The Annals of Applied Probability
2008, Vol. 18, No. 6, 2392–2420
DOI: 10.1214/08-AAP528
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THE CONTACT PROCESS IN A DYNAMIC RANDOM ENVIRONMENT

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We study a contact process running in a random environment in $\mathbb{Z}^d$ where sites flip, independently of each other, between blocking and nonblocking states, and the contact process is restricted to live in the space given by nonblocked sites. We give a partial description of the phase diagram of the process, showing in particular that, depending on the flip rates of the environment, survival of the contact process may or may not be possible for large values of the birth rate. We prove block conditions for the process that parallel the ones for the ordinary contact process and use these to conclude that the critical process dies out and that the complete convergence theorem holds in the supercritical case.

1. Introduction. We consider the following version of a contact process running in a dynamic random environment in $\mathbb{Z}^d$. The state of the process is represented by some $\eta \in \mathcal{X} = \{-1, 0, 1\}^{\mathbb{Z}^d}$, where sites in state 0 are regarded as vacant, sites in state 1 as occupied and sites in state -1 as blocked (that is, no births of 1's are allowed on that site). The process η_t is defined by the following transition rates:

$$\begin{array}{rcll} 0 & \longrightarrow & 1 & \text{at rate } \beta f_1 \\ 1 & \longrightarrow & 0 & \text{at rate } 1 \\ 0, 1 & \longrightarrow & -1 & \text{at rate } \alpha \\ -1 & \longrightarrow & 0 & \text{at rate } \alpha\delta \end{array}$$

where f_1 is the fraction of occupied neighbors at L^1 distance 1.

In words, the -1's define a random environment in which each site becomes blocked at rate α and flips back to being unblocked at rate $\alpha\delta$, while the 1's behave like a nearest neighbor contact process with birth rate β in the space left unblocked by the environment. Observe that when an occupied site becomes blocked, the particle is killed. This version is simpler than the alternative in which only 0's can turn to -1's (mainly because our process satisfies a self-duality relation, see Proposition 2.2). However, we feel that our choice is natural: If a site becomes uninhabitable, the particles living there will soon die.

Ever since it was introduced in Harris (1974), the contact process has been object of intensive study, and many extensions and modifications of the process have

Received February 2008; revised February 2008.
AMS 2000 subject classification. 60K35.
Key words and phrases. Contact process, random environment, complete convergence, block construction, interacting particle system.

been considered. In particular, the literature includes several different versions of contact processes in random environments. One class of these processes corresponds to contact processes where the birth and death rates are not homogeneous in space, and they are chosen according to some probability distribution, independently across sites, and remain fixed in time [see, e.g., Bramson, Durrett and Schonmann (1991), Liggett (1992), Andjel (1992) and Klein (1994)]. The main question for this class of processes is to determine conditions on the parameters that guarantee or preclude survival.

A different class of models, which are somehow closer to the process we consider, have two species with different parameters or ranges, but one of them behaves independently of the other while the second is restricted to live in the space left by the first. These processes were studied in Durrett and Swindle (1991), Durrett and Møller (1991) and Durrett and Schinazi (1993). The results in these papers (mainly bounds on critical parameters for coexistence and complete convergence theorems) are asymptotic, in the sense that they are proved when the range of one or both types is sufficiently large.

The process we consider differs from both of the types of examples mentioned above: The random environment is dynamic and it behaves independently across sites. An example of a spin system running in this type of environment was studied in Luo (1992), and corresponds to the Richardson model which would result from ignoring transitions from 1 to 0 in our process. Another example was studied recently in Broman (2007), where the author considers a process in which the environment changes the death rate of the contact process instead of blocking sites. The dynamics of the process $\Psi_{\delta_0,\delta_1}^{\gamma,p,A}$ introduced there are the same as those of our process if $\delta_1 = \infty$. The author considers this case as a tool in the study of his process, but the results of the paper focus on the case $\delta_1 < \infty$. We will use one of his results to give a bound on a part of the phase diagram of our process in Theorem 1.

As mentioned above, the -1's evolve independently of the 1's. They follow an "independent flip process" whose equilibrium is given by the product measure

$$\mu_\rho(\{\eta : \eta(x) = -1\}) = 1 - \mu_\rho(\{\eta : \eta(x) \neq -1\}) = \rho = \frac{1}{1+\delta} \qquad \forall x \in \mathbb{Z}^d.$$

This process is reversible, and its reversible measure is given by μ_ρ.

In Section 2.1, we will construct our process using the so-called graphical representation. A direct consequence of the construction will be that η_t satisfies some monotonicity properties analogous to those of the contact process. (Here and in the rest of the paper, when we refer to the contact process we mean the "ordinary" nearest-neighbor contact process in $\mathbb{Z}^d$.) We consider the following partial order on configurations:

$$\eta^1 \leq \eta^2 \quad \Leftrightarrow \quad \eta^1(x) \leq \eta^2(x) \qquad \forall x \in \mathbb{Z}^d. \tag{1.1}$$

With this order, our process has the following property: Given two initial states $\eta_0^1 \le \eta_0^2$, it is possible to couple two copies of the process η_t^1 and η_t^2 with these initial conditions in such a way that $\eta_t^1 \le \eta_t^2$ for all $t \ge 0$. We will refer to this property as *attractiveness* by analogy with the case of spin systems [this property is sometimes termed *monotonicity*, see Sections II.2 and III.2 of Liggett (1985) for a discussion of general monotone processes and of attractive spin systems, resp.].

For $A \subseteq \mathbb{Z}^d$, we define the following probability measure ν_A on $\mathcal{X}$: -1's are chosen first according to their equilibrium measure μ_ρ and then 1's are placed at every site in A that is not blocked by a -1. These measures are the initial conditions for η_t that are suitable for duality.

Let $\underline{\nu} = \nu_\varnothing$, which corresponds to having the -1's at equilibrium and no 1's. Let also $\overline{\nu}$ be the limit distribution of the process when starting at the configuration having all sites at state 1, which is obviously the largest configuration in the partial order (1.1). We will show in Proposition 2.1 that this limit is well defined and it is stationary, and that $\underline{\nu}$ and $\overline{\nu}$ are, respectively, the *lower* and *upper invariant measure* of the process (i.e., the smallest and largest stationary distribution of the process).

We will say that the process *survives* if there is an invariant measure ν such that

$$\nu\big(\{\eta : \eta(x) = 1 \text{ for some } x \in \mathbb{Z}^d\}\big) > 0,$$

or, equivalently, if $\underline{\nu} \ne \overline{\nu}$ (we remark that, as a consequence of Theorem 2, every invariant measure for the process is translation invariant, so the above probability is actually 1 whenever it is positive). Otherwise, we will say that the process *dies out*. We will see in Section 4 that this definition of survival is equivalent to the following condition: The process started with a single 1 at the origin and everything else at -1 contains 1's at all times with positive probability.

A second monotonicity property that will follow from the construction of η_t is monotonicity with respect to the parameters β and δ:

(i) If α and δ are fixed, and for some $\beta > 0$ the process survives, then the process also survives for any $\beta' > \beta$.

(ii) If α and β are fixed, and for some $\delta > 0$ the process survives, then the process also survives for any $\delta' > \delta$.

These properties follow easily from standard coupling arguments. We will denote by $\beta_c = \beta_c(\alpha, \delta) \in [0, \infty]$ the parameter value such that, fixing these α and δ, η_t survives for $\beta > \beta_c$ and dies out for $\beta < \beta_c$. We define $\delta_c = \delta_c(\alpha, \beta)$ analogously.

Our first result provides some bounds on the critical parameters for survival. Let β_c^{cp} be the critical value of the contact process in $\mathbb{Z}^d$ [here we are taking the birth rate β to be the total birth rate from each site, so each site sends births to each given neighbor at rate $\beta/(2d)$].

THEOREM 1.

(a) *If $\beta \le (\alpha + 1)\beta_c^{\mathrm{cp}}$, then the process dies out.*

(b) *There exists a* $\delta_p > 0$ *such that for any* $\delta < \delta_p$ *the process dies out* (*regardless of* α *and* β).

(c) *Let*

$$\overline{\lambda}(\alpha, \beta, \delta) = \frac{1}{2}\left[\beta + \alpha(1+\delta) - \sqrt{(\beta - \alpha(1+\delta))^2 + 4\alpha\beta}\right].$$

If $\overline{\lambda}(\alpha, \beta, \delta) > (\alpha+1)\beta_c^{\mathrm{cp}}$, *then the process survives.*

Part (a) of the theorem is trivial because the 1's die at rate $\alpha + 1$. For part (b), observe that if the complement of the set of sites at state -1 does not space-time percolate, then each 1 in the process is doomed to live in a finite space-time region, and then the process cannot have 1's at all times when started with finitely many occupied sites. We will show by adapting arguments in Meester and Roy (1996) that, with probability 1, no such space-time percolation occurs if δ is small enough. For part (c), we will use Broman's result to obtain a suitable coupling with a contact process with birth rate $\overline{\lambda}(\alpha, \beta, \delta)$ and death rate $\alpha + 1$.

In particular, Theorem 1 implies that if δ is large enough then $\beta_c(\alpha, \delta) < \infty$, and in fact $\delta > \frac{\alpha+1}{\alpha}\beta_c^{\mathrm{cp}}$ is enough. To see this, observe that

$$\lim_{\beta\to\infty} \overline{\lambda}(\alpha, \beta, \delta) = \alpha\delta > (\alpha+1)\beta_c^{\mathrm{cp}}$$

whenever the above condition on δ holds. Then part (c) of the theorem implies that the process survives for these choices of α and δ and large enough β. Another consequence is that $\delta_p \le \beta_c^{\mathrm{cp}}$. Indeed, if $\delta > \beta_c^{\mathrm{cp}}$, then $\delta > \frac{\alpha+1}{\alpha}\beta_c^{\mathrm{cp}}$ for large enough α, and the previous property implies that the process survives for these choices of α and δ and large enough β.

A significant difficulty in giving a more complete picture of the phase diagram of η_t is that we lack a result about monotonicity with respect to α analogous to the properties (i) and (ii) (monotonicity with respect to β and δ) mentioned above. Observe that the equilibrium density of nonblocked sites is independent of α, but the environment changes more quickly as α increases. Simulations suggest that if β and δ are given and the process dies out at some parameter value α, then it also dies out for any parameter value $\alpha' > \alpha$ (note that part (a) of Theorem 1 says that the process dies out at least for all α large enough). But the usual simple arguments based on coupling do not work in this case, since increasing α increases both the rate at which sites are blocked, which plays against survival, and the rate at which sites are unblocked, which plays in favor of survival, and we have not been able to find an alternative proof.

The second part of our study of η_t investigates the convergence of the process and the structure of its limit distributions. For $\eta \in \mathcal{X}$, we will write $\eta = (A, B)$, where

$$A = \{x \in \mathbb{Z}^d : \eta(x) = 1\} \quad \text{and} \quad B = \{x \in \mathbb{Z}^d : \eta(x) = -1\}.$$

$\eta_t^\mu = (A_t^\mu, B_t^\mu)$ will denote the process with initial distribution μ, and we will refer to B_t^μ (or B_t if no initial distribution is prescribed) as the *environment process*. Observe that the dynamics of the environment process are independent of the 1's in η_t.

THEOREM 2. *Denote by* $\tau = \inf\{t \geq 0 : A_t = \varnothing\}$ *the extinction time of the process. Then for every initial distribution* μ,

$$\eta_t^\mu \Longrightarrow \mathbb{P}^\mu(\tau < \infty)\underline{\nu} + \mathbb{P}^\mu(\tau = \infty)\overline{\nu},$$

where the limit is in the topology of weak convergence of probability measures.

This result, which is usually called a *complete convergence theorem*, implies that all limit distributions are convex combinations of $\underline{\nu}$ and $\overline{\nu}$. Thus, the only interesting nontrivial stationary distribution is $\overline{\nu}$.

The proof of Theorem 2 relies on extending for η_t the classical block construction for the contact process introduced in Bezuidenhout and Grimmett (1990), so that we are able to use the proof of complete convergence for the contact process to prove the corresponding convergence of the contact process part of η_t. As a consequence of this construction, we will obtain, just as for the contact process, the fact that the process dies out at the critical parameters β_c and δ_c (see Corollary 4.4). The arguments involved in this part will depend heavily on a duality relation which will be developed in Section 2.2.

The rest of the paper is devoted to the proofs of the two theorems. Section 2 describes the construction of η_t and presents some basic preliminary results. Theorem 1 is proved in Section 3. In Section 4 we obtain the block conditions for the survival of the process. Finally, in Section 5 we use duality and the conditions obtained in Section 4 to prove Theorem 2.

2. Preliminaries.

2.1. *Graphical representation and monotonicity.* The graphical representation is one of the basic and most useful tools in the study of the contact process and other interacting particle systems. It will allow us to construct our process from a collection of independent Poisson processes and obtain a single probability space in which copies of the process with arbitrary initial states can be coupled. We will give a rather informal description of this construction, which can be made precise by adapting the arguments of Harris (1972). We refer the reader to Section III.6 of Liggett (1985) for more details on this construction in the case of an additive spin system.

The construction is done by placing symbols in $\mathbb{Z}^d \times [0, \infty)$ to represent the different events in the process. For each ordered pair $x, y \in \mathbb{Z}^d$ at distance 1, let $N^{x,y}$ be a Poisson process with rate $\beta/(2d)$, and take the processes assigned to

different pairs to be independent. At each event time t of $N^{x,y}$, draw an arrow $\xrightarrow{1}$ in $\mathbb{Z}^d \times [0,\infty)$ from (x,t) to (y,t) to indicate the birth of a 1 sent from x to y (which will only take place if x is occupied and y is vacant at time t). Similarly, define a family of independent Poisson processes $(U^{1,x})_{x\in\mathbb{Z}^d}$ with rate 1 and for each event time t of $U^{1,x}$ place a symbol $*_1$ at (x,t) to indicate that a 1 flips to 0 (i.e., that a particle dies). To represent the environment, consider two families of independent Poisson processes $(V^x)_{x\in\mathbb{Z}^d}$ and $(U^{-1,x})_{x\in\mathbb{Z}^d}$ with rates α and $\alpha\delta$, respectively. For each event time t of V^x, place a symbol $\bullet_{-1}$ at (x,t) to indicate the birth of a -1 (i.e., the blocking of a site) and for each event time t of $U^{-1,x}$, place a symbol $*_{-1}$ to indicate that a -1 flips to 0 (i.e., the unblocking of a site).

We construct η_t from this percolation structure in the following way. Consider a deterministic initial condition η_0 and define the environment process B_t by setting $\eta_t(x) = -1$ when (x,t) lies between symbols $\bullet_{-1}$ and $*_{-1}$ (in that order) in the time line $\{x\} \times [0,\infty)$, and also if $\eta_0(x) = -1$ and there is no symbol $*_{-1}$ in that time line before time t. Having defined B_t, we say that there is an *active path* between (x,s) and (y,t) if there is a connected oriented path, moving along the time lines in the increasing direction of time and passing along arrows $\xrightarrow{1}$, which crosses neither symbols $*_1$ nor space-time points that were set to -1. The collection of active paths corresponds to the possible space-time paths along which 1's can move, so we define A_t by

$$A_t = \{y \in \mathbb{Z}^d : \exists x \in A_0 \text{ with an active path from } (x,0) \text{ to } (y,t)\}.$$

The arguments of Harris (1972) imply that this construction gives a well-defined Markov process with the right transition rates. Moreover, the same realization of this graphical representation can be used for different initial conditions, and this gives the coupling mentioned above [see Section III.6 in Liggett (1985) for more details on this coupling in the case of a spin system]. For the rest of the paper, we will implicitly use this "canonical" coupling every time we couple copies of η_t with different initial conditions. The attractiveness property mentioned in the Introduction follows directly from this construction, and the monotonicity properties with respect to β and δ can be obtained by a simple modification of this coupling (analogous to what is done for the contact process).

Recall the definition of the partial order on configurations given in (1.1). Clearly,

$$\eta^1 \le \eta^2 \quad \Leftrightarrow \quad A^1 \subseteq A^2 \text{ and } B^1 \supseteq B^2.$$

For probability measures on $\mathcal{X}$, which we endow with the product topology, we consider the usual ordering: $\mu_1 \le \mu_2$ if and only if $\int f\,d\mu_1 \le \int f\,d\mu_2$ for every continuous increasing $f : \mathcal{X} \longrightarrow \mathbb{R}$. We recall that the property $\mu_1 \le \mu_2$ is equivalent to the existence of a probability space in which a pair of random variables X_1 and X_2 with distributions μ_1 and μ_2 can be coupled in such a way that $X_1 \le X_2$ almost surely [see Theorem II.2.4 in Liggett (1985)]. We will use this fact repeatedly, and for simplicity we will say that X_2 *dominates* X_1 when this condition holds. We

will also use this term to compare two processes, so saying that η_t^2 dominates η_t^1 will mean that the two processes can be constructed in a single probability space in such a way that $\eta_t^1 \le \eta_t^2$ for all $t \ge 0$.

The attractiveness property allows us to obtain the lower and upper invariant measure of the process.

PROPOSITION 2.1. *Let $\chi_{\mathbb{Z}^d}$ be the probability distribution on $\mathcal{X}$ assigning mass* 1 *to the all* 1*'s configuration, and let $S(t)$ be the semigroup associated to the process. Define*

$$\overline{\nu} = \lim_{t\to\infty} \chi_{\mathbb{Z}^d} S(t),$$

where the limit is in the topology of weak convergence of probability measures. Then $\overline{\nu}$ is the upper invariant measure of the process, that is, $\overline{\nu}$ is invariant and every other invariant measure is stochastically smaller than $\overline{\nu}$. Moreover,

$$\overline{\nu} = \lim_{t\to\infty} \nu_{\mathbb{Z}^d} S(t).$$

Analogously,

$$\underline{\nu} = \nu_{\varnothing}$$

is the lower invariant measure of the process.

PROOF. Since μ_ρ is invariant for the environment and the empty state is a trap for the 1's, $\underline{\nu}$ is invariant. It is the lower invariant measure because every invariant measure has μ_ρ as its projection onto the environment, and $\nu_\varnothing$ is the smallest probability measure on $\mathcal{X}$ having μ_ρ as its marginal on the -1's.

For the other part, standard arguments imply that the limit defining $\overline{\nu}$ exists and is invariant [see, e.g., Sections I.1 and III.2 in Liggett (1985)]. Since $\chi_{\mathbb{Z}^d}$ is larger than any other measure on $\mathcal{X}$, it follows by attractiveness that $\overline{\nu}$ is the largest invariant measure.

Now let $\nu^* = \lim_{t\to\infty} \nu_{\mathbb{Z}^d} S(t)$. As above, ν^* is well defined and invariant, so to prove that $\nu^* = \overline{\nu}$ it is enough to prove that ν^* is larger than any other invariant measure. If ν is any invariant measure, its projection onto the -1's must be μ_ρ, so for any continuous increasing f,

$$\begin{aligned}\int f\,d\nu &= \mathbb{E}^\nu(f(\eta_0)) = \mathbb{E}^\nu(f(\eta_t)) \\ &\le \mathbb{E}^{\nu_{\mathbb{Z}^d}}(f(\eta_t)) = \int f\,d[\nu_{\mathbb{Z}^d} S(t)] \underset{t\to\infty}{\longrightarrow} \int f\,d\nu^*. \qquad \square\end{aligned}$$

2.2. *Duality.* The dual process $(\hat{\eta}_s^t)_{0\le s\le t} = (\hat{A}_s^t, \hat{B}_s^t)_{0\le s\le t}$ is constructed using the same graphical representation we used for constructing η_t. Our duality relation will require that the process be started with the environment at equilibrium. The

dual processes will also be started with measures of the form ν_C, for $C \subseteq \mathbb{Z}^d$, and the dual process started with this distribution will be denoted by $(\hat{\eta}_s^{\nu_C,t})_{0\le s\le t}$.

Fix $t > 0$, and start by choosing B_0 according to μ_ρ. Then run the environment process forward in time until t, using the graphical representation. This defines $(B_s)_{0\le s\le t}$. The dual environment is given by $\hat{B}_s^t = B_{t-s}$. Now place a 1 at time t at every site $x \in C \setminus \hat{B}_0^t$, that is, every site in C which is not blocked by the environment at time t. This defines $\hat{A}_0^{\nu_C,t}$, and by the stationarity of the environment process we get an initial condition $(\hat{A}_0^{\nu_C,t}, \hat{B}_0^t)$ for the dual chosen according to ν_C. Having defined $\hat{A}_0^{\nu_C,t}$ and $(\hat{B}_s^t)_{0\le s\le t}$, we define the 1-dual by

$$\hat{A}_s^{\nu_C,t} = \{y \in \mathbb{Z}^d : \exists x \in \hat{A}_0^{\nu_C,t} \text{ with an active path from } (y, t-s) \text{ to } (x,t)\}.$$

That is, the 1-dual is defined by running the contact process for the 1's backwards in time and with the direction of the arrows reversed. An active path in η_t from $(y, t-s)$ to (x,t) will be called a *dual active path* from (x,t) to $(y, t-s)$ in the dual process.

We could have defined the dual by simply choosing a random configuration at time t according to ν_C and then running the whole process backward. The idea of the preceding construction is to allow coupling the process and its dual in the same graphical representation in such a way that the initial state of the environment for η_s is the same as the final state of the environment for $\hat{\eta}_s^t$ (that is, $B_0 = \hat{B}_t^t$). This allows us to obtain the following duality result:

PROPOSITION 2.2. *For any $A, C, D \subseteq \mathbb{Z}^d$,*

$$\mathbb{P}^{\nu_A}(A_t \cap C \neq \varnothing, B_t \cap D \neq \varnothing) = \mathbb{P}^{\nu_C}(\hat{A}_t^t \cap A \neq \varnothing, \hat{B}_0^t \cap D \neq \varnothing). \tag{2.1}$$

Moreover, η_t satisfies the following self-duality relation: if A or C is finite, then

$$\mathbb{P}^{\nu_A}(A_t \cap C \neq \varnothing, B_t \cap D \neq \varnothing) = \mathbb{P}^{\nu_C}(A_t \cap A \neq \varnothing, B_0 \cap D \neq \varnothing). \tag{2.2}$$

PROOF. The first equality follows directly from coupling the process and its dual using the same realization of the graphical representation. Indeed, if we use this coupling then, by definition,

$$\mathbb{P}(\hat{B}_s^{\nu_C,t} = B_{t-s}^{\nu_A} \text{ for every } 0 \le s \le t) = 1.$$

Calling $\mathcal{E}$ the σ-algebra generated by the environment process, observe that our construction implies that

$$\mathbb{P}^{\nu_A}(A_t \cap C \neq \varnothing | \mathcal{E}) = \mathbb{P}^{\nu_C}(\hat{A}_t^t \cap A \neq \varnothing | \mathcal{E}).$$

Therefore,

$$\begin{aligned}
\mathbb{P}^{\nu_A}(A_t \cap C \neq \varnothing, B_t \cap D \neq \varnothing) &= \mathbb{E}^{\nu_A}\big(\mathbb{P}(A_t \cap C \neq \varnothing | \mathcal{E}), B_t \cap D \neq \varnothing\big) \\
&= \mathbb{E}^{\nu_C}\big(\mathbb{P}(\hat{A}_t^t \cap A \neq \varnothing | \mathcal{E}), \hat{B}_0^t \cap D \neq \varnothing\big) \\
&= \mathbb{P}^{\nu_C}(\hat{A}_t^t \cap A \neq \varnothing, \hat{B}_0^t \cap D \neq \varnothing).
\end{aligned}$$

Equation (2.2) is obtained from (2.1), the self-duality of the contact process, and the reversibility of the environment. $\square$

Taking A finite and $C = D = \mathbb{Z}^d$ in (2.2) and using the monotonicity of the event $\{A_t \neq \varnothing\}$ in t we obtain the following:

$$\mathbb{P}^{\nu_A}(A_t \neq \varnothing \ \forall t \geq 0) = \overline{\nu}(\{(E, F) : E \cap A \neq \varnothing\}).$$

Since $\overline{\nu}$ is translation invariant, the right-hand side of this equality is positive if and only if $A \neq \varnothing$ and η_t survives, that is, $\overline{\nu} \neq \underline{\nu}$. As a consequence, we deduce that the following condition is equivalent to the survival of the process:

(S1) For any (or, equivalently, some) finite $A \subseteq \mathbb{Z}^d$ with $A \neq \varnothing$, the process started at ν_A contains 1's for every $t \geq 0$ with positive probability.

2.3. *Positive correlations*. A second property that is central to the study of the contact process is *positive correlations*. Recall that a probability measure μ has positive correlations if for every f, g increasing,

$$\int fg \, d\mu \geq \int f \, d\mu \int g \, d\mu. \tag{2.3}$$

In the following lemma we prove a version of positive correlations for $\eta_t^{\nu_A}$ with respect to cylinder functions.

LEMMA 2.3. *Let f, g be increasing real-valued functions on $\mathcal{X}$ depending on finitely many coordinates. Then if μ_t denotes the distribution of $\eta_t^{\nu_A}$, (2.3) holds with $\mu = \mu_t$, that is,*

$$\mathbb{E}^{\nu_A}(f(\eta_t)g(\eta_t)) \geq \mathbb{E}^{\nu_A}(f(\eta_t))\mathbb{E}^{\nu_A}(g(\eta_t)). \tag{2.4}$$

The same inequality holds if ν_A is replaced by any deterministic initial condition.

PROOF. Since f and g depend on finitely many coordinates and every jump in our process is between states which are comparable in the partial order (1.1), a result of Harris [see Theorem II.2.14 in Liggett (1985)] and attractiveness imply that it is enough to show that the initial distribution of the process has positive correlations in the sense of the lemma. The result with ν_A replaced by a deterministic initial condition readily follows.

To show that ν_A is positively correlated, consider the process ς_t defined in $\mathcal{X}$ by $\varsigma_0 \equiv 1$ and independent transitions at each site given by

$$\left.\begin{array}{r} 0 \longrightarrow -1 \text{ at rate } \rho \\ -1 \longrightarrow \ \ 0 \text{ at rate } 1-\rho \end{array}\right\} \quad \text{for } x \notin A,$$

$$\left.\begin{array}{r} 1 \longrightarrow -1 \text{ at rate } \rho \\ -1 \longrightarrow \ \ 1 \text{ at rate } 1-\rho \end{array}\right\} \quad \text{for } x \in A.$$

It is clear that ς_t converges weakly to the measure ν_A. Since the initial distribution of ς_t has positive correlations (because it is deterministic), (2.3) holds for its limit ν_A, using again Harris' result. $\square$

3. Survival and extinction. In this section, we prove Theorem 1. Throughout the proof, we will implicitly use (S1) to characterize survival. We start with the easy part.

PROOF OF THEOREM 1, PART (A). Consider the process $\widetilde{\eta}_t$ defined by the following transition rates:

$$\begin{aligned} 0, -1 &\longrightarrow 1 \text{ at rate } \beta f_1 \\ 1 &\longrightarrow 0 \text{ at rate } 1 \\ 0, 1 &\longrightarrow -1 \text{ at rate } \alpha \\ -1 &\longrightarrow 0 \text{ at rate } \alpha\delta. \end{aligned}$$

This process corresponds to modifying η_t by ignoring the effect of blocked sites on births. It is easy to couple $\widetilde{\eta}_t$ and η_t using the graphical representation in such a way that if the initial states are the same, $\eta_t \le \widetilde{\eta}_t$ for all $t \ge 0$. Therefore, it is enough to show that $\widetilde{\eta}_t$ dies out, and this follows directly from the hypothesis because the 1's in $\widetilde{\eta}_t$ behave just like a contact process with birth rate β and death rate $\alpha + 1$. $\square$

The proof of part (b) is more involved, and it is based on adapting the techniques of Boolean models in continuum percolation [see Meester and Roy (1996)].

PROOF OF THEOREM 1, PART (B). The idea is to show that when δ is small, the set of unblocked sites in the environment process B_t does not "space-time percolate" with probability 1. By this we mean that there is no infinite path in $\mathbb{Z}^d \times [0, \infty)$ moving between nearest-neighbor sites in $\mathbb{Z}^d$ and along time lines in the increasing direction of time that uses only nonblocked sites. The conclusion follows directly from this fact, since in that case, every 1 will live in a finite space-time box, so it will not be able to contribute to the survival of the process.

By a simple time change, we can consider the environment process as having transitions given by

$$\begin{aligned} -1 &\longrightarrow 0 \text{ at rate } q \\ 0 &\longrightarrow -1 \text{ at rate } 1 - q, \end{aligned}$$

where $q = \delta/(1+\delta) \longrightarrow 0$ as $\delta \longrightarrow 0$. We still consider this process as defined by the graphical representation, though now the symbols $\bullet_{-1}$ and $*_{-1}$ appear at rate $1 - q$ and q, respectively.

Take the percolation structure given by the graphical representation and draw for every symbol $*_{-1}$ at a space-time point (x, t) a box of base $x + [-2/3, 2/3]^d$ spanning the interval in the time coordinate from t until the time corresponding

to the next symbol $\bullet_{-1}$ (i.e., these boxes span intervals where the sites are not blocked). Then since the environment process is translation invariant, the 0's will almost surely not space-time percolate if and only if

$$\mathbb{P}(|\mathcal{W}| = \infty) = 0, \tag{3.1}$$

where $\mathcal{W}$ denotes the connected component of the union of the boxes that contains the origin at time 0, and $|\mathcal{W}|$ denotes the number of boxes that form this cluster.

To prove (3.1), we compare this continuum percolation structure with a multitype branching process $X = (X_{n,i})_{n,i\in\mathbb{N}}$. The first step in the comparison is to stretch all the boxes so that their heights are all integer-valued. It is enough to show that (3.1) holds after this modification, since increasing the heights of the boxes increases the probability of space-time percolation of the unblocked sites. Assume that the origin is not blocked at time 0, and call $i_0 \in \mathbb{N}$ the (random) height of its associated box. For simplicity, assume further that all the neighbors of the origin are blocked at time 0, the extension to the general case being straightforward. We start defining X by saying that the 0th generation has only one member, and it is of type i_0 (i.e., $X_{0,j} = \mathbf{1}_{\{j=i_0\}}$). The box containing the origin at time 0 is possibly intersected by boxes placed at the $2d$ neighbors of the origin, and these boxes will constitute the children of the initial member: we let $X_{1,j}$ be the number of boxes of height j that intersect the original box. We define the subsequent generations of X inductively: $X_{n+1,j}$ is the number of boxes of height j that intersect boxes of the nth generation and which have not been counted up to generation $n-1$. Now let

$$X^\infty = \sum_{n=0}^{\infty}\sum_{i=1}^{\infty} X_{n,i},$$

and observe that every box in $\mathcal{W}$ is counted in X^∞, so

$$|\mathcal{W}| \le X^\infty \tag{3.2}$$

(recall that X is constructed from the stretched boxes).

Our goal is to show that $\mathbb{E}(X^\infty) < \infty$. To achieve this, we will couple X with another multitype branching process $Y = (Y_{n,i})_{n,i\in\mathbb{N}}$, which we define below. The details of this part can be adapted easily from the proof of Theorem 3.2 in Meester and Roy (1996), so we will only sketch the main ideas. Consider a box of height i based at $[x-2/3, x+2/3]^d \times \{t\}$, which we will denote by $B(x,t,i)$. The boxes of height j that intersect this box must all have bases of the form $[y-2/3, y+2/3] \times \{s\}$ for some y at distance 1 of x and some $s \in (0 \vee (t-j), t+i]$. The number of symbols $*_{-1}$ appearing in the piece $\{y\} \times (0 \vee (t-j), t+i]$ of the graphical representation above a given neighbor y of x is a Poisson random variable with mean $q[t+i-0\vee(t-j)] \le q[i+j]$, and each of these symbols corresponds to a box that intersects $B(x,t,i)$. Since the probability that any one of these (stretched) boxes is of height j is $p_j = \mathbb{P}(Z \in (j-1, j])$, where Z is an exponential random

variable with rate $1-q$, we deduce that the number of children of $B(x,t,i)$ of height j is a Poisson random variable with mean bounded by

$$2dp_jq[i+j] \le 4dqijp_j, \tag{3.3}$$

where we used the fact that $i+j \le 2ij$ for positive integers i and j. Now let Y be a multitype branching process where the number of children of type j of each individual of type i is a Poisson random variable with mean $4dqijp_j$ ($Y_{n,i}$ is the number of individuals of type i in generation n). Then a coupling argument and (3.3) imply that if $X_{0,i} = Y_{0,i}$ for all $i \ge 1$ then $X_{n,i}$ is dominated by $Y_{n,i}$ for each $n \ge 0$ and $i \ge 1$, and thus

$$\mathbb{E}(X^\infty | X_{0,k} = \mathbf{1}_{\{k=i_0\}}) \le \mathbb{E}\left(\sum_{n=0}^{\infty}\sum_{j=1}^{\infty} Y_{n,j} | Y_{0,k} = \mathbf{1}_{\{k=i_0\}}\right). \tag{3.4}$$

To bound this last sum, we recall a standard result in branching processes theory [see, e.g., Chapter V in Athreya and Ney (1972)]: the expected number of individuals of type j in the nth generation of Y when starting with one individual of type i_0 is given by

$$\mathbb{E}\big(Y_{n,j} | Y_{0,k} = \mathbf{1}_{\{k=i_0\}}\big) = (M^n)_{i_0,j}, \tag{3.5}$$

where M is the infinite matrix indexed by $\mathbb{N}$ with $M_{i,j}$ being the expected number of children of type j of an individual of type i. By definition of Y, $M_{i,j} = 4dqijp_j$, and from this we get inductively a bound for $(M^n)_{i_0,j}$:

$$(M^n)_{i_0,j} \le (4dq)^n i_0 \mathbb{E}(H^2)^{n-1}\mathbb{P}(H=j)j$$

for all $n \ge 1$, where H is a random variable with positive integer values and distribution given by $\mathbb{P}(H=j) = p_j$. Using this together with (3.4) and (3.5) gives

$$\begin{aligned}\mathbb{E}(X^\infty | X_{0,k} = \mathbf{1}_{k=i_0}) &\le 1 + i_0 \sum_{n=1}^{\infty}\left((4dq)^n \mathbb{E}(H^2)^{n-1}\sum_{j=1}^{\infty} p_j j\right) \\ &= 1 + 4dqi_0\mathbb{E}(H)\sum_{n=0}^{\infty}(4dq\mathbb{E}(H^2))^n.\end{aligned} \tag{3.6}$$

Observe that H is dominated by $Z+1$, so $\mathbb{E}(H^2) \le \frac{2(2-q)}{(1-q)^2} + 1$. Hence,

$$4dq\mathbb{E}(H^2) \le 4d\left(\frac{2q(2-q)}{(1-q)^2} + q\right) < 1 \tag{3.7}$$

for sufficiently small q, and then the last sum in (3.6) converges for such q. This implies by (3.2) that $\mathbb{E}(|\mathcal{W}|) < \infty$, so $\mathbb{P}(|\mathcal{W}| = \infty) = 0$. $\square$

Using (3.7) we can get explicit lower bounds for δ_p, but these turn out to be rather small (around 0.02 for $d=2$ and 0.01 for $d=3$).

Before proving the last part of Theorem 1, we need to introduce a result from Broman (2007). Let (J_t, X_t) be the process with state space $\{0,1\} \times \mathbb{N}$ defined as follows. J_0 is a Bernoulli random variable with $\mathbb{P}(J_0 = 1) = 1 - \mathbb{P}(J_0 = 0) = p$, and $X_0 = 0$. The evolution of the process is given by the following transition rates:

$$\text{for } J_t\colon \begin{cases} 0 \longrightarrow 1 \text{ at rate } \gamma p \\ 1 \longrightarrow 0 \text{ at rate } \gamma(1-p) \end{cases}$$
$$\text{for } X_t\colon k \longrightarrow k+1 \text{ at rate } \sigma_0(1-J_t) + \sigma_1 J_t$$

where $\gamma, \sigma_1 > 0$ and $0 \le \sigma_0 \le \sigma_1$. In words, J_t acts as the environment, starting at equilibrium and then flipping between states 0 and 1 independently of X_t, while X_t is a sort of Poisson process where the rate depends on J_t. The next lemma recovers the part of Theorem 1.4 in Broman (2007) that is relevant for our purposes. We observe that the original theorem is stated for $\sigma_0 > 0$, but the same proof works if $\sigma_0 = 0$.

LEMMA 3.1. *Let*

$$\overline{\sigma} = \tfrac{1}{2}\big[\sigma_0 + \sigma_1 + \gamma - \sqrt{(\sigma_1 - \sigma_0 - \gamma)^2 + 4\gamma(1-p)(\sigma_1 - \sigma_0)}\big]. \tag{3.8}$$

Then a Poisson process $N_t(\overline{\sigma})$ with rate $\overline{\sigma}$ can be coupled with (J_t, X_t) in such a way that if $N_t(\overline{\sigma})$ has an arrival at time T, then so does X_t. Moreover, $\overline{\sigma}$ is the largest rate such that this coupling is possible.

Recall that we denote

$$\overline{\lambda}(\alpha, \beta, \delta) = \tfrac{1}{2}\big[\beta + \alpha(1+\delta) - \sqrt{\big(\beta - \alpha(1+\delta)\big)^2 + 4\alpha\beta}\big].$$

The following result gives the coupling that we need to prove part (c) of Theorem 1. Its proof is very similar to that of Theorem 1.7 in Broman (2007); we include here a version based in the graphical representation.

PROPOSITION 3.2. *Let ξ_t denote the set of occupied sites of a contact process with birth rate $\overline{\lambda} = \overline{\lambda}(\alpha, \beta, \delta)$ and death rate $\alpha + 1$. Then the processes η_t and ξ_t can be coupled in such a way that if $\xi_0 \subseteq A_0$, then $\xi_t \subseteq A_t$ for all $t > 0$.*

PROOF. Consider the graphical representation used to construct η_t. Each time line defines an independent copy of the process J_t introduced above by identifying symbols $\bullet_{-1}$ and $*_{-1}$ with J_t flipping to 0 and 1, respectively, and setting $\gamma = \alpha(1+\delta)$ and $p = \delta/(1+\delta)$. Now consider the collection of arrows emanating from that time line ignoring arrows born at times where the site is blocked. By construction, this collection of arrows defines the arrival times of the process X_t associated to J_t, with $\sigma_0 = 0$ and $\sigma_1 = \beta$. By Lemma 3.1, we can construct a Poisson process $N_t(\overline{\lambda})$ [where $\overline{\lambda}$ comes from plugging in our parameters in (3.8)]

such that if this process has an arrival at time T, then there is an arrow at that time for η_t.

We repeat this construction at each time line, getting an i.i.d. collection of Poisson processes $(N_t^x(\overline{\lambda}))_{x\in\mathbb{Z}^d}$, and use this collection of processes and the graphical representation of η_t to construct the graphical representation of ξ_t: for each arrival time of $N_t^x(\overline{\lambda})$ put an arrow at that time from x to the site pointed by the corresponding arrow in the graphical representation of η_t, and for each symbol $*_1$ and each symbol $\bullet_{-1}$ for η_t put a death symbol for ξ_t. It is easy to see that this construction gives a graphical representation for the desired contact process ξ_t. Moreover, since only the arrows at nonblocked sites can carry births of 1's for η_t, the construction gives a coupling that satisfies the desired monotonicity property. These facts can be checked exactly as in the proof of Theorem 1.7 of Broman (2007) (there the processes Y_t and Y_t' correspond to A_t and ξ_t). $\square$

The proof of the remaining part of Theorem 1 is now straightforward.

PROOF OF THEOREM 1, PART (C). Since $\frac{\overline{\lambda}(\alpha,\beta,\delta)}{\alpha+1} > \lambda_c$ implies that the contact process ξ_t with birth rate $\overline{\lambda}(\alpha,\beta,\delta)$ and death rate $\alpha+1$ survives, the coupling achieved in Proposition 3.2 gives the survival of η_t. $\square$

4. Block construction. The aim of this section is to establish "block conditions" concerning the process in a finite space-time box that guarantee survival. This was first done in Bezuidenhout and Grimmett (1990). Here we will follow closely Section I.2 of Liggett (1999), together with the corrections to the book that can be found in the author's website.

Before getting started with the block construction, we need to obtain the equivalent condition for survival mentioned in the Introduction, which says that η_t survives if and only if the following condition holds:

(S2) The process started with a single 1 at the origin and everything else at -1 contains 1's at all times with positive probability.

The sufficiency of this condition is a consequence of (S1) and attractiveness. The necessity will be a consequence of the following stronger result, which is precisely what we will need in the proof of Lemma 4.2 below. Let χ_A denote the probability measure on $\mathcal{X}$ that assigns mass 1 to the configuration η with $\eta|_A \equiv 1$, $\eta|_{A^c} \equiv -1$.

LEMMA 4.1. *Suppose that the process survives. Then for any $\sigma > 0$ there is a positive integer n such that*

$$\mathbb{P}^{\chi_{[-n,n]^d}}(A_t \neq \varnothing\ \forall t \geq 0) > 1 - \sigma^2.$$

To obtain (S2) from this result observe that the process started with a single 1 at the origin has $[-n,n]^d$ fully occupied by time 1 with some positive probability,

so we can use the strong Markov property and attractiveness to restart the process at time 1 starting from $\chi_{[-n,n]^d}$ and obtain $\mathbb{P}^{\chi_{\{0\}}}(A_t \neq \varnothing\ \forall t \geq 0) > 0$. Observe that the lemma is a simple consequence of duality when the initial condition for η_t is $\nu_{[-n,n]^d}$ instead of $\chi_{[-n,n]^d}$. Indeed, using (2.2) with $D = \mathbb{Z}^d$ gives

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}^{\nu_{[-n,n]^d}}(A_t \neq \varnothing\ \forall t \geq 0) &= \lim_{n\to\infty}\lim_{t\to\infty} \mathbb{P}^{\nu_{[-n,n]^d}}(A_t \neq \varnothing) \\ &= \lim_{n\to\infty}\lim_{t\to\infty} \mathbb{P}^{\nu_{\mathbb{Z}^d}}(A_t \cap [-n,n]^d \neq \varnothing) \\ &= \lim_{n\to\infty} \overline{\nu}\big(\{(E,F) : E \cap [-n,n]^d \neq \varnothing\}\big) \\ &= \overline{\nu}\big(\{(E,F) : E \neq \varnothing\}\big).\end{aligned}$$

This last probability is 1 when η_t survives, so in this case given any $\varepsilon > 0$ we can choose a positive integer m such that

$$\mathbb{P}^{\nu_{[-m,m]^d}}(A_t \neq \varnothing\ \forall t \geq 0) > 1 - \varepsilon. \tag{4.1}$$

Recall that in Proposition 2.1 we showed that the limit distributions of the processes started at $\chi_{\mathbb{Z}^d}$ and at $\nu_{\mathbb{Z}^d}$ are the same. It is then reasonable to expect that the asymptotic behavior as $t \to \infty$ of the process started at $\chi_{[-n,n]^d}$ is similar to that of the process started at $\nu_{[-n,n]^d}$, at least for large enough n. This idea will allow us to derive the lemma from (4.1).

PROOF OF LEMMA 4.1. Let $\varepsilon > 0$ and choose m to be the positive integer obtained in (4.1). To extend this inequality to the process started at $\chi_{[-n,n]^d}$, we will consider two copies of the process η_t^1 and η_t^2 coupled using the graphical representation, with η_t^1 started at $\nu_{[-m,m]^d}$ and η_t^2 at $\chi_{[-n,n]^d}$ for some large $n > m$. For simplicity, we will write $Q(k) = [-k,k]^d$.

We want to obtain a space-time cone growing linearly in time such that $\bigcup_{t\geq 0}\{t\} \times A_t^{\nu_{Q(m)}}$ is contained in that cone with high probability. To achieve this, we compare $A_t^{\nu_{Q(m)}}$ with a branching random walk Z_t with branching rate $\beta/(2d)$ and no deaths [i.e., each particle in Z_t gives birth to a new particle at each neighbor at rate $\beta/(2d)$, and multiple particles per site are allowed]. Let $\{p_t(x,y)\}_{x,y\in\mathbb{Z}^d}$ be the transition probabilities of a simple random walk in $\mathbb{Z}^d$ that moves to each neighbor at rate $\beta/(2d)$ and let C_t be the set-valued process given by

$$C_t = \{x \in \mathbb{Z}^d : Z_t(x) > 0\}.$$

For $D \subseteq \mathbb{Z}^d$, Z_t^D and C_t^D will denote the processes started with all sites in D occupied by one particle and no particles outside D. It is not hard to see that for any $t > 0$ and any $x \in \mathbb{Z}^d$,

$$\mathbb{E}\big(Z_t^{\{0\}}(x)\big) = e^{\beta t} p_t(0,x)$$

[see, e.g., the proof of Proposition I.1.21 in Liggett (1999)]. Therefore, for any $D \subseteq \mathbb{Z}^d$,

$$\mathbb{E}\big(|C_t^{\{0\}} \cap D^{\mathrm{c}}|\big) \le \sum_{x \notin D} \mathbb{E}\big(Z_t^{\{0\}}(x)\big) = e^{\beta t} \sum_{x \notin D} p_t(0, x).$$

From this we get that if $k > m$ and $c > 0$ then

$$\mathbb{E}\big(|C_t^{Q(m)} \cap Q(k+ct)^{\mathrm{c}}|\big) \le (2m+1)^d e^{\beta t} \sum_{\|x\|_\infty > k-m+ct} p_t(0, x). \tag{4.2}$$

Now if X_t is the one dimensional random walk starting at 0 and moving to each neighbor at rate $\beta/(2d)$, Chebyshev's inequality gives

$$\begin{aligned}\mathbb{P}(|X_t| > k - m + ct) &= 2\mathbb{P}(X_t - k + m - ct > 0)\\ &\le 2\mathbb{E}(e^{X_t - k + m - ct}) = 2e^{-(k-m)} e^{-ct + (\beta/2d)(e + e^{-1} - 2)t}.\end{aligned}$$

The last equality can be obtained by seeing X_t as the difference between two independent Poisson random variables, each with mean $(\beta t)/(2d)$, and using the fact that the moment generating function of a Poisson random variable Y with mean λ is $\mathbb{E}(e^{sY}) = e^{\lambda(e^s - 1)}$. Applying this bound to each coordinate of the d-dimensional walk, we get that

$$\begin{aligned}\sum_{\|x\|_\infty > k-m+ct} p_t(0, x) &\le d\mathbb{P}(|X_t| > k - m + ct)\\ &\le 2de^{-(k-m)} e^{-ct + (\beta/2d)(e + e^{-1} - 2)t},\end{aligned}$$

and then using (4.2), we deduce that c can be taken large enough so that

$$\mathbb{E}\big(|C_t^{Q(m)} \cap Q(k+ct)^{\mathrm{c}}|\big) \le 2d(2m+1)^d e^{-(k-m)} e^{-t}.$$

Observe that, by the definition of Z_t, the process $A_t^{\nu_{Q(m)}}$ is dominated by $C_t^{Q(m)}$, so the last bound implies that

$$\begin{aligned}\mathbb{E}\left(\int_0^\infty |A_t^{\nu_{Q(m)}} \cap Q(k+ct)^{\mathrm{c}}|\, dt\right) &\le \int_0^\infty \mathbb{E}\big(|C_t^{Q(m)} \cap Q(k+ct)^{\mathrm{c}}|\big)\, dt\\ &\le 2d(2m+1)^d e^{-(k-m)}.\end{aligned} \tag{4.3}$$

We can use this inequality to estimate the probability that $A_t \subseteq Q(k+1+ct)$ for all $t \ge 0$. Observe that if $x \in A_t \cap Q(k+1+ct)^{\mathrm{c}}$, the particle at x survives at least until time $t + 2/c$ with probability $e^{-2\alpha(1+\delta)/c}$, and thus $x \in A_s \cap Q(k+cs)^{\mathrm{c}}$ for all $s \in [t + 1/c, t + 2/c]$ with at least that probability. We deduce that

$$\begin{aligned}&\mathbb{E}^{\nu_{Q(m)}}\left(\int_0^\infty |A_t \cap Q(k+ct)^{\mathrm{c}}|\, dt\right)\\ &\qquad \ge \mathbb{P}^{\nu_{Q(m)}}\big(A_t \cap Q(k+1+ct)^{\mathrm{c}} \ne \varnothing \text{ for some } t \ge 0\big) e^{-2\alpha(1+\delta)/c} \frac{1}{c}.\end{aligned}$$

Therefore, if we let

$$\mathrm{G}_1 = \{A_t^1 \subseteq Q(k+1+ct)\ \forall t \geq 0\},$$

(where A_t^1 denotes the set of 1's in the process η_t^1 started at $\nu_{Q(m)}$), we can use this bound together with (4.3) to get

$$\mathbb{P}(\mathrm{G}_1^{\mathrm{c}}) \leq 2cd(2m+1)^d e^{2\alpha(1+\delta)/c} e^{-(k-m)}.$$

Choosing now k large enough yields

$$\mathbb{P}(\mathrm{G}_1) > 1 - \varepsilon. \tag{4.4}$$

Now take $n > k$, $T > 0$, let $(t-T)^+ = (t-T) \vee 0$, and call G_2 the event that on the space-time region $\bigcup_{t\geq 0}\{t\} \times Q(n + c(t-T)^+)$ the environment for η_t^2 dominates the environment for η_t^1 [with respect to the order (1.1)]:

$$\mathrm{G}_2 = \{B_t^2 \subseteq B_t^1 \text{ on } Q(n + c(t-T)^+)\ \forall t \geq 0\}.$$

We want this space-time region to contain the region defining G_1, so we let $T = (n-k-1)/c$.

Observe that, since we are coupling the processes using the canonical coupling given by the graphical representation, once the environment is equal for both process at a given site, it stays equal at that site from that time on. In particular, B_t^2 dominates B_t^1 on $Q(n)$ for all $t \geq 0$. For any other site, any symbol $\bullet_{-1}$ or $*_{-1}$ leaves the environment equal for both process. Therefore,

$$\begin{aligned}
\mathbb{P}(\mathrm{G}_2^{\mathrm{c}}) &\leq \sum_{x \notin Q(n)} \mathbb{P}\big(\text{no } \bullet_{-1} \text{ or } *_{-1} \text{ at } x \text{ by time } T + (\|x\|_\infty - n)/c\big) \\
&= \sum_{j>n} |Q(j) \setminus Q(j-1)| e^{-\alpha(1+\delta)(T+(j-n)/c)} \\
&\leq e^{\alpha(1+\delta)(k+1)/c} \sum_{j>n} (2j+1)^d e^{-\alpha(1+\delta)j/c}.
\end{aligned}$$

By taking n large enough, we obtain

$$\mathbb{P}(\mathrm{G}_2) > 1 - \varepsilon. \tag{4.5}$$

Finally, let

$$\mathrm{G}_3 = \{A_t^1 \neq \varnothing\ \forall t \geq 0\}.$$

By (4.1), $\mathbb{P}(\mathrm{G}_3) > 1 - \varepsilon$. Observe that on the event $\mathrm{G}_1 \cap \mathrm{G}_2 \cap \mathrm{G}_3$, η_t^2 contains 1's at all times with probability 1. Therefore,

$$\begin{aligned}
\mathbb{P}^{\chi_{[-n,n]^d}}(A_t \neq \varnothing\ \forall t \geq 0) &\geq \mathbb{P}(\mathrm{G}_1 \cap \mathrm{G}_2 \cap \mathrm{G}_3) \\
&\geq 1 - \mathbb{P}(\mathrm{G}_1^{\mathrm{c}}) - \mathbb{P}(\mathrm{G}_2^{\mathrm{c}}) - \mathbb{P}(\mathrm{G}_3^{\mathrm{c}}) \\
&> 1 - 3\varepsilon,
\end{aligned}$$

and choosing ε small enough we get the result. $\square$

In the following lemma, we combine and extend for our process the results in Liggett (1999) that lead to the block conditions. Consider the process ${}_L\eta_t$, for $L > 0$, where no births are allowed outside of $(-L, L)^d$. Define $N_+(L, T)$ to be the maximal number of space-time points in

$$S_+(L, T) = \{(x, s) \in (\{L\} \times [0, L)^{d-1}) \times [0, T] : x \in {}_LA_s\}$$

such that each pair of these points having the same spatial coordinate have their time coordinates at distance at least 1.

LEMMA 4.2. *Suppose that the process survives. Then for any $\sigma > 0$, there is a positive integer n satisfying the following: For any given pair of positive integers N and M, there are choices of a positive integer L and a positive real number T such that*

$$\mathbb{P}^{\chi_{[-n,n]^d}}(|{}_LA_T \cap [0, L)^d| > N) \geq 1 - \sigma^{2^{-d}} \tag{4.6a}$$

and

$$\mathbb{P}^{\chi_{[-n,n]^d}}(N_+(L, T) > M) \geq 1 - \sigma^{2^{-d}/d}. \tag{4.6b}$$

PROOF. By Lemma 4.1, we can choose a large enough integer n such that

$$\mathbb{P}^{\chi_{[-n,n]^d}}(A_t \neq \varnothing \ \forall t \geq 0) > 1 - \sigma^2. \tag{4.7}$$

Having this, the proof of the lemma is a simple adaptation of the corresponding proofs for the contact process. To avoid repetition of published results, we will explain the main ideas involved and why the original proofs still work with the random environment, but refer the reader to Section I.2 of Liggett (1999) for the details.

We claim the following: For any finite $A \subseteq \mathbb{Z}^d$ and any $N \geq 1$,

$$\lim_{t\to\infty} \lim_{L\to\infty} \mathbb{P}^{\chi_A}(|{}_LA_t| \geq N) = \mathbb{P}^{\chi_A}(A_t \neq \varnothing \ \forall t \geq 0). \tag{4.8}$$

To see that this is true, we observe that

$$\lim_{L\to\infty} \mathbb{P}^{\chi_A}(|{}_LA_t| \geq N) = \mathbb{P}^{\chi_A}(|A_t| \geq N)$$

and then argue that, conditioned on survival, $|A_t| \longrightarrow \infty$ as $t \longrightarrow \infty$ with probability 1. This follows from the easy fact that there is an $\varepsilon_N > 0$ such that if $|A| \leq N$ then the process started with 1's at A becomes extinct with probability at least ε_N, so

$$\mathbb{P}^{\chi_A}(0 < |A_t| \leq N)\varepsilon_N \leq \mathbb{P}^{\chi_A}(t < \tau < \infty) \underset{t\to\infty}{\longrightarrow} 0.$$

The next step is to use positive correlations to localize estimates on the cardinality of ${}_L A_t$ to a specific orthant of $\mathbb{Z}^d$: For every $N \geq 1$ and $L \geq n$,

$$\mathbb{P}^{\chi_{[-n,n]^d}}\big(|{}_L A_t \cap [0, L)^d| \leq N\big) \leq \big[\mathbb{P}^{\chi_{[-n,n]^d}}(|{}_L A_t| \leq 2^d N)\big]^{2^{-d}}. \tag{4.9}$$

This relation follows easily from the positive correlations result in Lemma 2.3, and its proof is identical the proof of Proposition I.2.6 in Liggett (1999).

Observe that (4.7), (4.8) and (4.9) together suffice to obtain (4.6a). The preceding arguments can be modified to obtain similar estimates for $N_+(L, T)$, which in turn give (4.6b). The only detail remaining is getting the same L and T to work for both inequalities. This is done by obtaining sequences $L_j \nearrow \infty$ and $T_j \nearrow \infty$ such that (4.6a) holds with $L = L_j$ and $T = T_j$ for every $j \geq 1$, and then adapting the arguments above to show that (4.6b) must hold for some pair (L_j, T_j). We refer the reader to the proof of Theorem I.2.12 in Liggett's book for the details on how this is achieved, and remark that the argument depends only on properties such as positive correlations and the Feller property which are available both for η_t and the contact process. $\square$

We state now the block conditions that are equivalent to the survival of the process.

THEOREM 4.3. *The process survives if and only if for any given $\varepsilon > 0$ there are positive integers n and L and a positive real number T such that the following conditions* (BC) *are satisfied*:

$$\mathbb{P}^{\chi_{[-n,n]^d}}\big({}_{L+2n} A_{T+1} \supseteq x + [-n, n]^d \textit{ for some } x \in [0, L)^d\big) > 1 - \varepsilon \tag{BC1}$$

and

$$\begin{aligned}\mathbb{P}^{\chi_{[-n,n]^d}}\big({}_{L+2n} A_{t+1} \supseteq x + [-n, n]^d \textit{ for some } 0 \leq t \leq T \\ \textit{and some } x \in \{L + n\} \times [0, L)^{d-1}\big) > 1 - \varepsilon.\end{aligned} \tag{BC2}$$

Observe that these conditions correspond exactly to the conditions in Theorem I.2.12 of Liggett (1999). This will allow us to borrow the arguments from Liggett's book to prove that (BC) implies survival for η_t. The reason why we need the conditions (BC) starting η_t from $\chi_{[-n,n]^d}$ is because the proof of their sufficiency for survival (as well as their use in the proof of Theorem 2) demands obtaining repeatedly cubes fully occupied by 1's and, at each step, restarting the process at the lowest possible configuration having those cubes fully occupied.

PROOF OF THEOREM 4.3. The proof uses the exact same arguments as those in the proofs of Theorems I.2.12 and I.2.23 in Liggett (1999). As before, we will only make some remarks and refer the reader to Liggett's book for the details.

The necessity of (BC) follows from Lemma 4.2, by choosing the quantities N and M to be large enough to produce the desired boxes filled with 1's.

For the sufficiency of (BC), attractiveness and (S2) imply that it is enough to show that for some $n > 0$ the process started at $\chi_{[-n,n]^d}$ contains 1's at all times (by using, as above, the fact that for any given $n > 0$ the process started at $\chi_{\{0\}}$ has $[-n, n]^d$ fully occupied by time 1 with some positive probability). The proof of this fact relies on starting with a large enough cube fully occupied by 1's and then moving its center in an appropriate way. This is used to compare the process with supercritical oriented site percolation, and conclude that such boxes exist for all times with positive probability. $\square$

The following consequence of Theorem 4.3 is obtained in the same way as for the contact process, see Theorem I.2.25 in Liggett (1999) for the details.

COROLLARY 4.4. *If $\beta = \beta_c(\alpha, \delta)$ or $\delta = \delta_c(\alpha, \beta)$, then the process dies out.*

5. Complete convergence. We are ready now to use the block construction of Section 4 to prove Theorem 2. The key step in the proof will be to obtain the result in the special case where the initial distribution μ is a probability measure of the form ν_A, in which case we can use duality.

PROPOSITION 5.1. *For every $A \subseteq \mathbb{Z}^d$,*

$$\eta_t^{\nu_A} \Longrightarrow \mathbb{P}^{\nu_A}(\tau < \infty)\underline{\nu} + \mathbb{P}^{\nu_A}(\tau = \infty)\overline{\nu}.$$

To prove the proposition, we need a preliminary lemma. Both the proof of the proposition and this lemma are inspired by the proof Theorem 2 in Durrett and Møller (1991).

We will denote by $\mathbb{P}^{\nu_A,\nu_C}$ the probability measure associated to starting the process at ν_A and its dual at ν_C, using the same realization of the graphical representation, as explained in Section 2.2.

LEMMA 5.2. *For every finite $C \subseteq \mathbb{Z}^d$ and every $\varepsilon > 0$, if r is a positive real number and s is large enough, then*

$$\left|\mathbb{P}^{\nu_A,\nu_C}\left(\tau > \frac{s}{2}, \hat{A}_r^{r+s} \neq \varnothing, \hat{B}_0^{r+s} \cap D \neq \varnothing\right)\right.$$
$$\left. - \mathbb{P}^{\nu_A}\left(\tau > \frac{s}{2}\right)\mathbb{P}^{\nu_C}(\hat{A}_r^r \neq \varnothing, \hat{B}_0^r \cap D \neq \varnothing)\right| < \varepsilon.$$

Observe that for the (ordinary) contact process, the forward process and the dual are independent when they run on nonoverlapping time intervals, so this fact is trivial and holds with $s/2$ replaced by s.

PROOF OF LEMMA 5.2. Given r and ε, there is a $q = q(|C|)$ such that every dual active path in $(\hat{\eta}_u^{\nu_C,r})_{0\le u\le r}$ stays inside $C + [-q, q]^d$ with probability at least $1-\varepsilon$. To see this, observe that the number of particles in all such dual active paths is dominated by X_r, where $(X_r)_{r\ge 0}$ is a branching process starting with $|C|$ particles and with birth rate β and death rate 0 (we are ignoring deaths and coalescence of paths). By Markov's inequality, $\mathbb{P}(X_r > q) \le \mathbb{E}(X_r)/q \le \varepsilon$ for large enough q. Since any dual active path in $\hat{\eta}_t^{\nu_C,r}$ starts inside C, $X_r \le q$ implies that all dual active paths are contained inside $C + [-q, q]^d$ up to time r.

Now denote by $\eta_t^{(\mu_\rho,s/2)}$ and $\hat{\eta}_t^{(\mu_\rho,s/2),r}$ modifications of the process and its dual, constructed on the same graphical representation as the original ones, where the environment is reset at time $s/2$ to its equilibrium μ_ρ, independently of its state before $s/2$ (that is, at time $s/2$ we replace every -1 by a 0 and then flip every site to -1 with probability ρ, regardless of it being at state 0 or 1). Then for given r and q, if s is large enough, we have that

$$\begin{aligned}\mathbb{P}^{\nu_A,\nu_C}&\big(B_u = B_u^{(\mu_\rho,s/2)} \text{ on } C + [-q, q]^d \ \forall u \in [s, s + r]\big)\\ &\ge \big(1 - e^{-\alpha(1+\delta)s/2}\big)^{|C+[-q,q]^d|} > 1 - \varepsilon.\end{aligned} \tag{5.1}$$

Indeed, for any given $x \in C + [-q, q]^d$ the probability that B_u and $B_u^{(\mu_\rho,s/2)}$ ar equal at x for every $u \in [s, s + r]$ is bounded below by the probability that an exponential random variable with parameter $\alpha(1 + \delta)$ is smaller than $s/2$ [because any symbol $\bullet_{-1}$ or $*_{-1}$ above $(x, s/2)$ leaves the environment at that site equal for both processes from that time on].

The property discussed at the first paragraph of the proof together with (5.1) imply that

$$\begin{aligned}&\Big|\mathbb{P}^{\nu_A,\nu_C}\Big(\tau > \frac{s}{2}, \hat{A}_r^{r+s} \ne \varnothing, \hat{B}_0^{r+s} \cap D \ne \varnothing\Big)\\ &\quad - \mathbb{P}^{\nu_A,\nu_C}\Big(\tau > \frac{s}{2}, \hat{A}_r^{(\mu_\rho,s/2),r+s} \ne \varnothing, \hat{B}_0^{(\mu_\rho,s/2),r+s} \cap D \ne \varnothing\Big)\Big| < 2\varepsilon.\end{aligned}$$

The statement of the lemma follows now from the independence of disjoint parts of the graphical representation and the stationarity of B_t, since

$$\begin{aligned}&\mathbb{P}^{\nu_A,\nu_C}\Big(\tau > \frac{s}{2}, \hat{A}_r^{(\mu_\rho,s/2),r+s} \ne \varnothing, \hat{B}_0^{(\mu_\rho,s/2),r+s} \cap D \ne \varnothing\Big)\\ &\quad = \mathbb{P}^{\nu_A}\Big(\tau > \frac{s}{2}\Big)\mathbb{P}^{\nu_C}\big(\hat{A}_r^{(\mu_\rho,s/2),r+s} \ne \varnothing, \hat{B}_0^{(\mu_\rho,s/2),r+s} \cap D \ne \varnothing\big)\\ &\quad = \mathbb{P}^{\nu_A}\Big(\tau > \frac{s}{2}\Big)\mathbb{P}^{\nu_C}(\hat{A}_r^{r} \ne \varnothing, \hat{B}_0^{r} \cap D \ne \varnothing).\end{aligned}$$

□

PROOF OF PROPOSITION 5.1. The result is straightforward in the subcritical case. If the process survives, and since weak convergence in this setting corresponds to the convergence of the finite-dimensional distributions, it is enough

to prove that the following three properties hold for any two finite subsets C, D of $\mathbb{Z}^d$:

$$\text{(c1)} \qquad \mathbb{P}^{\nu_A}(A_t \cap C \neq \varnothing) \underset{t\to\infty}{\longrightarrow} \mathbb{P}^{\nu_A}(\tau = \infty)\overline{\nu}\big(\{(E, F) : E \cap C \neq \varnothing\}\big),$$

$$\text{(c2)} \qquad \begin{aligned} \mathbb{P}^{\nu_A}(B_t \cap D \neq \varnothing) \;&=\; \mathbb{P}^{\nu_A}(\tau < \infty)\underline{\nu}\big(\{(E, F) : F \cap D \neq \varnothing\}\big) \\ &\quad + \mathbb{P}^{\nu_A}(\tau = \infty)\overline{\nu}\big(\{(E, F) : F \cap D \neq \varnothing\}\big) \end{aligned}$$

and

$$\text{(c3)} \qquad \begin{aligned} &\mathbb{P}^{\nu_A}(A_t \cap C \neq \varnothing, B_t \cap D \neq \varnothing) \\ &\qquad \underset{t\to\infty}{\longrightarrow} \mathbb{P}^{\nu_A}(\tau = \infty)\overline{\nu}\big(\{(E, F) : E \cap C \neq \varnothing, F \cap D \neq \varnothing\}\big). \end{aligned}$$

Indeed, all the finite-dimensional distributions of the process are determined by these probabilities via the inclusion-exclusion formula. Observe that the right-hand side of (c2) is equal to $\mu_\rho(\{\eta : \eta(x) = -1 \text{ for some } x \in D\})$.

The convergence in (c1) follows from the same arguments used in Liggett (1999) for the contact process. Using duality (Proposition 2.2), the proof of Theorem I.1.12 in that book applies in the same way to obtain the fact that (c1) holds if and only if for every $x \in \mathbb{Z}^d$ and every $A \subseteq \mathbb{Z}^d$,

$$\text{(5.2a)} \qquad \mathbb{P}^{\nu_A}(\tau = \infty) = \mathbb{P}^{\nu_A}(x \in A_t \text{ i.o.})$$

and

$$\text{(5.2b)} \qquad \lim_{n\to\infty} \liminf_{t\to\infty} \mathbb{P}^{\nu_{[-n,n]^d}}(A_t \cap [-n, n]^d \neq \varnothing) = 1.$$

The analogous conditions are checked for the contact process in the proof of Theorem I.2.27 in Liggett (1999). The equality in (5.2a) follows from the same proof after some minor modifications, so we will skip the argument. For (5.2b), Theorem 4.3 allows us to use Liggett's arguments to get the desired limit when $\nu_{[-n,n]^d}$ is replaced by $\chi_{[-n,n]^d}$, so given any $\varepsilon > 0$ we can choose a large enough integer m such that

$$\text{(5.3)} \qquad \liminf_{t\to\infty} \mathbb{P}^{\chi_{[-m,m]^d}}(A_t \cap [-m, m]^d \neq \varnothing) > 1 - \varepsilon.$$

Given this m, we can choose a large enough n so that the process started at $\nu_{[-n,n]^d}$ contains at time 0 a fully occupied cube of side $2m+1$ (contained in $[-n, n]^d$) with probability at least $1 - \varepsilon$ (in fact, any translate of $[-m, m]^d$ contained in $[-n, n]^d$ is fully occupied by 1's with some probability $p > 0$, so we only need to choose n so that $[-n, n]^d$ contains enough disjoint translates of $[-m, m]^d$). On this event, we can restart the process by putting every site outside that cube at state -1 and use attractiveness, translation invariance and (5.3) to get

$$\liminf_{t\to\infty} \mathbb{P}^{\nu_{[-n,n]^d}}(A_t \cap [-n, n]^d \neq \varnothing) > (1 - \varepsilon)^2,$$

whence (5.2b) follows. There is only one detail to consider: in his book, Liggett only proves the condition analogous to (5.2b) in case $d \geq 2$, because it is simpler and the case $d = 1$ was already done in Liggett (1985). The difficulty in the one-dimensional case arises from the fact that certain block events are not independent. This can be overcome by comparing with k-dependent oriented site percolation instead of ordinary oriented site percolation [see Theorem B26 in Liggett (1999)]. We refer the reader to Section 5 of Durrett and Schonmann (1987), where the authors use a similar block construction to derive a complete convergence theorem for a general class of one-dimensional growth models.

The convergence in (c2) is trivial due to the stationarity of the environment process. To prove (c3), we start by observing that

$$\begin{aligned}\mathbb{P}^{\nu_A}&(A_{r+s} \cap C \neq \varnothing, B_{r+s} \cap D \neq \varnothing)\\ &= \mathbb{P}^{\nu_A,\nu_C}(A_s \cap \hat{A}_r^{r+s} \neq \varnothing, \hat{B}_0^{r+s} \cap D \neq \varnothing),\end{aligned} \tag{5.4}$$

which follows from constructing $(\eta_u^{\nu_A})_{0\leq u\leq r+s}$ and $(\hat{\eta}_u^{\nu_C,r+s})_{0\leq u\leq r+s}$ on the same copy of the graphical representation. On the other hand,

$$\begin{aligned}&|\mathbb{P}^{\nu_A,\nu_C}(A_s \cap \hat{A}_r^{r+s} \neq \varnothing, \hat{B}_0^{r+s} \cap D \neq \varnothing)\\ &\quad - \mathbb{P}^{\nu_A,\nu_C}(A_s \neq \varnothing, \hat{A}_r^{r+s} \neq \varnothing, \hat{B}_0^{r+s} \cap D \neq \varnothing)|\\ &\qquad = \mathbb{P}^{\nu_A,\nu_C}(A_s \neq \varnothing, \hat{A}_r^{r+s} \neq \varnothing, A_s \cap \hat{A}_r^{r+s} = \varnothing, \hat{B}_0^{r+s} \cap D \neq \varnothing)\\ &\qquad \leq \mathbb{P}^{\nu_A,\nu_C}(A_s \neq \varnothing, \hat{A}_r^{r+s} \neq \varnothing, A_s \cap \hat{A}_r^{r+s} = \varnothing)\\ &\qquad = \mathbb{P}^{\nu_A,\nu_C}(A_s \neq \varnothing, \hat{A}_r^{r+s} \neq \varnothing) - \mathbb{P}^{\nu_A,\nu_C}(A_s \cap \hat{A}_r^{r+s} \neq \varnothing).\end{aligned} \tag{5.5}$$

Observe that

$$\mathbb{P}^{\nu_A}(s/2 < \tau < \infty) \underset{s\to\infty}{\longrightarrow} 0.$$

Thus, for any given $D \subseteq \mathbb{Z}^d$ and $\varepsilon > 0$, and for large enough s, we can write

$$\begin{aligned}&|\mathbb{P}^{\nu_A,\nu_C}(A_s \neq \varnothing, \hat{A}_r^{r+s} \neq \varnothing, \hat{B}_0^{r+s} \cap D \neq \varnothing)\\ &\quad - \mathbb{P}^{\nu_A,\nu_C}(\tau > s/2, \hat{A}_r^{r+s} \neq \varnothing, \hat{B}_0^{r+s} \cap D \neq \varnothing)|\\ &\qquad = \mathbb{P}^{\nu_A,\nu_C}(s/2 < \tau \leq s, \hat{A}_r^{r+s} \neq \varnothing, \hat{B}_0^{r+s} \cap D \neq \varnothing)\\ &\qquad \leq \mathbb{P}^{\nu_A}(s/2 < \tau < \infty) < \frac{\varepsilon}{3}.\end{aligned} \tag{5.6}$$

Putting the previous observations together we get, for large enough s

$$\begin{aligned}&|\mathbb{P}^{\nu_A}(A_{r+s} \cap C \neq \varnothing, B_{r+s} \cap D \neq \varnothing)\\ &\quad - \mathbb{P}^{\nu_A,\nu_C}(A_s \neq \varnothing, \hat{A}_r^{r+s} \neq \varnothing, \hat{B}_0^{r+s} \cap D \neq \varnothing)|\\ &\qquad = |\mathbb{P}^{\nu_A,\nu_C}(A_s \cap \hat{A}_r^{r+s} \neq \varnothing, \hat{B}_0^{r+s} \cap D \neq \varnothing)\end{aligned}$$

$$
\begin{aligned}
&- \mathbb{P}^{\nu_A,\nu_C}(A_s \neq \varnothing, \hat{A}_r^{r+s} \neq \varnothing, \hat{B}_0^{r+s} \cap D \neq \varnothing)| && \text{by (5.4)}\\
&\leq \mathbb{P}^{\nu_A,\nu_C}(A_s \neq \varnothing, \hat{A}_r^{r+s} \neq \varnothing) - \mathbb{P}^{\nu_A,\nu_C}(A_s \cap \hat{A}_r^{r+s} \neq \varnothing) && \text{by (5.5)}\\
&\leq \frac{\varepsilon}{3} + |\mathbb{P}^{\nu_A,\nu_C}(\tau > s/2, \hat{A}_r^{r+s} \neq \varnothing) - \mathbb{P}^{\nu_A,\nu_C}(A_s \cap \hat{A}_r^{r+s} \neq \varnothing)| && \text{by (5.6)},
\end{aligned}
$$

where we used $D = \mathbb{Z}^d$ and the fact that $\mathbb{P}^{\nu_A,\nu_C}(\hat{B}_0^{r+s} \neq \varnothing) = 1$ in the application of (5.6). Using again this fact to apply Lemma 5.2 with $D = \mathbb{Z}^d$, and then using duality we get

$$
\begin{aligned}
&|\mathbb{P}^{\nu_A,\nu_C}(\tau > s/2, \hat{A}_r^{r+s} \neq \varnothing) - \mathbb{P}^{\nu_A,\nu_C}(A_s \cap \hat{A}_r^{r+s} \neq \varnothing)|\\
&\quad \leq \frac{\varepsilon}{3} + |\mathbb{P}^{\nu_A}(\tau > s/2)\mathbb{P}^{\nu_C}(\hat{A}_r^{r} \neq \varnothing) - \mathbb{P}^{\nu_A,\nu_C}(A_s \cap \hat{A}_r^{r+s} \neq \varnothing)|\\
&\quad = \frac{\varepsilon}{3} + |\mathbb{P}^{\nu_A}(\tau > s/2)\mathbb{P}^{\nu_{\mathbb{Z}^d}}(A_r \cap C \neq \varnothing) - \mathbb{P}^{\nu_A}(A_{r+s} \cap C \neq \varnothing)|
\end{aligned}
$$

for large enough s. By (c1), the last difference converges to 0 as $r, s \to \infty$, so we finally get

$$
\begin{aligned}
&|\mathbb{P}^{\nu_A}(A_{r+s} \cap C \neq \varnothing, B_{r+s} \cap D \neq \varnothing)\\
&\quad - \mathbb{P}^{\nu_A,\nu_C}(A_s \neq \varnothing, \hat{A}_r^{r+s} \neq \varnothing, \hat{B}_0^{r+s} \cap D \neq \varnothing)| < \varepsilon
\end{aligned}
$$

for large enough r, s.

This calculation implies that in order to prove (c3) it is enough to show that

$$
\begin{aligned}
&\mathbb{P}^{\nu_A,\nu_C}(A_s \neq \varnothing, \hat{A}_r^{r+s} \neq \varnothing, \hat{B}_0^{r+s} \cap D \neq \varnothing)\\
&\qquad \xrightarrow[r,s\to\infty]{} \mathbb{P}^{\nu_A}(\tau = \infty)\overline{\nu}\big((E, F) : E \cap C \neq \varnothing, F \cap D \neq \varnothing\big).
\end{aligned}
$$

Repeating the previous application of (5.6) and Lemma 5.2 we get that, for large enough s,

$$
\begin{aligned}
&|\mathbb{P}^{\nu_A,\nu_C}(A_s \neq \varnothing, \hat{A}_r^{r+s} \neq \varnothing, \hat{B}_0^{r+s} \cap D \neq \varnothing)\\
&\quad - \mathbb{P}^{\nu_A}(\tau > s/2)\mathbb{P}^{\nu_C}(\hat{A}_r^{r} \neq \varnothing, \hat{B}_0^{r} \cap D \neq \varnothing)|\\
&\qquad \leq \frac{\varepsilon}{2} + |\mathbb{P}^{\nu_A,\nu_C}(\tau > s/2, \hat{A}_r^{r+s} \neq \varnothing, \hat{B}_0^{r+s} \cap D \neq \varnothing)\\
&\qquad\qquad - \mathbb{P}^{\nu_A}(\tau > s/2)\mathbb{P}^{\nu_C}(\hat{A}_r^{r} \neq \varnothing, \hat{B}_0^{r} \cap D \neq \varnothing)|\\
&\qquad \leq \varepsilon.
\end{aligned}
$$

Therefore, we can finally reduce to proving that

$$
\begin{aligned}
&\mathbb{P}^{\nu_A}(\tau > s/2)\mathbb{P}^{\nu_C}(\hat{A}_r^{r} \neq \varnothing, \hat{B}_0^{r} \cap D \neq \varnothing)\\
&\qquad \xrightarrow[r,s\to\infty]{} \mathbb{P}^{\nu_A}(\tau = \infty)\overline{\nu}\big((E, F) : E \cap C \neq \varnothing, F \cap D \neq \varnothing\big).
\end{aligned}
$$

This follows easily from duality, since (2.1) yields

$$\begin{aligned}\mathbb{P}^{\nu_A}(\tau > s/2)\mathbb{P}^{\nu_C}(\hat{A}_r^r \neq \varnothing, \hat{B}_0^r \cap D \neq \varnothing)\\ = \mathbb{P}^{\nu_A}(\tau > s/2)\mathbb{P}^{\nu_{\mathbb{Z}^d}}(A_r \cap C \neq \varnothing, B_r \cap D \neq \varnothing),\end{aligned}$$

and this last term converges to the desired limit as $r, s \to \infty$. $\square$

We extend now Proposition 5.1 to the general case.

PROOF OF THEOREM 2. It is enough to show that

$$\lim_{t\to\infty} \mathbb{E}^{\mu}(f(\eta_t)) = \mathbb{P}^{\mu}(\tau < \infty)\int f\, d\underline{\nu} + \mathbb{P}^{\mu}(\tau = \infty)\int f\, d\overline{\nu} \tag{5.7}$$

for every f in the space of continuous increasing functions depending on finitely many coordinates of $\mathcal{X}$, which we will denote by $\mathcal{F}$. To see this, observe that given any two finite subsets C, D of $\mathbb{Z}^d$, the functions

$$\begin{aligned}&f_1(E, F) = \mathbf{1}_{E\cap C\neq\varnothing},\\ &f_2(E, F) = \mathbf{1}_{F\cap D=\varnothing} \quad \text{and} \quad f_3(E, F) = \mathbf{1}_{E\cap C\neq\varnothing, F\cap D=\varnothing}\end{aligned}$$

are all in $\mathcal{F}$ and (as in the proof of Proposition 5.1) all the finite-dimensional distributions of the process can be obtained from $\mathbb{E}^{\mu}(f_1(\eta_t))$, $\mathbb{E}^{\mu}(f_2(\eta_t))$, and $\mathbb{E}^{\mu}(f_3(\eta_t))$ by the inclusion-exclusion formula.

Let f be a function in $\mathcal{F}$ and observe that, in particular, f is bounded. One inequality in (5.7) is easy: by the Markov property and attractiveness, given $0 < s < t$ we have that

$$\begin{aligned}\mathbb{E}^{\mu}(f(\eta_t)) &= \mathbb{E}^{\mu}\big(f(\eta_t), \tau < s\big) + \mathbb{E}^{\mu}\big(f(\eta_t), \tau \geq s\big)\\ &= \mathbb{E}^{\mu}\big(\mathbb{E}^{\eta_s}(f(\eta_{t-s})), \tau < s\big) + \mathbb{E}^{\mu}\big(\mathbb{E}^{\eta_s}(f(\eta_{t-s})), \tau \geq s\big)\\ &\leq \mathbb{E}(f(\eta_{t-s}^0))\mathbb{P}^{\mu}(\tau < s) + \mathbb{E}^{\chi_{\mathbb{Z}^d}}(f(\eta_{t-s}))\mathbb{P}^{\mu}(\tau \geq s),\end{aligned}$$

where η_t^0 denotes the process started at the configuration $\eta \equiv 0$. Since $\eta_t^0 \Longrightarrow \mu_\rho = \underline{\nu}$ and $\eta_t^{\chi_{\mathbb{Z}^d}} \Longrightarrow \overline{\nu}$, we get

$$\limsup_{t\to\infty} \mathbb{E}^{\mu}(f(\eta_t)) \leq \mathbb{P}^{\mu}(\tau < s)\int f\, d\underline{\nu} + \mathbb{P}^{\mu}(\tau \geq s)\int f\, d\overline{\nu},$$

and now taking $s \to \infty$ we deduce that

$$\limsup_{t\to\infty} \mathbb{E}^{\mu}(f(\eta_t)) \leq \mathbb{P}^{\mu}(\tau < \infty)\int f\, d\underline{\nu} + \mathbb{P}^{\mu}(\tau = \infty)\int f\, d\overline{\nu}. \tag{5.8}$$

To obtain the other inequality in (5.7), we will begin by considering the case $\mu = \chi_{[-n,n]^d}$ and showing that, given any $\varepsilon > 0$ and any $x \in \mathbb{Z}^d$,

$$\liminf_{t\to\infty} \mathbb{E}^{\chi_{x+[-n,n]^d}}\big(f(\eta_t), \tau = \infty\big) \geq \int f\, d\overline{\nu} - \varepsilon \tag{5.9}$$

for large enough n. By the translation invariance of η_t and $\overline{\nu}$, it is enough to consider the case $x = 0$. To show (5.9), we will use the construction introduced in the proof of Lemma 4.1. Using the notation of that proof, recall that we showed that, given any $\gamma > 0$, there are positive integers $n > k > m$ such that

$$\mathbb{P}(\mathrm{G}_1 \cap \mathrm{G}_2 \cap \mathrm{G}_3) > 1 - 3\gamma.$$

This means that the processes η_t^1 (started at $\nu_{[-m,m]^d}$) and η_t^2 (started at $\chi_{[-n,n]^d}$) can be coupled in such a way that, with probability at least $1 - 3\gamma$, for all $t \geq 0$ we have that $A_t^1 \neq \varnothing$, $A_t^2 \neq \varnothing$, $A_t^1 \subseteq Q(k+1+ct)$ and $B_t^2 \subseteq B_t^1$ inside $Q(k+1+ct)$.

Let $\mathrm{G} = \mathrm{G}_1 \cap \mathrm{G}_2 \cap \mathrm{G}_3$ and $\gamma > 0$ and choose $n > k > m$ so that $\mathbb{P}(\mathrm{G}) > 1 - 3\gamma$. We will denote by τ^1 and τ^2 the extinction times of the processes η_t^1 and η_t^2, respectively. Define

$$\check{\eta}_t = (A_t^1, B_t^2)$$

and observe that, on the event G, $\check{\eta}_t$ defines an $\mathcal{X}$-valued process and, moreover, $\eta_t^2 \geq \check{\eta}_t$ for all $t \geq 0$. Therefore, since f is increasing and $\{\tau^2 = \infty\} \supseteq \mathrm{G}$,

$$\mathbb{E}\big(f(\eta_t^2), \tau^2 = \infty\big) \geq \mathbb{E}(f(\eta_t^2), \mathrm{G}) \geq \mathbb{E}(f(\check{\eta}_t), \mathrm{G}) \tag{5.10}$$

for all $t \geq 0$. Now observe that, trivially,

$$\mathbb{E}(f(\check{\eta}_t), \mathrm{G}) = \mathbb{E}\big(f(\check{\eta}_t), \tau^2 = \infty\big) - \mathbb{E}\big(f(\check{\eta}_t), \tau^2 = \infty, \mathrm{G}^c\big),$$

and

$$\mathbb{E}\big(f(\check{\eta}_t), \tau^2 = \infty, \mathrm{G}^c\big) \leq \|f\|_\infty \mathbb{P}(\mathrm{G}^c) < 3\gamma \|f\|_\infty,$$

so

$$\mathbb{E}(f(\check{\eta}_t), \mathrm{G}) > \mathbb{E}\big(f(\check{\eta}_t), \tau^2 = \infty\big) - 3\gamma \|f\|_\infty \tag{5.11}$$

for all $t \geq 0$. On the other hand,

$$\big|\mathbb{E}\big(f(\check{\eta}_t), \tau^2 = \infty\big) - \mathbb{E}\big(f(\eta_t^1), \tau^2 = \infty\big)\big| \underset{t\to\infty}{\longrightarrow} 0. \tag{5.12}$$

To see this, observe that since f depends on finitely many coordinates, then given any $q > 0$, $f(\check{\eta}_s) = f(\eta_s^1)$ for all $s \geq t$ with probability at least $1 - q$ if t is large enough. Indeed, if $K \subseteq \mathbb{Z}^d$ is the finite set of coordinates of $\mathcal{X}$ on which f depends, then repeating the calculations that led to (4.5) we get that

$$\begin{aligned}
&\mathbb{P}\big(B_s^1(x) \neq B_s^2(x) \text{ for some } x \in K \text{ and some } s \geq t\big) \\
&\quad \leq \sum_{x \in K} \mathbb{P}(\text{no } \bullet_{-1} \text{ or } *_{-1} \text{ at } x \text{ by time } t) = |K| e^{-\alpha(1+\delta)t} \underset{t\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Therefore, given any $q > 0$,

$$\big|\mathbb{E}\big(f(\check{\eta}_t), \tau^2 = \infty\big) - \mathbb{E}\big(f(\eta_t^1), \tau^2 = \infty\big)\big| \leq \mathbb{E}\big(|f(\check{\eta}_t) - f(\eta_t^1)|\big) \leq 2q\|f\|_\infty$$

for large enough t, and we get (5.12). Finally, we have that

$$\mathbb{E}\big(f(\eta_t^1), \tau^2 = \infty\big) = \mathbb{E}\big(f(\eta_t^1), \tau^1 = \infty\big) - \big(\mathbb{E}\big(f(\eta_t^1), \tau^1 = \infty\big) - \mathbb{E}(f(\eta_t^1), \mathrm{G})\big) - \big(\mathbb{E}(f(\eta_t^1), \mathrm{G}) - \mathbb{E}(f(\eta_t^1), \tau^2 = \infty\big),$$

and since $\mathrm{G} \subseteq \{\tau^1 = \infty\} \cap \{\tau^2 = \infty\}$,

$$\big|\mathbb{E}\big(f(\eta_t^1), \tau^i = \infty\big) - \mathbb{E}(f(\eta_t^1), \mathrm{G})\big| \le \|f\|_\infty \mathbb{P}(\mathrm{G}^c) < 3\gamma \|f\|_\infty$$

for $i = 1, 2$. Thus, Proposition 5.1 implies that

$$\liminf_{t\to\infty} \mathbb{E}\big(f(\eta_t^1), \tau^2 = \infty\big) > \mathbb{P}(\tau^1 = \infty) \int f\, d\overline{\nu} - 6\gamma \|f\|_\infty,$$

and since $\mathbb{P}(\tau^1 = \infty) \ge \mathbb{P}(\mathrm{G}) > 1 - 3\gamma$, we obtain

$$\liminf_{t\to\infty} \mathbb{E}\big(f(\eta_t^1), \tau^2 = \infty\big) > \int f\, d\overline{\nu} - 9\gamma \|f\|_\infty. \tag{5.13}$$

Putting (5.10), (5.11), (5.12) and (5.13) together, we deduce that

$$\liminf_{t\to\infty} \mathbb{E}\big(f(\eta_t^2), \tau^2 = \infty\big) \ge \int f\, d\overline{\nu} - 12\gamma \|f\|_\infty,$$

and choosing γ appropriately we obtain (5.9).

Getting back to the proof of the remaining inequality in (5.7), let $\varepsilon > 0$ and choose $n \in \mathbb{N}$ so that (5.9) holds. Define

$$N = \inf\{k \in \mathbb{N} : \eta_k \supseteq x + [-n, n]^d \text{ for some } x \in \mathbb{Z}^d\}$$

and let $p = \mathbb{P}^{\chi_{\{0\}}}(A_1 \supseteq x + [-n, n]^d \text{ for some } x \in \mathbb{Z}^d) > 0$. Observe that for any $k \ge 0$, if $A_k \ne \varnothing$ then A_{k+1} contains some translate of $[-n, n]^d$ with probability at least p (by attractiveness and translation invariance) and, therefore, since the Poisson processes used in the graphical representation for disjoint time intervals are independent, we deduce that

$$\{\tau = \infty\} \subseteq \{N < \infty\}. \tag{5.14}$$

When $N < \infty$, we will denote by X the center of the corresponding fully occupied box. If there is more than one point x such that $x + [-n, n]^d$ is fully occupied by 1's at time N, we pick X to be the one minimizing $\phi(x)$, where ϕ is any fixed bijection between $\mathbb{Z}^d$ and $\mathbb{N}$ (this ensures that the events $\{X = x\}$ are disjoint for different x). Then given $m \in \mathbb{N}$, the Markov property and attractiveness imply that

$$\begin{aligned}
\mathbb{E}^\mu\big(f(\eta_t), \tau = \infty\big) &\ge \sum_{k=0}^m \mathbb{E}^\mu\big(f(\eta_t), \tau = \infty, N = k\big) \\
&= \sum_{k=0}^m \mathbb{E}^\mu\big(\mathbb{E}^{\eta_k}\big(f(\eta_{t-k}), \tau = \infty\big), N = k\big) \\
&\ge \sum_{k=0}^m \sum_{x\in\mathbb{Z}^d} \mathbb{E}^\mu\big(\mathbb{E}^{\chi_{x+[-n,n]^d}}\big(f(\eta_{t-k}), \tau = \infty\big), N = k, X = x\big)
\end{aligned}$$

for $t \geq m$. Since f is bounded, (5.9) implies that

$$\liminf_{t\to\infty} \mathbb{E}^{\mu}\big(f(\eta_t), \tau = \infty\big) \geq \left(\int f\, d\overline{\nu} - \varepsilon\right) \sum_{k=0}^{m} \sum_{x\in\mathbb{Z}^d} \mathbb{P}^{\mu}(N = k, X = x)$$
$$= \left(\int f\, d\overline{\nu} - \varepsilon\right) \mathbb{P}^{\mu}(N \leq m).$$

Taking now $m \to \infty$, we get by (5.14) that

$$\liminf_{t\to\infty} \mathbb{E}^{\mu}\big(f(\eta_t), \tau = \infty\big) \geq \left(\int f\, d\overline{\nu} - \varepsilon\right) \mathbb{P}^{\mu}(N < \infty)$$
$$\geq \left(\int f\, d\overline{\nu} - \varepsilon\right) \mathbb{P}^{\mu}(\tau = \infty)$$

if $\varepsilon < \int f\, d\overline{\nu}$, and taking $\varepsilon \to 0$ we deduce that

$$\liminf_{t\to\infty} \mathbb{E}^{\mu}\big(f(\eta_t), \tau = \infty\big) \geq \mathbb{P}^{\mu}(\tau = \infty) \int f\, d\overline{\nu}.$$

On the other hand, by arguments similar to those that led to (5.8) (using attractiveness to compare with the process started at $\chi_{\varnothing}$), we get

$$\liminf_{t\to\infty} \mathbb{E}^{\mu}\big(f(\eta_t), \tau < \infty\big) \geq \mathbb{P}^{\mu}(\tau < \infty) \int f\, d\underline{\nu}.$$

We finally deduce that

$$\liminf_{t\to\infty} \mathbb{E}^{\mu}(f(\eta_t)) \geq \mathbb{P}^{\mu}(\tau < \infty) \int f\, d\overline{\nu} + \mathbb{P}^{\mu}(\tau = \infty) \int f\, d\overline{\nu},$$

and the proof is ready. □

Acknowledgments. I am grateful to my advisor, Rick Durrett, for his guidance during the work in this paper. I thank him especially for his careful reading of several versions of the manuscript and the thorough comments and suggestions he provided, which led to a substantial improvement of the paper. I also wish to thank Ted Cox for reviewing part of an earlier version of the manuscript. Finally, I thank an anonymous referee for many detailed comments and suggestions which greatly improved the exposition of the paper.

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