LEVEL–WISE APPROXIMATION OF A MARKOV PROCESS ASSOCIATED TO THE BOUNDARY OF AN INFINITE TREE

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Abstract. We study an approximation of a Markov process associated to the boundary of an infinite rooted tree. This approximation is constructed by projecting the infinitesimal generator of the original process (defined in the boundary) onto the spaces associated to the filtration spanned by the successive levels of the rooted tree.

1. Introduction

Every (finite or locally-finite) rooted tree can be associated with a *tree matrix*, constructed by assigning increasing values to the successive levels of the tree. This tree matrix turns out to be an *ultrametric matrix*, a class of matrices, first introduced in [MMM94] for the finite case, corresponding to those matrices $U = (U_{ij} : i, j \in I)$ satisfying the ultrametric inequality:

$$U_{ij} \geq \min\{U_{ik}, U_{kj}\} \quad \text{for each } i, j, k \in I.$$

In [DMM96] it was shown that every finite ultrametric matrix has a minimal extension corresponding to a tree matrix and the result is directly extended to infinite matrices in [DMM04].

In the finite setting it was shown in [MMM94] and [NV94] that the inverse of a non-singular ultrametric matrix U is a diagonal dominant Stieltjes matrix. Thus U is proportional to the potential of a discrete-time sub-markovian kernel P, that is $U = \kappa \sum_{n\geq 0} P^n$. The graph of the kernel P is closely related to the minimal tree extension of U (see [DMM96]).

The infinite setting is less simple. In [DMM04] it was shown that each column of an infinite tree matrix U is the sum of a potential and a harmonic function (non-trivial except in the special *recurrent* case). In this case there appears a continuous-time sub-markovian kernel that replaces the kernel P of the finite case. The matrix U allows to construct a stochastic integral operator W acting on the boundary ∂_∞ of the tree, which in turn is the generator of a Markov process Ξ defined on the boundary. The underlying ultrametricity of the tree matrices is crucial for these results, which, under certain conditions, can be extended to general infinite ultrametric matrices. This class of operators were already considered in [Lyo90] and [Lyo92], where some of its potential properties (as the dimension and capacity of the boundary) are fully studied.

This work is devoted to constructing an approximation of this process Ξ in a manner such that the n^{th} process of the approximating sequence is in some way a projection of Ξ onto the n^{th} level of the tree. The next section presents the notions that we need for our work, mainly taken from [DMM04].

2. Basic Notions and Notation

2.1. **Trees and tree matrices.** In what follows, I will denote a countable infinite set and $T \subseteq I \times I$ will denote a locally-finite tree with set of nodes I and root r.

In [Car72] the following properties are shown. Given two points $i \neq j$ in I there exists a unique path of minimal length in the tree that connects them, which we will call $\text{geod}(i, j)$, the *geodesic between* i *and* j. We say that $j \preccurlyeq i$ if $j \in \text{geod}(i, r)$. Thus, given two points i and k in I, there exists a unique point, denoted by $i \curlywedge k$ (the *minimum between* i *and* k), that satisfies $i \curlywedge k \preccurlyeq i$, $i \curlywedge k \preccurlyeq k$ and $[(j \preccurlyeq i,\ j \preccurlyeq k) \Rightarrow j \preccurlyeq i \curlywedge k]$.

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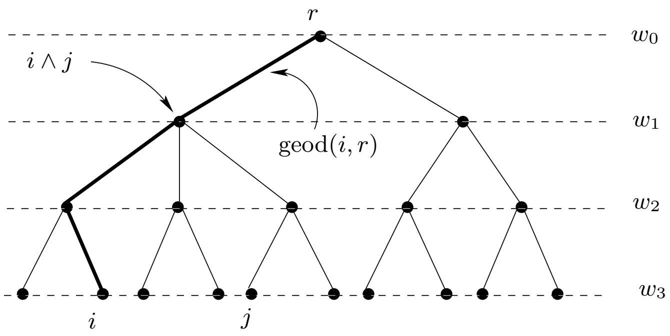

FIGURE 1. Levels, geodesics and minimums on a tree.

$i \curlywedge k$ corresponds to the farthest point from the root r that belongs both to geod(i,r) and geod(k,r). We can also define the *level function*

$$|i| = [\textit{length of geod}(i,r)],$$

which is an increasing function in i. (Figure 1 sketches these concepts).

Definition 2.1. Given the rooted tree (I,T) and a non-negative non-decreasing function $w : \mathbb{N} \to \mathbb{R}$, the matrix U defined by

$$U_{ij} = w_{|i \curlywedge j|}$$

is called the *tree matrix* associated to (I,T) and w.

It is evident that every tree matrix is ultrametric.

2.2. **Compactification of the tree.** In this part we define the boundary ∂_∞ of the tree T. An *infinite path* in the tree is a sequence $(i_n \in I : n \in \mathbb{N})$ such that $(i_n, i_{n+1}) \in T$ for each $n \in \mathbb{N}$. If every i_n is different, the path is called an *infinite chain.* We define the following relation over the set of chains:

$$(i_n \in I : n \in \mathbb{N}) \sim_{\partial_\infty} (j_n \in I : n \in \mathbb{N}) \Leftrightarrow |\{i_n : n \in \mathbb{N}\} \cap \{j_n : n \in \mathbb{N}\}| = \infty.$$

It is clear that $\sim_{\partial_\infty}$ is an equivalence relation.

Definition 2.2. The *boundary of the tree* T is the set ∂_∞ corresponding to the quotient between the set of infinite chains of T and the relation $\sim_{\partial_\infty}$.

Given $i \in I$ and $\xi \in \partial_\infty$, the *geodesic* geod(i,ξ) between i and ξ is the unique chain with origin i that belongs to the equivalence class of ξ (see figure 2). It is easy to define also geod(ξ,η) for $\xi,\eta \in \partial_\infty$. For $n \in \mathbb{N}$, $\xi(n)$ will be the unique point in geod(r,ξ) such that $|\xi(n)| = n$.

$\preccurlyeq$ and $\curlywedge$ can also be extended to ∂_∞:

$$i \preccurlyeq \xi \Leftrightarrow i \in \text{geod}(r,\xi) \quad \text{for } i \in I, \xi \in \partial_\infty, \tag{2.1}$$

$$\xi \curlywedge \eta = i, \text{ where } |i| = \max\{|j| : j \in \text{geod}(r,\xi) \cap \text{geod}(r,\eta)\}, \quad \text{for } \xi,\eta \in I \cup \partial_\infty. \tag{2.2}$$

Then $\xi \curlywedge \xi = \xi$ and if $\xi \neq \eta$ then $\xi \curlywedge \eta \in I$. In this last case, $\xi \curlywedge \eta = i$ if and only if $\xi(|i|) = \eta(|i|)$ and $\xi(n) \neq \eta(n)$ for $n > |i|$.

We need some more notation. For $i \in I$, $\xi \in \partial_\infty$ and $n \in \mathbb{N}$, we denote

$$[i,\infty] = \{z \in I \cup \partial_\infty : i \preccurlyeq z\}, \qquad [i,\infty) = [i,\infty] \cap I,$$
$$\partial_\infty(i) = [i,\infty] \cap \partial_\infty, \qquad C^n(\xi) = \partial_\infty(\xi(n)),$$
$$I^n = \{i \in I : |i| \leq n\}, \qquad B^n = \{i \in I : |i| = n\}.$$

We endow the set $I \cup \partial_\infty$ with the topology Λ generated by the base formed by the open-closed sets in the family $\mathcal{A} = \{[i,\infty] : i \in I\}$. $(I \cup \partial_\infty, \Lambda)$ turns out to be compact, metrizable and totally discontinuous, and for $\xi \in \partial_\infty$ it holds that $\lim_{n\to\infty} \xi(n) = \xi$ and $\partial_\infty(\xi(n)) = \{\eta \in \partial_\infty ; |\xi \curlywedge \eta| \geq n\}$ (see [Car72]).

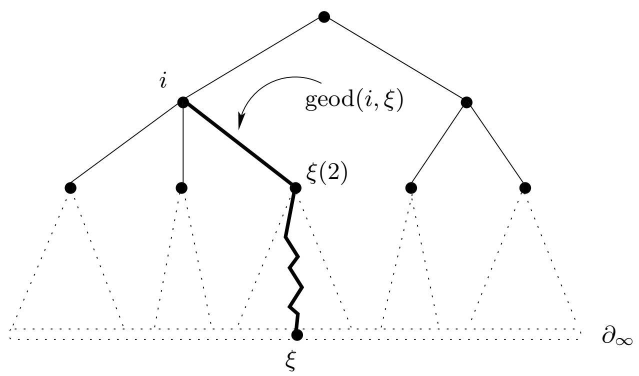

FIGURE 2. ∂_∞ and the geodesic between $i \in I$ and $\xi \in \partial_\infty$.

2.3. **The boundary operator.** In [DMM04] it was shown that, given a tree matrix U, there exists a matrix Q such that $UQ = QU = -\mathbb{I}$ (this infinite product is well defined because all the columns of Q have finitely many non-zero entries, so the sums involved are finite). This matrix is the generator of a continuous-time Markov process $(X_t)_{t\geq 0}$ that lives on the tree. Q is conservative except at the root r, so we add an absorbing state ∂_r connected to r. We denote by ζ the lifetime of X, so that $X_\zeta = \partial_r$ or $X_\zeta \in \partial_\infty$.

We say that the tree matrix is *transient* if $\mathbb{P}_r\{X_\zeta \in \partial_\infty\} > 0$. Otherwise, we say that the matrix is *recurrent.* This last case is rather easy to analyze, since the unique bounded harmonic function h of the tree (that is, such that $Qh = 0$) is $h \equiv 0$ (see [DMM04]), so in what follows we assume that the matrix is transient. This allows us to define the conditional measure μ over ∂_∞ by

$$\mu(\cdot) = \mathbb{P}_r\{X_\zeta \in \cdot \,|\, X_\zeta \in \partial_\infty\}. \tag{2.3}$$

With this measure and this notion of harmonicity, ∂_∞ turns out to be isomorphic to the Martin boundary of the tree (see [Car72]).

Using (2.2) we can extend U to $I \cup \partial_\infty$ in the following way:

$$\text{for } \xi, \eta \in \partial_\infty, \quad U_{\xi\eta} = \begin{cases} w_{|\xi \curlywedge \eta|} & \text{if } \xi \neq \eta, \\ \lim_{n\to\infty} U_{\xi(n)\xi(n)} & \text{if } \xi = \eta, \end{cases}$$
$$\text{and for } i \in I,\ \xi \in \partial_\infty, \quad U_{i\xi} = w_{|i \curlywedge \xi(|i|)|}.$$

This extension is continuous in both variables: $U_{\xi\eta} = \lim_{n,m\to\infty} U_{\xi(n)\eta(m)}$ for $\xi, \eta \in \partial_\infty$.

In [DMM04] the set ∂_∞^r of the *regular points* is introduced as a mean to avoid some technical complications. We will not enter into that discussion here, let us just mention that ∂_∞^r has full measure, so any integral computed over ∂_∞ can be also computed over ∂_∞^r.

Definition 2.3. Given a (positive) bounded and measurable function $f : \partial_\infty \to \mathbb{R}$ we define

$$Wf\,(\xi) = \int_{\partial_\infty} U_{\xi\eta} f(\eta)\, \mu(d\eta).$$

W is self adjoint and Wf is bounded if f is so. $W\mathbf{1} = \alpha = w_0/\mathbb{P}_r\{X_\zeta \in \partial_\infty\}$ (where w_0 is the value associated to the level zero of the tree, i.e. the root), W is well defined as an operator in $L^p(\mu)$ and $\|W\|_p = \alpha$.

A useful calculation made in [DMM04] leads to the spectral decomposition of W, which we present now. For every $n \in \mathbb{N}$, the collection $\{C^n(\xi) : \xi \in \partial_\infty\}$ forms a finite partition of ∂_∞, and we write $\mathcal{F}_n$ for the σ-algebra associated to that partition. This gives us a filtration $(\mathcal{F}_n)_{n\geq 0}$ whose limit is $\mathcal{F} = \sigma(\Lambda)$. We denote $\Delta_k[w] = w_k - w_{k-1}$ for $k \in \mathbb{N}$ and $\Delta_{-1}[w] = 0$ and define the sequence $(G_n)_{n\geq 0}$ by

$$G_n(\xi) = \sum_{k\geq n} \Delta_k[w]\mu(C^k(\xi)) \quad \text{for } \xi \in \partial_\infty \text{ and } n \in \mathbb{N}.$$

Observe that $G_0 \equiv W\mathbf{1} = \alpha$, so G_0 is convergent. Moreover, $G_n > 0$ for all n, and the sequence is decreasing, predictable and convergent to 0. The spectral decomposition of W (that shows that it is a stochastic integral operator) amounts to

$$W = \sum_{n\in\mathbb{N}} G_n \left(\mathbb{E}_\mu(\cdot \,|\, \mathcal{F}_n) - \mathbb{E}_\mu(\cdot \,|\, \mathcal{F}_{n-1})\right),$$

where we assume $\mathbb{E}_\mu(\cdot|\mathcal{F}_{-1}) \equiv 0$.

It is possible to construct an inverse of W over $\mathrm{Im}(W)$. We will write W^{-1} for that inverse and assume implicitly that its domain is $\mathrm{Im}(W)$, so

$$W^{-1} = \sum_{n\in\mathbb{N}} G_n^{-1} \left(\mathbb{E}_\mu(\cdot|\,\mathcal{F}_n) - \mathbb{E}_\mu(\cdot|\,\mathcal{F}_{n-1})\right).$$

2.4. **The process on the boundary.** We will write $Z \sim \exp[\lambda]$ to indicate that Z is a random variable with exponential distribution of parameter λ (and mean $1/\lambda$) and $B \sim \mathrm{Ber}[p]$ to indicate that B is a random variable with Bernoulli distribution of parameter p (that is, $\mathbb{P}\{B = 1\} = 1 - \mathbb{P}\{B = 0\} = p$).

The following theorem, proved in [DMM04], is the first thing that we will have to replicate for our level-wise approximation.

Theorem 2.4. *Consider the symmetric kernel*

$$p(t,\xi,\eta) = \sum_{n=0}^{|\xi\curlywedge\eta|} \frac{e^{-t/G_n(\xi)} - e^{-t/G_{n+1}(\xi)}}{\mu(C^n(\xi))} \qquad \xi,\eta \in \partial_\infty,\ t > 0. \tag{2.4}$$

This kernel is sub-markovian with total mass $e^{-t/G_0} = \int_{\partial_\infty} p(t,\xi,\eta)\,\mu(d\eta)$*. The markovian semigroup*

$$P_t f\,(\xi) = \int_{\partial_\infty} p(t,\xi,\eta) f(\eta)\,\mu(d\eta)$$

induced by this kernel in $L^2(\mu)$ *satisfies*

$$P_t f = \sum_{n\geq 0} e^{-t/G_n} \left(\mathbb{E}_\mu(f|\,\mathcal{F}_n) - \mathbb{E}_\mu(f|\,\mathcal{F}_{n-1})\right),$$

and the infinitesimal generator of P_t *is an extension of* $-W^{-1}$*. The potential of the semigroup is* W ($\int_0^\infty P_t f\,dt = Wf$)*, and its Green kernel is* U ($\int_0^\infty p(t,\xi,\eta)\,dt = U_{\xi\eta}$)*.*

Definition 2.5. The Markov process $(\Xi_t)_{0\leq t\leq \Upsilon}$ is the process over ∂_∞ associated to the semigroup (P_t), where $\Upsilon = \inf\{t > 0 : \Xi_t \notin \partial_\infty\}$ is its lifetime. The coffin state of Ξ will be denoted by $\dagger$ ($\Xi_\Upsilon = \dagger$). Ξ^ν will denote a copy of the process with initial distribution ν in ∂_∞, and $\Xi^\xi = \Xi^{\delta_\xi}$ indicates that the process starts in $\xi \in \partial_\infty$.

3. Approximation of the Process Ξ Onto the n^{th} Level

3.1. **Construction of the approximation of the process on ∂_∞.** Let $n \geq 0$ be the level of the tree onto which we are going to project Ξ. We write $\mathbb{E}_m = \mathbb{E}_\mu(\cdot|\mathcal{F}_m)$ and define the *approximate operator* $W^{(n)} : L^2(\mu,\mathcal{F}_\infty) \to L^2(\mu,\mathcal{F}_n)$ by

$$W^{(n)} = W\,\mathbb{E}_n.$$

Evidently, using the spectral decomposition of W, it holds that

$$W^{(n)} = \sum_{m=0}^{n} G_m[\mathbb{E}_m - \mathbb{E}_{m-1}].$$

We can define the following operator, that corresponds to the inverse of $W^{(n)}$ over $L^2(\mu,\mathcal{F}_n)$:

$$W^{(-n)} = W^{-1}\,\mathbb{E}_n = \sum_{m=0}^{n} G_m{}^{-1}[\mathbb{E}_m - \mathbb{E}_{m-1}].$$

The semigroup associated to this approximation is

$$P_t^{(n)} = e^{-tW^{(-n)}} = \sum_{m=0}^{n} e^{-t/G_m}[\mathbb{E}_m - \mathbb{E}_{m-1}].$$

To our approximation we associate a truncation of the coefficients G_m: we define $G_m^{(n)}$ as G_m if $m \leq n$ and 0 if $m > n$. Note that $e^{-t/G_{n+1}^{(n)}} = 0$ for $t > 0$.

Proposition 3.1.

$$P_t^{(n)}f(\xi) = \int_{\partial_\infty} p^{(n)}(t,\xi,\eta)f(\eta)\,\mu(d\eta), \quad \forall \xi \in \partial_\infty,\ \forall t>0, \tag{3.1}$$

where the kernel $p^{(n)}$ corresponds to

$$p^{(n)}(t,\xi,\eta) = \sum_{m=0}^{|\xi\curlywedge\eta|\wedge n} \frac{e^{-t/G_m^{(n)}(\xi)} - e^{-t/G_{m+1}^{(n)}(\xi)}}{\mu(C^m(\xi))}. \tag{3.2}$$

Proof. We can assume that f is $\mathcal{F}_n$-measurable, since $P_t^{(n)}f = P_t^{(n)}\mathbb{E}_n f$. Thus, it is enough to restrict the proof to the case $f = \mathbf{1}_{\partial_\infty(i)}$ for $i \in B^n$.

$$\begin{aligned}
(P_t^{(n)}\mathbf{1}_{\partial_\infty(i)})(\xi) &= \sum_{m=0}^{n} e^{-t/G_m(\xi)}\left[\mathbb{E}_m\mathbf{1}_{\partial_\infty(i)} - \mathbb{E}_{m-1}\mathbf{1}_{\partial_\infty(i)}\right](\xi)\\
&= \sum_{m=0}^{n}\left(e^{-t/G_m^{(n)}(\xi)} - e^{-t/G_{m+1}^{(n)}(\xi)}\right)\mathbb{E}_m\mathbf{1}_{\partial_\infty(i)}(\xi)\\
&= \sum_{m=0}^{n}\left(e^{-t/G_m^{(n)}(\xi)} - e^{-t/G_{m+1}^{(n)}(\xi)}\right)\frac{\mu(\partial_\infty(i))}{\mu(\partial_\infty(i(m)))}\mathbf{1}_{\partial_\infty(i(m))}(\xi)\\
&= \mu(\partial_\infty(i))\sum_{m=0}^{|\xi\curlywedge i|}\left(e^{-t/G_m^{(n)}(\xi)} - e^{-t/G_{m+1}^{(n)}(\xi)}\right)\frac{1}{\mu(\partial_\infty(i(m)))}.
\end{aligned}$$

Now, for $\eta \in \partial_\infty(i)$ and $m \le |\xi\curlywedge i| \wedge n$, it holds that $\mu(\partial_\infty(i(m))) = \mu(C^m(\xi))$ and $|\xi\curlywedge i| = |\xi\curlywedge\eta|\wedge n$. To get the last equality, observe that if $|\xi\curlywedge i| < n$, then $\xi \notin \partial_\infty(i)$ and thus $\xi\curlywedge\eta = \xi\curlywedge i$, while if $|\xi\curlywedge i| = n$, then $\xi \in \partial_\infty(i)$, and as $\eta \in \partial_\infty(i)$, we get $|\xi\curlywedge\eta| \ge n$, so $|\xi\curlywedge\eta|\wedge n = n = |\xi\curlywedge i|$. Hence,

$$\begin{aligned}
(P_t^{(n)}\mathbf{1}_{\partial_\infty(i)})(\xi) &= \int_{\partial_\infty(i)}\sum_{m=0}^{|\xi\curlywedge\eta|\wedge n}\left(e^{-t/G_m^{(n)}(\xi)} - e^{-t/G_{m+1}^{(n)}(\xi)}\right)\frac{1}{\mu(C^m(\xi))}\,\mu(d\eta)\\
&= \int_{\partial_\infty} p^{(n)}(t,\xi,\eta)\mathbf{1}_{\partial_\infty(i)}(\eta)\,\mu(d\eta).
\end{aligned}$$

□

The following proposition describes the Green kernel of this semigroup, that corresponds to the function $U^{(n)} : \partial_\infty \times \partial_\infty \to \mathbb{R}$ given by

$$\begin{aligned}
U_{\xi\eta}^{(n)} &= \int_0^\infty p^{(n)}(t,\xi,\eta)\,dt\\
&= \sum_{m=0}^{|\xi\curlywedge\eta|\wedge n}\left(G_m^{(n)}(\xi) - G_{m+1}^{(n)}(\xi)\right)\frac{1}{\mu(C^m(\xi))}.
\end{aligned}$$

Proposition 3.2.

$$U_{\xi\eta}^{(n)} = \frac{1}{\mu(C^n(\eta))}\int_0^\infty \mathbb{P}_{\mu(\cdot|C^n(\xi))}\{\Xi_t \in C^n(\eta)\}\,dt \tag{3.3}$$

$$= U_{\xi(n)\eta(n)} + \frac{1}{\mu(C^n(\xi))}G_{n+1}(\xi)\,\mathbf{1}_{C^n(\xi)}(\eta) \tag{3.4}$$

$$= \begin{cases} U_{\xi\eta} & \text{if } |\xi\curlywedge\eta| < n,\\ w_n + \frac{1}{\mu(C^n(\xi))}G_{n+1}(\xi) & \text{if } |\xi\curlywedge\eta| \ge n.\end{cases} \tag{3.5}$$

Moreover, if $\sigma \in C^n(\xi)$ and $\tau \in C^n(\eta)$, then $U_{\sigma\tau}^{(n)} = U_{\xi\eta}^{(n)}$.

Proof. On one hand, it holds that

$$
\begin{aligned}
U_{\xi\eta}^{(n)} &= \sum_{m=0}^{|\xi\curlywedge\eta|\wedge n} \left(G_m^{(n)}(\xi) - G_{m+1}^{(n)}(\xi)\right) \frac{1}{\mu(C^m(\xi))} \\
&= \sum_{m=0}^{|\xi\curlywedge\eta|\wedge n} G_m^{(n)}(\xi) \left[\frac{1}{\mu(C^m(\xi))} - \frac{1}{\mu(C^{m-1}(\xi))}\right] - \frac{G^{(n)}_{|\xi\curlywedge\eta|\wedge n+1}(\xi)}{\mu(C^{|\xi\curlywedge\eta|\wedge n})(\xi)},
\end{aligned} \tag{3.6}
$$

where, by convention, we put $1/\mu(C^{-1}(\xi)) = 0$. On the other hand, using that $\mathbf{1}_{C^n(\xi)}$ is $\mathcal{F}_n$-measurable, it holds that

$$
\begin{aligned}
<W\mathbf{1}_{C^n(\xi)}, \mathbf{1}_{C^n(\eta)}>_{L^2(\mu)} &=< \sum_{m\geq 0} G_m \left[\mathbb{E}_m \mathbf{1}_{C^n(\xi)} - \mathbb{E}_{m-1}\mathbf{1}_{C^n(\xi)}\right], \mathbf{1}_{C^n(\eta)}>_{L^2(\mu)} \\
&= \sum_{m=0}^{n} <G_m \left[\mathbb{E}_m \mathbf{1}_{C^n(\xi)} - \mathbb{E}_{m-1}\mathbf{1}_{C^n(\xi)}\right], \mathbf{1}_{C^n(\eta)}>_{L^2(\mu)} \\
&= \sum_{m=0}^{n} \int_{C^n(\eta)} G_m(\sigma) \left[\mathbb{E}_m \mathbf{1}_{C^n(\xi)} - \mathbb{E}_{m-1}\mathbf{1}_{C^n(\xi)}\right](\sigma)\, \mu(d\sigma)
\end{aligned}
$$

($G_m(\sigma) = G_m(\eta)$ for $\sigma \in C^n(\eta)$ y $m \leq n$)

$$
= \sum_{m=0}^{n} G_m(\eta) < \left[\mathbb{E}_m \mathbf{1}_{C^n(\xi)} - \mathbb{E}_{m-1}\mathbf{1}_{C^n(\xi)}\right], \mathbf{1}_{C^n(\eta)}>_{L^2(\mu)} .
$$

Now $\mathbb{E}_\mu(\mathbf{1}_{C^n(\xi)}|\mathcal{F}_m) = \frac{\mu(C^n(\xi))}{\mu(C^m(\xi))}\mathbf{1}_{C^m(\xi)}$, and $C^m(\xi)\cap C^n(\eta)$ equals $C^n(\eta)$ if $m \leq |\xi\curlywedge\eta|$ and is empty otherwise. Thus

$$
\begin{aligned}
&< \left[\mathbb{E}_m \mathbf{1}_{C^n(\xi)} - \mathbb{E}_{m-1}\mathbf{1}_{C^n(\xi)}\right], \mathbf{1}_{C^n(\eta)}>_{L^2(\mu)}= \\
&\qquad \begin{cases} \mu(C^n(\xi))\mu(C^n(\eta)) \left[\frac{1}{\mu(C^m(\xi))} - \frac{1}{\mu(C^{m-1}(\xi))}\right] & \text{if } m \leq |\xi\curlywedge\eta|, \\ -\mu(C^n(\xi))\mu(C^n(\eta))/\mu(C^{|\xi\curlywedge\eta|}(\xi)) & \text{if } m = |\xi\curlywedge\eta|+1, \\ 0 & \text{if } m > |\xi\curlywedge\eta|+1 \end{cases}
\end{aligned}
$$

and then,

$$
\begin{aligned}
\frac{<W\mathbf{1}_{C^n(\xi)}, \mathbf{1}_{C^n(\eta)}>_{L^2(\mu)}}{\mu(C^n(\xi))\mu(C^n(\eta))} = \Bigg[&\sum_{m=0}^{|\xi\curlywedge\eta|\wedge n} G_m^{(n)}(\eta) \left[\frac{1}{\mu(C^m(\xi))} - \frac{1}{\mu(C^{m-1}(\xi))}\right] \\
&- G^{(n)}_{|\xi\curlywedge\eta|\wedge n+1}(\eta) \frac{1}{\mu(C^{|\xi\curlywedge\eta|\wedge n}(\xi))}\Bigg].
\end{aligned}
$$

But in the previous formula $G_m^{(n)}(\eta)$ can be replaced by $G_m^{(n)}(\xi)$ since it appears only for $m \leq |\xi\curlywedge\eta|+1$, in which case both terms coincide. Thus, this formula recovers (3.6), and then

$$
\begin{aligned}
\mu(C^n(\xi))\mu(C^n(\eta))U_{\xi\eta}^{(n)} &=<W\mathbf{1}_{C^n(\xi)}, \mathbf{1}_{C^n(\eta)}>_{L^2(\mu)}= \int_{C^n(\eta)} (W\mathbf{1}_{C^n(\xi)})(\sigma)\,\mu(d\sigma) \\
&= \int_{C^n(\eta)}\int_{C^n(\xi)} U_{\sigma\tau}\,\mu(d\tau)\,\mu(d\sigma)\,dt = \int_0^\infty \int_{C^n(\eta)}\int_{C^n(\xi)} p(t,\sigma,\tau)\,\mu(d\tau)\,\mu(d\sigma)\,dt \\
&= \int_0^\infty \int_{C^n(\xi)}\int_{C^n(\eta)} p(t,\tau,\sigma)\,\mu(d\sigma)\,\mu(d\tau)\,dt = \int_0^\infty \int_{C^n(\xi)} \mathbb{P}_\tau\{\Xi_t \in C^n(\eta)\}\,\mu(d\tau)\,dt \\
&= \mu(C^n(\xi)) \int_0^\infty \int_{\partial_\infty} \mathbb{P}_\tau\{\Xi_t \in C^n(\eta)\}\,\mu(d\tau|C^n(\xi))\,dt.
\end{aligned}
$$

Hence,

$$
U_{\xi\eta}^{(n)} = \frac{1}{\mu(C^n(\eta))} \int_0^\infty \mathbb{P}_{\mu(\cdot|C^n(\xi))}\{\Xi_t \in C^n(\eta)\}\,dt.
$$

On the other hand, from the previous calculations one obtains that

$$
U_{\xi\eta}^{(n)} = \frac{1}{\mu(C^n(\xi))}\frac{1}{\mu(C^n(\eta))} \int_{C^n(\eta)}\int_{C^n(\xi)} U_{\sigma\tau}\,\mu(d\tau)\,\mu(d\sigma).
$$

Then, if $\eta \notin C^n(\xi)$, it holds that $U^{(n)}_{\xi\eta} = U_{\xi\eta}$. Besides, if $\sigma \in C^n(\xi)$ and $\tau \in C^n(\eta)$, then $U^{(n)}_{\sigma\tau} = U^{(n)}_{\xi\eta}$. So we have only $U^{(n)}_{\xi\xi}$ left to calculate:

$$
\begin{aligned}
U^{(n)}_{\xi\xi} &= \sum_{m=0}^{n} \left(G^{(n)}_m(\xi) - G^{(n)}_{m+1}(\xi) \right) \frac{1}{\mu(C^m(\xi))} = \sum_{m=0}^{n-1} \Delta_m[w] + \frac{G_n(\xi)}{\mu(C^n(\xi))} \\
&= w_{n-1} + \frac{G_n(\xi)}{\mu(C^n(\xi))} = w_n + \frac{1}{\mu(C^n(\xi))} \left[(w_{n-1} - w_n)\mu(C^n(\xi)) + G_n(\xi) \right] \\
&= w_n + \frac{1}{\mu(C^n(\xi))} G_{n+1}(\xi),
\end{aligned}
$$

from where the formula proposed holds. □

Using the previous proposition it is easy to see that $U^{(n)}$ turns out to be an ultrametric matrix.

For the kernel $p^{(n)}$ we have a formula similar to (3.4), which is proved in an analogous way:

Proposition 3.3.

$$
p^{(n)}(t,\xi,\eta) = p(t,\xi(n),\eta(n)) + \frac{1}{\mu(C^n(\xi))} e^{-t/G_{n+1}(\xi)} \mathbf{1}_{C^n(\xi)}(\eta), \tag{3.7}
$$

where $p(t,\xi(n),\eta(n))$ is understood as the same formula as $p(t,\xi,\eta)$, save truncating the sum at n:

$$
p(t,\xi(n),\eta(n)) = \sum_{m=0}^{|\xi\curlywedge\eta|\wedge n} \frac{e^{-t/G_m(\xi)} - e^{-t/G_{m+1}(\xi)}}{\mu(C^m(\xi))}.
$$

Besides, if $\sigma \in C^n(\xi)$ and $\tau \in C^n(\eta)$, then $p^{(n)}(t,\sigma,\tau) = p^{(n)}(t,\xi,\eta)$, i.e., the values of $p^{(n)}$ are constant within the atoms of $\mathcal{F}_n$.

The following proposition describes the relation between $W^{(n)}$, $U^{(n)}$ and $P^{(n)}$, and finishes recovering theorem 2.4:

Proposition 3.4. *Let $f \in L^2(\mu)$. Then:*

$$
\begin{aligned}
W^{(n)} f(\xi) &= \int_{\partial_\infty} U^{(n)}_{\xi\eta} f(\eta)\, \mu(d\eta) && (3.8) \\
&= \int_0^\infty P^{(n)}_t f(\xi)\, dt. && (3.9)
\end{aligned}
$$

That is, $U^{(n)}$ is the Green kernel of $P^{(n)}$ and $W^{(n)}$ is its potential.

Proof.

$$
\begin{aligned}
W^{(n)} f(\xi) &= \sum_{m=0}^{n} G_m(\xi) \left[\mathbb{E}_m f(\xi) - \mathbb{E}_{m-1} f(\xi) \right] \\
&= \sum_{m=0}^{n} [G_m(\xi) - G_{m+1}(\xi)] \mathbb{E}_m f(\xi) + G_{n+1} \mathbb{E}_n f(\xi) \\
&= \sum_{m=0}^{n} \Delta_m[w] \mu(C^m(\xi)) \mathbb{E}_m f(\xi) + G_{n+1} \mathbb{E}_n f(\xi) \\
&= \sum_{m=0}^{n} \Delta_m[w] \int_{C^m(\xi)} f\, d\mu + \frac{G_{n+1}}{\mu(C^n(\xi))} \int_{C^n(\xi)} f\, d\mu \\
&= \sum_{m=0}^{n-1} w_m \int_{C^m(\xi)\setminus C^{m+1}(\xi)} f\, d\mu + w_n \int_{C^n(\xi)} f\, d\mu + \frac{G_{n+1}}{\mu(C^n(\xi))} \int_{C^n(\xi)} f\, d\mu \\
&= \sum_{m=0}^{n-1} w_m \int_{C^m(\xi)\setminus C^{m+1}(\xi)} f\, d\mu + \left[w_n + \frac{G_{n+1}}{\mu(C^n(\xi))} \right] \int_{C^n(\xi)} f\, d\mu \\
&= \int_{\partial_\infty} U^{(n)}_{\xi\eta} f(\eta)\, \mu(d\eta).
\end{aligned}
$$

The other equality results from

$$\int_0^\infty P_t^{(n)} f(\xi)\,dt = \int_{\partial_\infty}\int_0^\infty p^{(n)}(t,\xi,\eta) f(\eta)\,dt\,\mu(d\eta) = \int_{\partial_\infty} U_{\xi\eta}^{(n)} f(\eta)\,\mu(d\eta).$$

□

In what follows we will denote by $\Pi^{(n)}$ the Markov process associated to the semigroup $P^{(n)}$. Observe that μ turns out to be a quasi-stationary measure for $\Pi^{(n)}$:

$$\mathbb{P}_\mu\{\Pi_t^{(n)} \in A\} = e^{-t/G_0}\mu(A). \tag{3.10}$$

This happens also with Ξ (see [DMM04]), the proofs being identical (see [Rem04]).

3.2. **Relation between Ξ and $\Pi^{(n)}$.** To begin, observe that

$$\int_{\partial_\infty} p^{(n)}(t,\xi,\eta)\,\mu(d\eta) = (P_t^{(n)}\mathbf{1})(\xi) = e^{-t/G_0},$$

thus the lifetime of $\Pi^{(n)}$, which we will denote by $\Gamma^{(n)}$, has distribution $\exp[1/G_0]$, the same as the distribution of the lifetime Υ of Ξ.

The following proposition shows that both processes have the same distribution over the atoms of $\mathcal{F}_n$, while the subsequent theorem studies the convergence of $\Pi^{(n)}$.

Proposition 3.5. *Let $\xi,\eta \in \partial_\infty$. Then:*

$$\mathbb{P}_\xi\{\Pi_t^{(n)} \in C^n(\eta)\} = \mathbb{P}_\xi\{\Xi_t \in C^n(\eta)\}.$$

Proof. Suppose first that $|\xi \curlywedge \eta| < n$. Then:

$$\begin{aligned}\mathbb{P}_\xi\{\Pi_t^{(n)} \in C^n(\eta)\} &= \int_{C^n(\eta)} p^{(n)}(t,\xi,\sigma)\,\mu(d\sigma) = \int_{C^n(\eta)} p(t,\xi,\sigma)\,\mu(d\sigma)\\ &= \mathbb{P}_\xi\{\Xi_t \in C^n(\eta)\}.\end{aligned}$$

Now suppose that $|\xi \curlywedge \eta| \geq n$. In this case, $C^n(\xi) = C^n(\eta)$, thus it is enough to restrict the proof to the case $\xi = \eta$:

$$\begin{aligned}\mathbb{P}_\xi\{\Pi_t^{(n)} \in C^n(\xi)\} &= 1 - \mathbb{P}_\xi\{\Pi_t^{(n)} \in \partial_\infty \backslash C^n(\xi)\} - \mathbb{P}_\xi\{\Gamma^{(n)} \leq t\}\\ &= 1 - \mathbb{P}_\xi\{\Xi_t \in \partial_\infty\backslash C^n(\xi)\} - (1 - e^{-t/G_0})\\ &= e^{-t/G_0} - \mathbb{P}_\xi\{\Xi_t \in \partial_\infty\backslash C^n(\xi)\} = \mathbb{P}_\xi\{\Upsilon > t\} - \mathbb{P}_\xi\{\Xi_t \in \partial_\infty\backslash C^n(\xi)\}\\ &= \mathbb{P}_\xi\{\Xi_t \in C^n(\xi)\}.\end{aligned}$$

□

Observe that, by the previous proposition, the following is also true:

$$\mathbb{P}_\xi\{\Pi_t^{(n)} \in C^m(\eta)\} = \mathbb{P}_\xi\{\Xi_t \in C^m(\eta)\} \qquad \text{for } 0 \leq m \leq n.$$

Theorem 3.6. *The sequence of processes $(\Pi^{(n)})_{n\geq 0}$ converges in distribution to Ξ,*

$$\Pi^{(n)} \xrightarrow[n\to\infty]{dist.} \Xi.$$

Proof. Since ∂_∞ is compact, it suffices to prove the convergence of the finite-dimensional laws of $\Pi^{(n)}$ to those of Ξ. That is, we must prove that, given $0 < t_1 < \cdots < t_k < \infty$ and $A_1,\ldots,A_k \in \mathcal{F}_\infty$, it holds that:

$$\mathbb{P}_\xi\{\Pi_{t_i}^{(n)} \in A_i,\, i = 1,\ldots,k\} \xrightarrow[n\to\infty]{} \mathbb{P}_\xi\{\Xi_{t_i} \in A_i,\, i=1,\ldots,k\}.$$

It is well known that in this context we can restrict the proof of the preceding property to atoms $A_1,\ldots,A_k$ of the generating algebra $\bigcup_{n=0}^\infty \mathcal{F}_n$ (see, for example, [Bil68, Theorem 2.2]). Moreover, we can restrict the proof to the case in which every atom is defined on the same level (otherwise it suffices to work in the lowest level in which an atom is defined and decompose the other atoms with respect to that level). We will only present the case $k = 2$, the rest being analogous. Observe that $A_1 = A_2$ or $A_1 \cap A_2 = \varnothing$.

We first assume that $\xi \notin A_1$. If $A_1 \cap A_2 = \varnothing$, we have that

$$\mathbb{P}_\xi\{\Pi_{t_1}^{(n)} \in A_1, \Pi_{t_2}^{(n)} \in A_2\} = \int_{A_2}\int_{A_1} p^{(n)}(t_1,\xi,\xi_1)p^{(n)}(t_2 - t_1,\xi_1,\xi_2)\,\mu(d\xi_1)\mu(d\xi_2),$$

and the fact that the terms $p^{(n)}(t_1,\xi,\xi_1)$ and $p^{(n)}(t_2-t_1,\xi_1,\xi_2)$ are constant for n sufficiently big, and thus equal to the corresponding terms replacing $p^{(n)}$ with p, implies that the convergence follows. If $A_1=A_2$ we define $\mathcal{C}_1^{(n)}=\{\partial_\infty(i),\, i\in B^n\}\backslash A_1$, the family of atoms on the level n except A_1, so

$$\begin{aligned}\mathbb{P}_\xi\{\Pi^{(n)}_{t_1}\in A_1,\Pi^{(n)}_{t_2}\in A_1\} &= \mathbb{P}_\xi\{\Pi^{(n)}_{t_2}\in A_1\}-\sum_{A_1'\in\mathcal{C}_1^{(n)}}\mathbb{P}_\xi\{\Pi^{(n)}_{t_1}\in A_1',\Pi^{(n)}_{t_2}\in A_1\}\\ &= \mathbb{P}_\xi\{\Xi_{t_2}\in A_1\}-\sum_{A_1'\in\mathcal{C}_1^{(n)}}\mathbb{P}_\xi\{\Xi_{t_1}\in A_1',\Xi_{t_2}\in A_1\}\\ &= \mathbb{P}_\xi\{\Xi_{t_1}\in A_1,\Xi_{t_2}\in A_1\},\end{aligned}$$

due to the previous case and proposition 3.5.

Now if $\xi\in A_1$, and for any pair A_1,A_2, we have

$$\begin{aligned}\mathbb{P}_\xi\{\Pi^{(n)}_{t_1}\in A_1,\Pi^{(n)}_{t_2}\in A_2\} &= \mathbb{P}_\xi\{\Pi^{(n)}_{t_2}\in A_2\}-\sum_{A_1'\in\mathcal{C}_1^{(n)}}\mathbb{P}_\xi\{\Pi^{(n)}_{t_1}\in A_1',\Pi^{(n)}_{t_2}\in A_2\}\\ &= \mathbb{P}_\xi\{\Xi_{t_1}\in A_1,\Xi_{t_2}\in A_2\},\end{aligned}$$

using the case $\xi\notin A_1$.

□

In this way, the process $\Pi^{(n)}$ approximates the original process Ξ.

Observe that we also have that

$$U^{(n)}\xrightarrow[n\to\infty]{} U\quad \mu\text{-a.s.}\qquad\text{and}\qquad p^{(n)}\xrightarrow[n\to\infty]{} p\quad \mu\text{-a.s.}$$

This follows from (3.4), (3.7) and the fact that $\frac{G_{n+1}}{\mu(C^n(\xi))}$ and $\frac{e^{-t/G_{n+1}}}{\mu(C^n(\xi))}$ both converge to 0 if $\mu(\{\xi\})>0$.

3.3. **Interpretation of the process $\Pi^{(n)}$.** We will start by giving an interpretation for the potential associated to $\Pi^{(n)}$. Remember that the Green kernel of $\Pi^{(n)}$ is given by

$$U^{(n)}_{\xi\eta}=U_{\xi(n)\eta(n)}+\frac{G_{n+1}(\xi)}{\mu(C^n(\xi))}\mathbf{1}_{C^n(\xi)}(\eta)$$

(see (3.4)). The potential of $\Pi^{(n)}$ corresponds to $W^{(n)}$. From (3.8) and (3.9) we obtain that, given $\xi,\eta\in\partial_\infty$,

$$\begin{aligned}W^{(n)}\mathbf{1}_{C^n(\eta)}(\xi) &= [\textit{total time spent by } \Pi^{(n)} \textit{ in } C^n(\eta)\textit{, starting from } \xi]\\ &= \int_{C^n(\eta)}U^{(n)}_{\xi\sigma}\,\mu(d\sigma)\\ &= U^{(n)}_{\xi\eta}\mu(C^n(\eta)).\end{aligned}\tag{3.11}$$

Observe that, except in the case $\xi=\eta$, it holds that

$$U^{(n)}_{\xi\eta}\xrightarrow[n\to\infty]{} U_{\xi\eta}.$$

In fact, when ξ and η are in different atoms of $\mathcal{F}_n$ (that is, if $|\xi\curlywedge\eta|<n$), it holds that $U^{(n)}_{\xi\eta}=U_{\xi\eta}$, and then $W^{(n)}\mathbf{1}_{C^n(\eta)}(\xi)=W\mathbf{1}_{C^n(\eta)}(\xi)$. In the opposite case, $|\xi\curlywedge\eta|\geq n$, we have $\mu(C^n(\eta))=\mu(C^n(\xi))$, so

$$\begin{aligned}U^{(n)}_{\xi\eta}\mu(C^n(\eta)) &= \mu(C^n(\xi))w_n+G_{n+1}(\xi)\\ &= \mu(C^n(\xi))w_n+(G_{n+1}(\xi)-G_n(\xi))+G_n(\xi)\\ &= \mu(C^n(\xi))\,(w_n-\Delta_n[w])+G_n(\xi)\\ &= \mu(C^n(\xi))w_{n-1}+G_n(\xi).\end{aligned}\tag{3.12}$$

To give an interpretation for this term we introduce the *shifted process*, which we will denote by $\overline{\Xi}^{(n)}$, and that lives in the subtree that is born of $\xi(n)$ (see figure 3). To each level of this subtree we associate the same value as before, that is, to the level m of the subtree we associate the coefficient $\overline{w}^{(n)}_m=w_{n+m}$ for $n\geq 0$ and we put $\overline{w}^{(n)}_{-1}=0$. In this way, the matrix $\overline{U}^{(n)}$ corresponding to this restriction is simply the restriction of

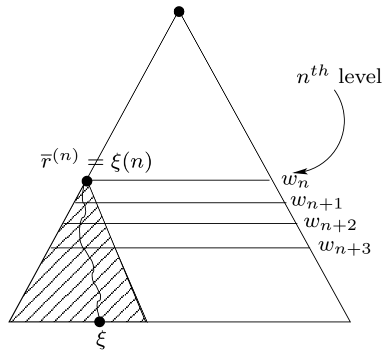

FIGURE 3. Shifted process, that corresponds to the process defined over the restriction of the whole problem to $[\xi(n),\infty)$.

the matrix U to $[\xi(n),\infty]$. We must consider the measure μ conditioned to the atom corresponding to the subtree, that is, $\overline{\mu}^{(n)}(\cdot)=\mu(\cdot|C^n(\xi))$. The associated operator turns out to be

$$\overline{W}^{(n)}f(\xi)=\int_{C^n(\xi)}U_{\xi\eta}f(\eta)\,\overline{\mu}^{(n)}(d\eta).$$

The resulting process is analogous to the original process Ξ. Observe that for $m\ge 1$ it holds that $\overline{G}^{(n)}_{n+m}(\xi)=G_m(\xi)/\mu(C^n(\xi))$. Nevertheless, this is not true for $m=0$ (since in the shifted process we put $\overline{w}^{(n)}_{-1}$ as 0 instead of w_{n-1}):

$$\begin{aligned}\overline{G}^{(n)}_0&=\overline{W}^{(n)}\mathbf{1}=\int_{C^n(\xi)}U_{\xi\eta}\,\overline{\mu}^{(n)}(d\eta)\\&=\sum_{m\ge n}\int_{C^m(\xi)\setminus C^{m+1}(\xi)}U_{\xi\eta}\,\overline{\mu}^{(n)}(d\eta)=\sum_{m\ge n}w_m\left(\overline{\mu}^{(n)}(C^m(\xi))-\overline{\mu}^{(n)}(C^{m+1}(\xi))\right)\\&=\frac{1}{\mu(C^n(\xi))}\left[\sum_{m\ge n}\Delta_m[w]\mu(C^m(\xi))+w_{n-1}\mu(C^n(\xi))\right].\end{aligned}$$

Thus,

$$\overline{G}^{(n)}_0=w_{n-1}+\frac{G_n(\xi)}{\mu(C^n(\xi))}. \tag{3.13}$$

Recall that $\overline{G}^{(n)}_0$ corresponds to the mean of the lifetime of the process $\Xi^{(n)}$, which we will denote by $\overline{\Upsilon}^{(n)}$. Using (3.12) and (3.13), we have obtained that

$$W^{(n)}\mathbf{1}_{C^n(\eta)}(\xi)=\mu(C^n(\xi))\mathbb{E}(\overline{\Upsilon}^{(n)}). \tag{3.14}$$

This means that the mean time spent by $\Pi^{(n)}$ in $C^n(\eta)$ when starting from a point $\xi\in C^n(\eta)$ corresponds to the mean of the lifetime of the shifted process $\overline{\Xi}^{(n)}$, normalized by the measure of the relevant atom. This result shows that the measure μ contains a big part of the global information of the process Ξ. Observe that the process $\Pi^{(n)}$ is constructed by truncating W to the space associated to the σ-algebra $\mathcal{F}_n$, that is, forgetting everything that happens in the tree after the n^{th} level, while $\overline{\Xi}^{(n)}$ is constructed by shifting all the problem to make it start from the n^{th} level, that is, forgetting everything before the n^{th} level. The only global information that both processes share, the measure μ, is enough for their behavior to be related.

Regarding the transition probabilities of the truncated process $\Pi^{(n)}$ it is possible to give a very precise interpretation of it in terms of the process Ξ. The equality

$$U^{(n)}_{\xi\eta}=\int_0^\infty p^{(n)}(t,\xi,\eta)\,dt=\frac{1}{\mu(C^n(\eta))}\int_0^\infty\mathbb{P}_{\mu(\cdot|C^n(\xi))}\{\Xi_t\in C^n(\eta)\}\,dt$$

gives an idea that is shown to be correct by the following theorem:

Theorem 3.7. *Given $\xi,\eta\in\partial_\infty$, it holds that*

$$p^{(n)}(t,\xi,\eta)=\frac{1}{\mu(C^n(\eta))}\mathbb{P}_{\mu(\cdot|C^n(\xi))}\{\Xi_t\in C^n(\eta)\}.$$

Proof.

$$\begin{aligned}\mathbb{P}_{\mu(\cdot|C^n(\xi))}\{\Xi_t \in C^n(\eta)\} &= \frac{1}{\mu(C^n(\xi))}\int_{C^n(\xi)} \mathbb{P}_\sigma\{\Xi_t \in C^n(\eta)\}\,\mu(d\sigma) \\ &= \frac{1}{\mu(C^n(\xi))}\int_{C^n(\xi)} \mathbb{P}_\sigma\{\Pi^{(n)}_t \in C^n(\eta)\}\,\mu(d\sigma)\end{aligned}$$

(but $\sigma \in C^n(\xi) \Rightarrow \mathbb{P}_\sigma\{\Pi^{(n)}_t \in C^n(\eta)\} = \mathbb{P}_\xi\{\Pi^{(n)}_t \in C^n(\eta)\}$)

$$= \mathbb{P}_\xi\{\Pi^{(n)}_t \in C^n(\eta)\} = \int_{C^n(\eta)} p^{(n)}(t,\xi,\sigma)\,\mu(d\sigma)$$

(and using again that $p^{(n)}(t,\xi,\sigma)$ is constant for σ located in the same atom of $\mathcal{F}_n$)

$$= \mu(C^n(\eta))p^{(n)}(t,\xi,\eta).$$

□

To end this part we will show that the distribution of the exit time of $\Pi^{(n)}$ from an atom $C^m(\xi)$ for $m \le n$ and $\xi \in \partial_\infty$ is equal to that of Ξ. To prove this it will be necessary to replicate for $\Pi^{(n)}$ the way of constructing a copy of Ξ given in [DMM04].

Given $n \ge 0$, fix $m \le n$, $\xi^* \in \partial^r_\infty$ and consider the subtree that is born of $\xi^*(m)$. Define the conditional measure ${}^m\mu = \mu(\cdot\,|\,C^m(\xi^*))$ and the weight function given by

$${}^m w_{-1} = 0 \text{ and } \Delta_k[{}^m w] = \mu(C^m(\xi^*))\Delta_{k+m}[w] \text{ for } k \ge 1.$$

Define the level function ${}_m|\cdot| = |\cdot| - m$ and the atoms ${}^mC^n(\cdot) = C^{n+m}(\cdot)$. With these definitions, ${}^mG_k = G_{k+m}$. We further define

$${}^mG^{(n)}_k = \begin{cases} {}^mG_k & \text{if } k+m \le n \\ 0 & \text{if } k+m > n. \end{cases}$$

Then ${}^mG^{(n)}_k = G^{(n)}_{k+m}$. The associated operator in this case is given by

$${}^mW^{(n)} = \sum_{k=0}^{n} {}^mG_k\left[\mathbb{E}_{{}^m\mu}(\cdot\,|\,\mathcal{F}_k) - \mathbb{E}_{{}^m\mu}(\cdot\,|\,\mathcal{F}_{k-1})\right],$$

and the kernel by

$${}^mp^{(n)}(t,\xi,\eta) = \sum_{k=0}^{|\xi\curlywedge\eta|\wedge(n-m)} \frac{e^{-t/{}^mG^{(n)}_k(\xi)} - e^{-t/{}^mG^{(n)}_{k+1}(\xi)}}{{}^m\mu({}^mC^k(\xi))}.$$

The process induced by the generator $-({}^mW^{(n)})^{-1}$ will be denoted by ${}^m\Pi^{(n)}$. Its lifetime satisfies ${}^m\Gamma^{(n)} \sim \exp[1/G_m]$.

Theorem 3.8. *Fix $n \ge 1$. Let $m \le n$ and $\xi \in \partial^r_\infty$. Let $(B_1,\dots,B_m)$ be a vector of independent random variables with $B_k \sim Ber[1 - G_k(\xi)/G_{k+1}(\xi)]$. Then, under the measure $\mathbb{P}_\xi$, the Markov process defined by*

$$\left[{}^m\widetilde{\Pi}^{(n)}_t\right]^\xi = \begin{cases} \left[{}^m\Pi^{(n)}_t\right]^\xi & \text{if } t < {}^m\Gamma^{(n)}, \\ \dagger & \text{if } t \ge {}^m\Gamma^{(n)} \text{ and } B_k = 0 \text{ for } 1 \le k \le m, \\ \left[{}^k\widetilde{\Pi}^{(n)}_{t-{}^m\Gamma^{(n)}}\right]^{{}^k\mu} & \text{if } t \ge {}^m\Gamma^{(n)} \text{ and } B_{k+1} = 1,\ B_p = 0 \text{ for } k+1 < p \le m, \end{cases} \tag{3.15}$$

is a copy ${\Pi^{(n)}}^\xi$.

Proof. Using the definitions and formulas for $\Pi^{(n)}$, the proof is identical to that of the construction of ${}^n\Xi$ in [DMM04] (see [Rem04] for more details). □

Proposition 3.9. *The exit time of $\Pi^{(n)}$, starting from a point $\xi^* \in \partial_\infty$, from any atom associated to a level $m \le n$, satisfies $\mathcal{R}^{(n)}_m \sim exp[\beta_m(\xi^*)]$, where*

$$\beta_m(\xi^*) = \mu(C^m(\xi^*))\left[\frac{1}{G_0(\xi^*)} + \sum_{k=1}^{m} \frac{\mu(C^{k-1}(\xi^*)\backslash C^k(\xi^*))}{\mu(C^k(\xi^*))\mu(C^{k-1}(\xi^*))G_k(\xi^*)}\right].$$

Hence that exit time has the same distribution as that of Ξ. (See [DMM04]*).*

Proof. The same proof given for the exit time of Ξ in [DMM04] works in this case. $\square$

4. A Different Construction of the Process $\Pi^{(n)}$

The definition of the approximated process $\Pi^{(n)}$ could have been also done considering it as defined in B^n. This gives another Markov process, which we will denote by $Y^{(n)}$, that satisfies a set of properties analogous to those of $\Pi^{(n)}$. The elements used in this construction are:

$$U_{ij}^{(n)} = U_{ij} \quad \text{for } i,j \in B^n,$$
$$W_{ij}^{(n)} = U_{ij}^{(n)} \mu(\partial_\infty(j)) \quad \text{for } i,j \in B^n,$$
$$\text{and } p^{(n)}(t,i,j) = \sum_{m=0}^{|i \curlywedge j|} \left(e^{-t/G_m^{(n)}(i)} - e^{-t/G_{m+1}^{(n)}(i)} \right) \frac{\mu(\partial_\infty(j))}{\mu(\partial_\infty(i(m)))} \quad \text{for } i,j \in B^n,\ t>0,$$

where in the last line $G_m^{(n)}(i) = G_m^{(n)}(\xi)$ for any $\xi \in \partial_\infty(i)$. (See [Rem04] for details).

In this section we will show how to obtain $\Pi^{(n)}$ from $Y^{(n)}$, allowing us to conclude the understanding of our approximation. To do this we will construct another Markov process in ∂_∞, starting from $Y^{(n)}$, with the same finite-dimensional laws of $\Pi^{(n)}$.

Denote by $T_1, T_2, \ldots$ the jump times of $Y^{(n)}$ and by $Z^{(n)}$ its skeleton. Suppose that $\Pi^{(n)}$ starts from $\xi \in \partial_\infty$, so we make $Y^{(n)}$ start from $\xi(n)$. Denote by u_m a random variable uniformly distributed according to μ in $\partial_\infty(Z_m^{(n)})$. Define the process $\widehat{\Pi^{(n)}}$ by

$$\widehat{\Pi^{(n)}}_t = \begin{cases} \xi & \text{if } t < T_1, \\ u_m & \text{if } T_m \le t < T_{m+1}. \end{cases}$$

To show that $\widehat{\Pi^{(n)}}$ is a Markov process we will prove that for every pair of measurable and bounded functions F and G it holds that

$$\mathbb{E}(F(X_{t_{m+1}})G(X_{t_m},\ldots,X_{t_1})) = \mathbb{E}(\mathbb{E}(F(X_{t_{m+1}})|X_{t_m})G(X_{t_m},\ldots,X_{t_1})).$$

We can restrict the proof to F and G of the form $\mathbf{1}_A$, $A \in \mathcal{F}_\infty$. Then we must show that, given $A_1,\ldots,A_{m+1} \in \mathcal{F}_\infty$,

$$\begin{aligned}
&\mathbb{E}_\xi(\widehat{\Pi^{(n)}}_{t_1} \in A_1,\ldots,\widehat{\Pi^{(n)}}_{t_{m+1}} \in A_{m+1}) \\
&\qquad = \mathbb{E}_\xi(\widehat{\Pi^{(n)}}_{t_1} \in A_1,\ldots,\widehat{\Pi^{(n)}}_{t_m} \in A_m\, \mathbb{E}_{\widehat{\Pi^{(n)}}_{t_m}}(\widehat{\Pi^{(n)}}_{t_{m+1}-t_m} \in A_{m+1})) \\
&\qquad = \mathbb{E}_\xi(\widehat{\Pi^{(n)}}_{t_1} \in A_1,\ldots,\widehat{\Pi^{(n)}}_{t_m} \in A_m)\mathbb{E}_{\widehat{\Pi^{(n)}}_{t_m}}(\widehat{\Pi^{(n)}}_{t_{m+1}-t_m} \in A_{m+1}).
\end{aligned}$$

We can restrict the proof further to the case when $A_1,\ldots,A_{m+1}$ are each contained in one atom of $\mathcal{F}_n$. Denote by $\partial_\infty(A_m)$ the atom of $\mathcal{F}_n$ that contains A_m and by $i(A_m)$ the corresponding node in B^n.

$$\begin{aligned}
&\mathbb{P}_\xi\{\widehat{\Pi^{(n)}}_{t_1} \in A_1,\ldots,\widehat{\Pi^{(n)}}_{t_{m+1}} \in A_{m+1}\} \\
&\qquad = \mathbb{P}_{\xi(n)}\{Y_{t_1}^{(n)} = i(A_1),\ldots,Y_{t_{m+1}}^{(n)} = i(A_{m+1})\} \frac{\mu(A_1)}{\mu(\partial_\infty(A_1))} \cdots \frac{\mu(A_{m+1})}{\mu(\partial_\infty(A_{m+1}))} \\
&\qquad = \mathbb{P}_{\xi(n)}\{Y_{t_1}^{(n)} = i(A_1),\ldots,Y_{t_m}^{(n)} = i(A_m)\}\mathbb{P}_{Y_{t_m}^{(n)}}\{Y_{t_{m+1}-t_m}^{(n)} = i(A_{m+1})\} \\
&\qquad\qquad \frac{\mu(A_1)}{\mu(\partial_\infty(A_1))} \cdots \frac{\mu(A_{m+1})}{\mu(\partial_\infty(A_{m+1}))} \\
&\qquad = \mathbb{P}_{\xi(n)}\{\widehat{\Pi^{(n)}}_{t_1} \in A_1,\ldots,\widehat{\Pi^{(n)}}_{t_m} \in A_m\}\mathbb{P}_{Y_{t_m}^{(n)}}\{\widehat{\Pi^{(n)}}_{t_{m+1}-t_m} \in A_{m+1}\}.
\end{aligned}$$

Thus, $\widehat{\Pi^{(n)}}$ is a Markov process.

We must prove now that the finite-dimensional laws of $\widehat{\Pi^{(n)}}$ and $\Pi^{(n)}$ coincide. Due to the compactness of ∂_∞ we may restrict ourselves to prove the equality of the one-dimensional laws. Again we restrict the proof to sets contained in some atom of $\mathcal{F}_n$.

$$\mathbb{P}_\xi\{\widehat{\Pi^{(n)}}_t \in A\} = \mathbb{P}_{\xi(n)}\{Y_t^{(n)} = i(A)\}\frac{\mu(A)}{\mu(\partial_\infty(A))} = p^{(n)}(t,\xi(n),i(A))\frac{\mu(A)}{\mu(\partial_\infty(A))}$$

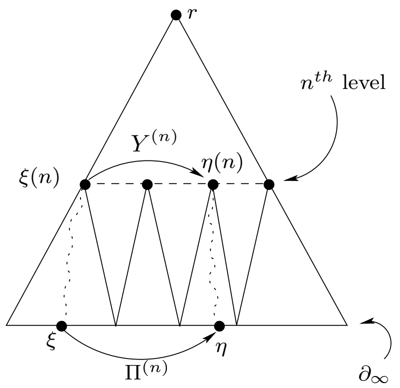

FIGURE 4. Sketch of the process $\Pi^{(n)}$ constructed from $Y^{(n)}$.

$$
\begin{aligned}
&= \sum_{m=0}^{|\xi(n)\curlywedge i(A)|} \left(e^{-t/G_m^{(n)}(\xi)} - e^{-t/G_{m+1}^{(n)}(\xi)}\right) \frac{\mu(\partial_\infty(i(A)))}{\mu(C^n(\xi))} \frac{\mu(A)}{\mu(\partial_\infty(A))} \\
&= \int_A p^{(n)}(t,\xi,\eta)\,\mu(d\eta) \\
&= \mathbb{P}_\xi\{\Pi_t^{(n)} \in A\}.
\end{aligned}
$$

We have proven the following theorem:

Theorem 4.1. *The process $\Pi^{(n)}$ coincides with $\widehat{\Pi^{(n)}}$. In other words, $\Pi^{(n)}$ corresponds to the process in ∂_∞ that jumps between the atoms of $\mathcal{F}_n$ according to the jumps of the process $Y^{(n)}$, and that upon the arrival to each atom chooses uniformly a point where it stays until the next jump (see figure 4).*

From the preceding theorem it follows directly the following corollary that finishes the description of $\Pi^{(n)}$:

Corollary 4.2. *The trajectories of the process $\Pi^{(n)}$ do not move within the atoms of $\mathcal{F}_n$, they only make jumps between different atoms. In other words,*

$$
\mathbb{P}_\xi\{\Pi_t^{(n)} \neq \xi \,|\, t < T_{\partial_\infty \setminus C^n(\xi)}\} = 0,
$$

where $T_{\partial_\infty \setminus C^n(\xi)}$ is the exit time from $C^n(\xi)$.

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