrm{GOE}}$, and thus also
$\sup_{u \in \mathbb{R}} \left\{ \mathscr{A}_2(u) - u^2 \right\} \overset{(d)}{=} 4^{1/3} \zeta_{\mathrm{GOE}}$, follow as a corollary.

The proof

There are formulas for the distribution of $\max_{t \in [0,1]} B_N(t)$ in the literature (obtained by path-integral techniques) but they do not make apparent any connection to a random matrix ensemble.

Instead we derive a new formula for the distribution of $\max_{t \in [0,1]} B_N(t)$, by a different method, which is suggestive of such a connection.

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Our proof is done at the level of Dyson Brownian motion. It has two steps:

1. Derive an expression for $\mathbb{P}\Big(\lambda_N(t) \le r\cosh(t)\ \forall\, t \in \mathbb{R}\Big)$.
2. Show that the result coincides with $F_{\mathrm{LOE},N}(2r^2)$.

Extended Hermite kernel

Let $\lambda_N(t)$ be the top line in Dyson Brownian motion. Then for $t_1 < t_2 < \ldots < t_n$ and $r_1, \ldots, r_n \in \mathbb{R}$,

$$\mathbb{P}\big(\lambda_N(t_j) \leq r_j,\, j = 1, \ldots, n\big) = \det\big(I - \mathrm{f}\, H_N^{\mathrm{ext}}\, \mathrm{f}\big)_{L^2(\{t_1,\ldots,t_n\}\times\mathbb{R})},$$

where $\mathrm{f}(t_j, x) = \mathbf{1}_{x\in(r_j,\infty)}$ and

$$H_N^{\mathrm{ext}}(s,x;t,y) = \begin{cases} \displaystyle\sum_{n=0}^{N-1} e^{n(s-t)}\varphi_n(x)\varphi_n(y) & \text{if } s \geq t, \\ \displaystyle -\sum_{n=N}^{\infty} e^{n(s-t)}\varphi_n(x)\varphi_n(y) & \text{if } s < t. \end{cases}$$

Here the φ_n are the Hermite or harmonic oscillator functions $\varphi_n(x) = e^{-x^2/2} p_n(x)$ with p_n the n-th normalized Hermite polynomial. They satisfy $\int_{\mathbb{R}} dx\, \varphi_n(x)\varphi_m(x) = \mathbf{1}_{n=m}$.

Extended Hermite kernel

Let $\lambda_N(t)$ be the top line in Dyson Brownian motion. Then for $t_1 < t_2 < \ldots < t_n$ and $r_1, \ldots, r_n \in \mathbb{R}$,

$$\mathbb{P}\big(\lambda_N(t_j) \leq r_j, j = 1, \ldots, n\big) = \det\big(I - \mathrm{f}\, H_N^{\mathrm{ext}}\, \mathrm{f}\big)_{L^2(\{t_1,\ldots,t_n\}\times\mathbb{R})},$$

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Note that we need to take $n \to \infty$. This was done for the Airy$_2$ process in [Corwin-Quastel-R '13].

Path integral kernel for DBM

Let $H_N(x,y) = \sum_{n=0}^{N-1} \varphi_n(x)\varphi_n(y)$ and $D = -\frac{1}{2}[\Delta - x^2 + 1]$.

Then

$$\mathbb{P}\big(\lambda_N(t_j) \le r_j,\, j = 1,\dots,n\big)$$
$$= \det\big(I - H_N + \bar{P}_{r_1} e^{(t_1-t_2)D} \bar{P}_{r_2} e^{(t_2-t_3)D} \cdots \bar{P}_{r_n} e^{(t_n-t_1)D} H_N\big)_{L^2(\mathbb{R})}.$$

[Borodin-Corwin-R '15]

where $\bar{P}_a f(x) = \mathbf{1}_{x\le a} f(x)$.

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For $g \in H^1([\ell, r])$, letting $r_i = g(t_i)$ and taking $n \to \infty$, one gets

$$\mathbb{P}\big(\lambda_N(t) \le g(t) \text{ for } t \in [\ell, r]\big) = \det\big(I - H_N + \Theta^g_{[\ell,r]} e^{(r-\ell)D} H_N\big),$$

where $\Theta^g_{[\ell_1,\ell_2]} f(x) = u(\ell_2, x)$ is the solution operator at time ℓ_2 of the boundary value problem

$$\begin{cases} \partial_t u + Du = 0 & x < g(t) \\ u(t,x) = 0 & x \ge g(t) \end{cases} \quad \text{with } u(\ell_1, x) = f(x).$$

Consider $\left\{\begin{array}{lr} \partial_t u - \frac{1}{2}(\partial_x^2 - x^2 + 1)u = 0 & x < g(t) \\ u(t,x) = 0 & x \geq g(t) \end{array}\right\}$ with $u(\ell_1, x) = f(x)$.

Setting $u(t,x) = e^{x^2/2+t} v(\tau, z)$ and $\alpha = \frac{1}{4}e^{\ell_1}$, $\beta = \frac{1}{4}e^{\ell_2}$, $\tau = \frac{1}{4}e^{2t}$, $z = e^t x$, leads to the standard heat equation

$$\left\{\begin{array}{lr} \partial_\tau v - \partial_z^2 v = 0 & z < \sqrt{4\tau}\, g(\log(4\tau)/2) \\ v(\tau, z) = 0 & z \geq \sqrt{4\tau}\, g(\log(4\tau)/2) \end{array}\right\}$$

with $v(\alpha, z) = e^{-\frac{1}{8\alpha}z^2 - \frac{1}{2}\log(4\alpha)} f(\frac{1}{\sqrt{4\alpha}} z) \mathbf{1}_{z < \sqrt{4\tau}\, g(\frac{1}{2}\log(4\tau))}$.

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By Feynman-Kac this gives

$$\Theta^g_{[\ell_1,\ell_2]}(x,y) = e^{\frac{1}{2}(y^2 - x^2) + \ell_2} \frac{e^{-(e^{\ell_1}x - e^{\ell_2}y)^2/(4(\beta-\alpha))}}{\sqrt{4\pi(\beta-\alpha)}}$$

$$\times \mathbb{P}_{\hat{b}(\alpha) = e^{\ell_1}x,\, \hat{b}(\beta) = e^{\ell_2}y}\left(\hat{b}(t) \leq \sqrt{4t}\, g\left(\tfrac{1}{2}\log(4t)\right) \ \forall\, t \in [\alpha, \beta]\right).$$

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$$\times\,\mathbb{P}_{\hat{b}(\alpha)=e^{\ell_1}x,\,\hat{b}(\beta)=e^{\ell_2}y}\Big(\hat{b}(t) \leq \sqrt{4t}\,g\big(\tfrac{1}{2}\log(4t)\big)\ \forall\, t \in [\alpha,\beta]\Big).$$

Now if $g(t) = r\cosh(t)$ we get $\sqrt{4t}\,g\big(\frac{1}{2}\log(4t)\big) = 2rt + \frac{1}{2}r$

$\longrightarrow$ the probability is explicit (by the reflection principle).

The result is

$$\Theta^{r\cosh(t)}_{[-L,L]} = \bar{P}_{r\cosh(L)} \left(e^{-2LD} - R^{(r)}_L \right) \bar{P}_{r\cosh(L)}$$

with $R^{(r)}_L(x,y) = \frac{1}{\sqrt{4\pi(\beta-\alpha)}} e^{\frac{1}{2}(y^2-x^2)+L-r(e^L y-e^{-L}x)+r^2(\beta-\alpha)-\frac{1}{4(\beta-\alpha)}(e^{-L}x+e^L y-2r(\alpha+\beta)-r)^2}$.

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$$\mathbb{P}\left(\lambda_N(t) \le r\cosh(t)\ \forall\, t \in \mathbb{R}\right) = \lim_{L\to\infty} \det\left(I - H_N + \Theta^{r\cosh(t)}_{[-L,L]} e^{2LD} H_N \right)$$

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&= \lim_{L\to\infty} \det\left(I - e^{LD} H_N R^{(r)}_L e^{LD} H_N + \text{error} \right).
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- [CQR '13] proved, using similar arguments, that
$$\mathbb{P}\left(\mathscr{A}_2(t) \le t^2 + r\ \forall\, t \in \mathbb{R}\right) = \det\big(I - K_{\mathrm{Ai}} \varrho_r K_{\mathrm{Ai}}\big).$$

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On the other hand, it is not clear *a priori* what $\det\left(I - H_N \varrho_r H_N\right)$ is, nor what it has to do with LOE.

Connection with LOE

Correlation kernels for orthogonal ensembles in RMT are not as simple as in the unitary case. To get around this we use a fact from [Forrester-Rains '04].

Take two independent LOE matrices, put all the $2N$ eigenvalues together in increasing order, and let $\bar{\lambda}(1) < \cdots < \bar{\lambda}(N)$ be the ones with *even labels*. Then the *superimposed ensemble* $(\bar{\lambda}(i))_{i=1,\ldots,N}$ has a simple correlation kernel:

$$\widetilde{L}_N(x,y) = -\frac{\partial}{\partial x}\int_0^y du\, L_N(x,u)$$

with

$$L_N(x,y) = \sum_{n=0}^{N-1} \psi_n(x)\psi_n(y).$$

Here the ψ_n are the Laguerre functions $\quad \psi_n(x) = e^{-x/2} q_n(x)$
with q_n the n-th normalized Laguerre polynomial. They satisfy

$$\int_0^\infty dx\, \psi_n(x)\psi_m(x) = \mathbf{1}_{n=m}.$$

This implies

$$\mathbb{P}(\lambda_{\mathrm{LOE}}(N) \le 2r^2)^2 = \mathbb{P}(\bar{\lambda}(N) \le 2r^2) = \det\left(I - P_{2r^2}\widetilde{L}_N P_{2r^2}\right).$$

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But $\widetilde{L}_N$ is a finite rank operator, and thus the Fredholm determinant can be written as a the determinant of a finite matrix.

we get

$$\mathbb{P}(\lambda_{\mathrm{LOE}}(N) \le 2r^2)^2 = \det[I - G + R_1 R_2^{\mathsf{T}}]$$

with

$$G_{ij} = \int_{2r^2}^{\infty} dx\, \psi_i(x)\psi_j(x), \quad (R_1)_i = \psi_i(2r^2) \quad \text{and} \quad (R_2)_i = \int_0^{2r^2} du\, \psi_i(u).$$

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Similarly,

$$\det\big(I - H_N \varrho_r H_N\big) = \det[I - M] \quad \text{with} \quad M_{ij} = \int_{\mathbb{R}} dx\, \varphi_i(x)\varphi_j(x).$$

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(A somewhat similar formula was obtained in [Rambeau-Schehr '10])

So we need to show that

$$\det[I - M]^2 = \det[I - G + R_1 R_2^{\mathsf{T}}].$$

The proof is relatively long. The key step is the following:

Lemma

Let $\widetilde{M}_{ij} = (-1)^N \left(\psi_{i+j-N}(2r^2) - \psi_{i+j-N+1}(2r^2)\right)$ *for* $i, j \in \{0, \ldots, N-1\}$. *Then:*

(1) $\det[I - M] = \det[I - \widetilde{M}]$.

(2) $(\widetilde{M})^2 = G$.

(3) $\widetilde{M}^{-1} R_1$ *and* $(I - \widetilde{M})^{-1} R_2$ *are explicit and simple.*

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The proof uses generating functions and contour integral formulas for Hermite and Laguerre polynomials and several ad-hoc combinatorial identities involving them.

Formulas for Brownian bridges on the half-line

We can also consider non-intersecting Brownian bridges on a half-line, with either absorbing or reflecting boundary conditions (corresponding to Brownian excursions and reflected Brownian motions).

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We can also consider non-intersecting Brownian bridges on a half-line, with either absorbing or reflecting boundary conditions (corresponding to Brownian excursions and reflected Brownian motions).

The story is analogous, with the following modifications:

- The Hermite kernels get replaced by

$$K^{\text{odd}}_{\text{Herm},N}(x,y) = \sum_{n=0}^{N-1} \varphi_{2n+1}(x)\varphi_{2n+1}(y) \qquad \text{in the abs. case}$$

$$K^{\text{even}}_{\text{Herm},N}(x,y) = \sum_{n=0}^{N-1} \varphi_{2n}(x)\varphi_{2n}(y) \qquad \text{in the refl. case}$$

- The boundary value PDE is solved in $[0,\infty)$, with an additional Dirichlet boundary condition in the abs. case.
- Feynman-Kac gives formulas in terms of reflected Brownian motion.

Let

$$\varrho_r^{\mathrm{be}} f(x) = 2\sum_{k=1}^{\infty} f(2kr-x) \qquad \text{and} \qquad \varrho_r^{\mathrm{rbb}} f(x) = 2\sum_{k=1}^{\infty} (-1)^{k+1} f(2kr-x).$$

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Theorem

$$\mathbb{P}\left(\max_{t\in[0,1]} \sqrt{2} B_N^{\text{be}}(t) \le r\right) = \det\left(\mathsf{I} - K_{\text{Herm},N}^{\text{odd}} \varrho_r^{\text{be}} K_{\text{Herm},N}^{\text{odd}}\right)_{L^2(\mathbb{R})}$$

and

$$\mathbb{P}\left(\max_{t\in[0,1]} \sqrt{2} B_N^{\text{rbb}}(t) \le r\right) = \det\left(\mathsf{I} - K_{\text{Herm},N}^{\text{even}} \varrho_r^{\text{rbb}} K_{\text{Herm},N}^{\text{even}}\right)_{L^2(\mathbb{R})}.$$

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$$\varrho_r^{\mathrm{be}} f(x) = 2\sum_{k=1}^{\infty} f(2kr - x) \qquad \text{and} \qquad \varrho_r^{\mathrm{rbb}} f(x) = 2\sum_{k=1}^{\infty} (-1)^{k+1} f(2kr - x).$$

Theorem

$$\mathbb{P}\left(\max_{t\in[0,1]} \sqrt{2} B_N^{\mathrm{be}}(t) \le r\right) = \det\left(\mathrm{I} - K^{\mathrm{odd}}_{\mathrm{Herm},N} \varrho_r^{\mathrm{be}} K^{\mathrm{odd}}_{\mathrm{Herm},N}\right)_{L^2(\mathbb{R})}$$

and

$$\mathbb{P}\left(\max_{t\in[0,1]} \sqrt{2} B_N^{\mathrm{rbb}}(t) \le r\right) = \det\left(\mathrm{I} - K^{\mathrm{even}}_{\mathrm{Herm},N} \varrho_r^{\mathrm{rbb}} K^{\mathrm{even}}_{\mathrm{Herm},N}\right)_{L^2(\mathbb{R})}.$$

In particular, this yields

$$\lim_{N\to\infty} \mathbb{P}\left(2^{7/6} N^{1/6} (\mathscr{M}_N^{\mathrm{be}} - \sqrt{2N}) \le r\right) = F_{\mathrm{GOE}}(4^{1/3} r)$$

$$\lim_{N\to\infty} \mathbb{P}\left(2^{7/6} N^{1/6} (\mathscr{M}_N^{\mathrm{rbb}} - \sqrt{2N}) \le r\right) = F_{\mathrm{GOE}}(4^{1/3} r)$$