A level set approach for computing solutions to incompressible two-phase flow

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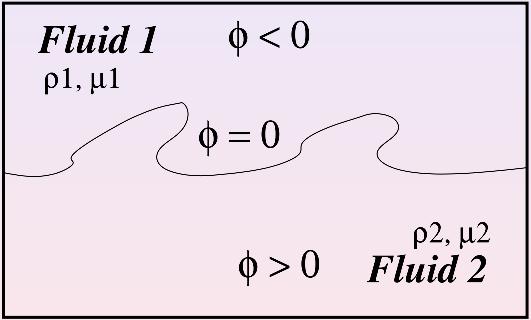
The physical problem



Ω: bounded domain in $\mathbb{R}^2$, $\partial\Omega = \Gamma$. For $t \in [0, T]$, set

$$\overline{\Omega} = \overline{\Omega_1(t)} \cup \overline{\Omega_2(t)}, \quad \Omega_1(t) \cap \Omega_2(t) = \emptyset.$$

Equations of the model (1/2)

$$\begin{cases} \rho_i \left(\dfrac{\partial \mathbf{u}^{(i)}}{\partial t} + (\mathbf{u}^{(i)} \cdot \nabla)\mathbf{u}^{(i)} \right) = \operatorname{div} \sigma^{(i)} + \rho_i \mathbf{g}, & \mathbf{x} \in \Omega_i(t), \\ \operatorname{div} \mathbf{u}^{(i)} = 0, & \mathbf{x} \in \Omega_i(t), \end{cases} \tag{1}$$

$\mathbf{u}^{(i)} = (u^{(i)}, v^{(i)})^t$: velocity, $\mathbf{g}$: gravitational acceleration. The stress tensor is defined by

$$\sigma^{(i)} = -p^{(i)}\mathbf{I} + 2\mu_i \mathbf{D}(\mathbf{u}^{(i)}),$$

where $p^{(i)}$: pressure, μ_i: viscosity and

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Equations of the model (2/2)

$$D_{\alpha,\beta}(\mathbf{u}^{(i)}) = \frac{1}{2}\left(\frac{\partial u_\alpha^{(i)}}{\partial x_\beta} + \frac{\partial u_\beta^{(i)}}{\partial x_\alpha}\right)$$

is the deformation tensor.

$$\left.\begin{aligned} \frac{D\rho}{Dt} &= \frac{\partial \rho}{\partial t} + \mathbf{u}\cdot\nabla\rho = 0, \\ \frac{D\mu}{Dt} &= \frac{\partial \mu}{\partial t} + \mathbf{u}\cdot\nabla\mu = 0. \end{aligned}\right\} \text{in } \Omega\times(0,T).$$

Boundary and initial conditions

We assume the free-slip condition on solid walls:

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \times (0, T).$$

The effect of surface tension is to balance the jump of the normal stress along the fluid interface:

$$(\sigma^{(1)} - \sigma^{(2)})\mathbf{n} = \tau\kappa\mathbf{n} \quad \text{on } I(t), \tag{2}$$

τ: surface tension coefficient (constant),
κ: mean curvature of the interface $I(t)$.
Continuity condition for the velocity:

$$\mathbf{u}^{(1)} = \mathbf{u}^{(2)} \quad \text{on } I(t).$$

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x) \quad \text{in } \Omega.$$

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Introducing the level set function

Let $\phi = \phi(\mathbf{x}, t)$ be a smooth function s.t.

$$\phi(\cdot, t) < 0 \text{ in } \Omega_1(t) \quad \text{and } \phi(\cdot, t) > 0 \text{ in } \Omega_2(t).$$

Thus

$$I(t) = \{\mathbf{x} = (x, y) \in \Omega \,/\, \phi(\mathbf{x}, t) = 0\}, \quad \text{for } t \in [0, T].$$

Therefore, ϕ satisfies

$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = 0, \quad \text{en } \Omega \times (0, T). \tag{3}$$

We initialize ϕ to be the signed normal distance from the interface:

$$\phi_0 = \begin{cases} -d & \text{in } \Omega_1(0), \\ 0 & \text{on } I(t), \\ +d & \text{in } \Omega_2(0), \end{cases}$$

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Reformulation of the physical problem (1/3)

We shall consider the fluid motion for rising air bubbles in water. Denote the density and viscosity inside the bubble by ρ_b and μ_b, resp., and for the continuous phase by ρ_c and μ_c.

With this, it can be proven that eqs. (1), plus the boundary and initial conditions, can be reformulated as follow:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} = L(\mathbf{u}, \phi) - \dfrac{\nabla p}{\rho}, & \text{in } \Omega \times (0, T), \\ \operatorname{div} \mathbf{u} = 0, & \text{in } \Omega \times (0, T), \\ \mathbf{u} \cdot \mathbf{n} = 0, & \text{on } \Gamma \times (0, T), \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), & \text{in } \Omega. \end{cases}$$

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where

$$L(\mathbf{u},\phi) = -(\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{g}_u + \frac{1}{\rho}\left(\frac{1}{\mathrm{Re}}\,\mathrm{div}\,(2\mu\mathbf{D}(\mathbf{u})) + \frac{1}{B}\kappa(\phi)\nabla\phi\,\delta(\phi)\right).$$

The curvature $\kappa(\phi)$ is computed as follow:

$$\kappa(\phi) = -\mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right) = -\frac{\phi_y^2\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + \phi_x^2\phi_{yy}}{(\phi_x^2+\phi_y^2)^{3/2}}.$$

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Reformulation of the physical problem (3/3)

Moreover, assuming that ρ_1 and ρ_2 (resp. μ_1 and μ_2) are constants in Ω_1 and Ω_2 resp., then

$$\rho = \rho_1 + (\rho_2 - \rho_1)H(\phi),$$
$$\mu = \mu_1 + (\mu_2 - \mu_1)H(\phi),$$

where $H(x)$ is the Heaviside 1-D function:

$$H(x) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

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Parameters of the model

The key parameters are

$$\begin{aligned}
\rho_b/\rho_c &: \quad \text{dimensionless density inside the bub.,} \\
\mu_b/\mu_c &: \quad \text{dimensionless viscosity inside the bub.,} \\
\mathrm{Re} = (2R)^{3/2}\sqrt{g}\rho_c/\mu_c &: \quad \text{Reynolds number,} \\
B = 4\rho_c g R^2/\tau &: \quad \text{Bond number,} \\
R &: \quad \text{the initial radius of the bubble,} \\
\mathbf{g}_u &: \quad \text{a unit gravitational force.}
\end{aligned}$$

The dimensionless density and viscosity outside the bubble are equal to 1.

Projection (1/2)

Let $\mathbf{V} = L(\mathbf{u}, \phi)$. It is known that $\exists!$ decomposition of the form:

$$\mathbf{V} = \mathbf{V}_d + \nabla p,$$

where $\mathbf{V}_d$ is div. free. We define a density weighted inner prod., such that

$$\mathbf{V} = \mathbf{V}_d + \nabla p / \rho.$$

Thus, given $\mathbf{V}$, we define the projection as

$$P_\rho(\mathbf{V}) = \mathbf{V}_d.$$

Thus, since this decomposition is unique and $\mathbf{u}_t$ is div. free, we have that $\mathbf{u}_t = P_\rho(L(\mathbf{u}, \phi))$.

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Projection (2/2)

For $\mathbf{u} = (u, v)^t$, we define $\mathrm{curl}(\mathbf{u}) = -v_x + u_y$. Thus, in order to compute the proj., we take the curl of both sides of the eq.

$$\rho \mathbf{V} = \rho \mathbf{V}_d + \nabla p$$

to obtain

$$\mathrm{curl}\ (\rho \mathbf{V}) = \mathrm{curl}\ (\rho \mathbf{V}_d).$$

Given any div. free vector $\mathbf{V}_d$, $\exists$ a stream function Ψ s.t. $\mathbf{V}_d = \nabla^{\perp}\Psi$. Thus, the above eq. can be written as

$$-\mathrm{div}(\rho \nabla \Psi) = \mathrm{curl}\,(\rho L(\mathbf{u}, \phi)). \tag{4}$$

We consider problems obeying the free-slip condition $\mathbf{u} \cdot \mathbf{n} = 0$ on Γ, hence $\Psi = 0$ on Γ.

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Smoothing (1/3)

Since ρ is discontinuous, then the solution of eq. (4) will yield instabilities at the interface. In order to prevent this, we smooth $\rho(\phi)$ at the interface as follow:

$$\begin{aligned} \bar{\rho} &= (\rho_b + \rho_c)/(2\rho_c), \\ \Delta\rho &= (\rho_c - \rho_b)/(2\rho_c), \\ \rho_\alpha(\phi) &= \begin{cases} 1, & \text{if } \phi > \alpha, \\ \rho_b/\rho_c, & \text{if } \phi < -\alpha, \\ \bar{\rho} + \Delta\rho \sin(\pi\phi/(2\alpha)), & \text{otherwise}, \end{cases} \end{aligned} \tag{5}$$

where, α is the thickness of the interface, with $\alpha = O(h)$. Implicit in the above eq. is that ϕ is a distance function. If we maintain ϕ as a distance function, then we approximate $\delta(\phi)$ by:

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$$\delta_\alpha(\phi) = \begin{cases} \frac{1}{2\alpha}(1+\cos(\pi\phi/\alpha)) & \text{if } |\phi| < \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

The contribution of the surface tension to $\mathrm{curl}(\rho L(\mathbf{u}, \phi))$ is

$$-\frac{1}{B}[(\kappa\delta_\alpha(\phi)\phi_y)_x - (\kappa\delta_\alpha(\phi)\phi_x)_y].$$

Thus, if we write $\delta_\alpha(\phi)$ as $\frac{\mathrm{d}H_\alpha(\phi)}{\mathrm{d}\phi}$, the above eq. reduces to

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Smoothing (3/3)

The eq. for H is

$$H_\alpha(\phi) = \begin{cases} \frac{1}{2} & \text{if } \phi > \alpha, \\ -\frac{1}{2} & \text{if } \phi < -\alpha, \\ \frac{1}{2}\left(\frac{\phi}{\alpha} + \frac{1}{\pi}\sin(\pi\phi/\alpha)\right) & \text{otherwise.} \end{cases}$$

Even if eq. (3) will move the level set $\phi = 0$ at the correct velocity, ϕ will no longer be a distance function (ie. $|\nabla\phi| \neq 1$). Consider ϕ_0 whose zero level set is the air-liquid interface (ϕ_0 need not be a distance function). We shall construct ϕ, with the properties that

$$\begin{aligned} \{\phi = 0\} &= \{\phi_0 = 0\}, \\ \phi & \quad \text{is the signed normal distance to the interface.} \end{aligned}$$

This is achieved by solving the following problem to steady state:

$$\begin{aligned} \frac{\partial\phi}{\partial t} &= S(\phi_0)(1 - |\nabla\phi|), \qquad \text{in } \Omega \times (0, T), \\ \phi(\mathbf{x}, 0) &= \phi_0(\mathbf{x}), \qquad \text{in } \Omega, \end{aligned} \tag{7}$$

where S is the sign function, which is smoothed as:

$$S_\varepsilon(\phi_0) = \frac{\phi_0}{\sqrt{\phi_0^2 + \varepsilon^2}}. \tag{8}$$

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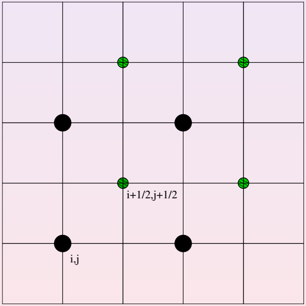
We solve the problem in $\Omega = (0, 7R) \times (0, 7R)$, with R the initial radius of the bubble. The discretization is based on a staggered mesh.



$\mathbf{u}$, ρ, ϕ are computed at the black points of the mesh, while $\mathrm{div}\,\mathbf{u}$, Ψ at the green points.

For the discretization in time, we use a second-order A-B method. For the discretization in space we use the following methods:

- We use a second-order upwinded finite difference scheme for the convection terms.
- In order to define the discrete approx. of the projection, we define discrete divergence and gradient operators and a discrete ρ-weighted inner product.
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- Finally, for the reinitialization procedure, we use a characteristic type method.

Discretization in time

We use a second-order Adams-Bashforth method to evolve the equations in time:

$$
\begin{aligned}
\mathbf{u}^{n+1} &= \mathbf{u}^n + \frac{\Delta t}{2}\left(3P_{\rho^n}(L\mathbf{u}^n) - P_{\rho^{n-1}}(L\mathbf{u}^{n-1})\right) \\
\phi^{n+1} &= \phi^n - \frac{\Delta t}{2}\left(3\mathbf{u}^n \cdot \nabla\phi^n - \mathbf{u}^{n-1} \cdot \nabla\phi^{n-1}\right)
\end{aligned}
$$

Discretization of convective terms (1/5)

We use a second-order ENO (Essentially Non-Oscilatory) method to approximate the convectives terms. Since $\mathbf{u}$ is div. free, we have that

$$\begin{aligned} \mathbf{u}\cdot\nabla\phi &= (u\phi)_x + (v\phi)_y \\ (\mathbf{u}\cdot\nabla)\mathbf{u} &= \mathbf{f}_x + \mathbf{g}_y, \end{aligned}$$

where $\mathbf{f} = (u^2, uv)^T, \quad \mathbf{g} = (uv, v^2)^T$. For the eq. of ϕ, we have

$$\begin{aligned} (u\phi)_x + (v\phi)_y &\approx ((u\phi)_{i+1/2,j} - (u\phi)_{i-1/2,j} \\ &\quad + (v\phi)_{i,j+1/2} - (v\phi)_{i,j-1/2})/(2h) \\ &= (u_{i+1/2,j} + u_{i-1/2,j})(\phi_{i+1/2,j} - \phi_{i-1/2,j})/(2h) \\ &\quad + (v_{i,j+1/2} + v_{i,j-1/2})(\phi_{i,j+1/2} - \phi_{i,j-1/2})/(2h) \\ &\quad + (\phi_{i+1/2,j} + \phi_{i-1/2,j})(u_{i+1/2,j} - u_{i-1/2,j})/(2h) \end{aligned}$$

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Discretization of convective terms (2/5)

For smooth data, we have

$$\phi_{i+1/2,j} + \phi_{i-1/2,j} \approx \phi_{i,j+1/2} + \phi_{i,j-1/2}.$$

Since $\mathbf{u}$ is div. free, we have that

$$u_{i+1/2,j} - u_{i-1/2,j} \approx -(v_{i,j+1/2} - v_{i,j-1/2}).$$

Thus, we find the following appr.

$$\begin{aligned}(u\phi)_x + (v\phi)_y \quad &\approx \quad (u_{i+1/2,j} + u_{i-1/2,j})(\phi_{i+1/2,j} - \phi_{i-1/2,j})/(2h) \\ &\quad +(v_{i,j+1/2} + v_{i,j-1/2})(\phi_{i,j+1/2} - \phi_{i,j-1/2})/(2h).\end{aligned}$$

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Discretization of convective terms (3/5)

Similarly, we have

$$
\begin{aligned}
(f_1)_x + (g_1)_y &\approx (u_{i+1/2,j} + u_{i-1/2,j})(u_{i+1/2,j} - u_{i-1/2,j})/(2h) \\
&\quad + (v_{i,j+1/2} + v_{i,j-1/2})(u_{i,j+1/2} - u_{i,j-1/2})/(2h). \\
(f_2)_x + (g_2)_y &\approx (u_{i+1/2,j} + u_{i-1/2,j})(v_{i+1/2,j} - v_{i-1/2,j})/(2h) \\
&\quad + (v_{i,j+1/2} + v_{i,j-1/2})(v_{i,j+1/2} - v_{i,j-1/2})/(2h).
\end{aligned}
$$

Discretization of convective terms (4/5)

To compute $u_{i+1/2,j}$ (similarly for $u_{i,j+1/2}$, $\phi_{i+1/2,j}$, etc.), we use a second-order ENO scheme.

Define

$$\mathrm{minmod}(a,b) = \begin{cases} a & \text{if } |a| \leq |b|, \\ b & \text{in another case} \end{cases}$$

Let

$$\begin{aligned} u_L &= u_{i,j} + \frac{1}{2}\mathrm{minmod}(u_{i+1,j} - u_{i,j}, u_{i,j} - u_{i-1,j}) \\ u_R &= u_{i,j} - \frac{1}{2}\mathrm{minmod}(u_{i+1,j} - u_{i,j}, u_{i,j} - u_{i-1,j}) \\ u_L &= [(u_{1,j} + u_{2,j})/2 \; ; \; u_L] \\ u_R &= [u_R \; ; \; (u_{N-1,j} + u_{N,j})/2] \end{aligned}$$

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u_R &= u_{i,j} - \frac{1}{2}\mathrm{minmod}(u_{i+1,j} - u_{i,j}, u_{i,j} - u_{i-1,j}) \\
u_L &= [(u_{1,j} + u_{2,j})/2\ ;\ u_L] \\
u_R &= [u_R\ ;\ (u_{N-1,j} + u_{N,j})/2]
\end{aligned}$$

Discretization of convective terms (5/5)

$$u_M = \frac{1}{2}(u_L + u_R)$$

With this, we can define

$$u_{i+1/2,j} = \begin{cases} u_M & \text{if} \quad u_L \leq 0 \text{ and } u_R \geq 0 \\ u_R & \text{if} \quad u_M \leq 0 \text{ and } u_R \leq 0 \\ u_L & \text{if} \quad u_M \geq 0 \text{ and } u_L \geq 0 \end{cases}$$

Viscous and curvature terms

We approximate the components of the viscous stress tensor D using centered differentiating formulae:

$$
\begin{aligned}
(u_x)_{i+1/2,j+1/2} &\approx (u_{i+1,j} - u_{i,j} + u_{i+1,j+1} - u_{i,j+1})/(2h) \\
(u_y)_{i+1/2,j+1/2} &\approx (u_{i+1,j+1} - u_{i+1,j} + u_{i,j+1} - u_{i,j})/(2h)
\end{aligned}
$$

Similarly for v_x, v_y.

The divergence of D is computed as follow:

$$
\begin{aligned}
((\mu D^{m,n})_x)_{i,j} &\approx ((\mu D^{m,n})_{i+1/2,j+1/2} - (\mu D^{m,n})_{i-1/2,j+1/2} \\
&\quad +(\mu D^{m,n})_{i+1/2,j-1/2} - (\mu D^{m,n})_{i-1/2,j-1/2})/(2h) \\
((\mu D^{m,n})_y)_{i,j} &\approx ((\mu D^{m,n})_{i+1/2,j+1/2} - (\mu D^{m,n})_{i+1/2,j-1/2} \\
&\quad +(\mu D^{m,n})_{i-1/2,j+1/2} - (\mu D^{m,n})_{i-1/2,j-1/2})/(2h)
\end{aligned}
$$

where

$$\mu_{i+1/2,j+1/2} = (\mu_{i,j} + \mu_{i+1,j} + \mu_{i,j+1} + \mu_{i+1,j+1})/4.$$

The curvature is discretized in the same fashion as before.

Given $\mathbf{V} = L\mathbf{u}^n$, we decompose $\mathbf{V}$ into the form

$$\mathbf{V} = \mathbf{V}_d + \nabla p/\rho,$$

where $\mathbf{V}_d$ is div. free and define $P_{\rho^n}(L\mathbf{u}^n) = \mathbf{V}_d$ and $\nabla p^n = \nabla p$.
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For the divergence, we have

$$
\begin{aligned}
(\nabla \cdot \mathbf{U})_{i+1/2,j+1/2} &\approx (D\mathbf{U})_{i+1/2,j+1/2} \\
&\equiv (u_{i+1,j+1} - u_{i,j+1} + u_{i+1,j} - u_{i,j})/(2h) \\
&\quad +(v_{i+1,j+1} - v_{i+1,j} + v_{i,j+1} - v_{i,j})/(2h)
\end{aligned}
$$

For the gradient, we have

$$(\nabla\Phi)_{i,j} \approx (\mathbf{G}\Phi)_{i,j} \equiv ((G_x\Phi)_{i,j}, (G_y\Phi)_{i,j})$$

$$
\begin{aligned}
(G_x\Phi)_{i,j} &\equiv (\Phi_{i+1/2,j+1/2} - \Phi_{i-1/2,j+1/2} \\
&\quad +\Phi_{i+1/2,j-1/2} - \Phi_{i-1/2,j-1/2})/(2h) \\
(G_y\Phi)_{i,j} &\equiv (\Phi_{i+1/2,j+1/2} - \Phi_{i+1/2,j-1/2} \\
&\quad +\Phi_{i-1/2,j+1/2} - \Phi_{i-1/2,j-1/2})/(2h)
\end{aligned}
$$

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$$\begin{aligned}(\nabla \cdot \mathbf{U})_{i+1/2,j+1/2} &\approx (D\mathbf{U})_{i+1/2,j+1/2} \\ &\equiv (u_{i+1,j+1} - u_{i,j+1} + u_{i+1,j} - u_{i,j})/(2h) \\ &\quad +(v_{i+1,j+1} - v_{i+1,j} + v_{i,j+1} - v_{i,j})/(2h)\end{aligned}$$

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$$\begin{aligned}(G_x\Phi)_{i,j} &\equiv (\Phi_{i+1/2,j+1/2} - \Phi_{i-1/2,j+1/2} \\ &\quad +\Phi_{i+1/2,j-1/2} - \Phi_{i-1/2,j-1/2})/(2h) \\ (G_y\Phi)_{i,j} &\equiv (\Phi_{i+1/2,j+1/2} - \Phi_{i+1/2,j-1/2} \\ &\quad +\Phi_{i-1/2,j+1/2} - \Phi_{i-1/2,j-1/2})/(2h)\end{aligned}$$

For the inner product, we have

$$(\mathbf{V}_1, \mathbf{V}_2)_\rho \equiv \sum_{i,j=1}^{N} (\mathbf{V}_{1,ij} \cdot \mathbf{V}_{2,ij}) \rho_{i,j}$$

We note that the div. oper. and Φ are defined at the points $(i + 1/2, j + 1/2)$ in terms of the velocities at the neighboring points.

With the above definitions for D and $\mathbf{G}$, the discrete operators are skew-adjoints, i.e. $\mathbf{G} = -D^T$.

Using this defs., we have that discretely divergence free vector fields with zero normal components are orthogonal to discrete vector fields of the form $\mathbf{G}\Phi/\rho$. Then, we can uniquely decompose any discrete vector field into $\mathbf{U} + \mathbf{G}\Phi/\rho$, where $D\mathbf{U} = 0$.

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Using this defs., we have that discretely divergence free vector fields with zero normal components are orthogonal to discrete vector fields of the form $\mathbf{G}\Phi/\rho$. Then, we can uniquely decompose any discrete vector field into $\mathbf{U} + \mathbf{G}\Phi/\rho$, where $D\mathbf{U} = 0$.

In 2-D, a div. free vector can be written as the orthogonal gradient of a scalar field Ψ. Define a discrete function Ψ at the points $(i + 1/2, j + 1/2)$. Moreover, define $\mathbf{G}^{\perp}\Phi$ as the discrete orthogonal gradient operator, e.g. $\mathbf{G}^{\perp}\Phi = (-G_y\Phi, G_x\Phi)$ and curl $(\mathbf{U})$ as the discrete curl, i.e. $\mathrm{curl}\,(\mathbf{U}) = -G_x(U_2) + G_y(U_1)$.
Then we have

$$\begin{aligned} \mathbf{G}^{\perp}\Psi + \mathbf{G}p/\rho &= \mathbf{V} \\ \mathrm{curl}\,(\rho\mathbf{G}^{\perp}\Psi) &= \mathrm{curl}\,(\rho\mathbf{V}) \\ -G_x(\rho G_x\Psi) - G_y(\rho G_y\Psi) &= G_y(\rho V_1) - G_x(\rho V_2). \end{aligned}$$

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$$
\begin{aligned}
\mathbf{G}^{\perp}\Psi + \mathbf{G}p/\rho &= \mathbf{V} \\
\mathrm{curl}\,(\rho\mathbf{G}^{\perp}\Psi) &= \mathrm{curl}\,(\rho\mathbf{V}) \\
-G_x(\rho G_x\Psi) - G_y(\rho G_y\Psi) &= G_y(\rho V_1) - G_x(\rho V_2).
\end{aligned}
$$

A centered five-points formula for the above eq. is

$$-\left[\rho_{i,j}\Psi_{i-1/2,j-1/2}+\rho_{i,j+1}\Psi_{i-1/2,j+3/2}+\rho_{i+1,j}\Psi_{i+3/2,j-1/2}\right.$$
$$\left.+\rho_{i+1,j+1}\Psi_{i+3/2,j+3/2}-(\rho_{i,j}+\rho_{i+1,j}+\rho_{i,j+1}+\rho_{i+1,j+1})\Psi_{i+1/2,j+1/2}\right]$$
$$=G_y(\rho V_1)-G_x(\rho V_2).$$

We note that the matrix of the above system is positive defined. Once Ψ is known, we can set $\mathbf{U}\equiv\mathbf{G}^{\perp}\Psi$.

Given $\phi_0(x)$, we want to construct ϕ s.t. its zero-levet set coincides with the zero-level set of $\phi_0(x)$ and s.t. ϕ be the normal distance to the interface.

This is achieved by solving the following problem *to steady state:*

$$\frac{\partial \phi}{\partial t} = S(\phi_0)(1 - |\nabla \phi|), \quad (9)$$

$$\phi(\mathbf{x}, 0) = \phi_0(\mathbf{x}), \quad (10)$$

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We can rewrite eq. (9) as follow:

$$\frac{\partial \phi}{\partial t} + \mathbf{w} \cdot \nabla \phi = S(\phi_0),$$

where $\mathbf{w} = S(\phi_0)(\nabla \phi / |\nabla \phi|)$. The characteristics curves of the above eq. are given by $\mathbf{w}$, which is a unit vector pointing outward from the zero-level set $\{\phi = 0\}$.

A possible discretization is as follow. Define

$$a = D_-^x \phi_{i,j}, \quad b = D_+^x \phi_{i,j},$$
$$c = D_-^y \phi_{i,j}, \quad d = D_+^y \phi_{i,j},$$

$$S_\varepsilon(\phi)_{i,j} = \phi_{i,j} / \sqrt{\phi_{i,j}^2 + \varepsilon^2}.$$

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$$S_\varepsilon(\phi)_{i,j} = \phi_{i,j} / \sqrt{\phi_{i,j}^2 + \varepsilon^2}.$$

$$G(\phi)_{i,j} = \begin{cases} \sqrt{\max((a^+)^2,(b^-)^2) + \max((c^+)^2,(d^-)^2)} - 1, & \text{if } \phi^0_{i,j} > 0 \\ \sqrt{\max((a^-)^2,(b^+)^2) + \max((c^-)^2,(d^+)^2)} - 1, & \text{if } \phi^0_{i,j} < 0 \\ 0, & \text{i.a.c.} \end{cases}$$

where the superscript $+$ denotes the positive part and $-$ the negative part.

Eq. (9) is then discretizaded using

$$\phi^{n+1}_{i,j} = \phi^n_{i,j} - \Delta t S_\varepsilon(\phi^0_{i,j}) G(\phi^n_{i,j})$$

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Summary

Step 1. Initialize $\phi(\mathbf{x}, 0)$ to be signed normal distance to the front.

Step 2. Solve

$$\mathbf{u}_t = P_\rho(L\mathbf{u}), \quad \phi_t + \mathbf{u} \cdot \nabla\phi = 0.$$

for one time step with $\rho(\phi)$ given by (5). Denote the updated ϕ by $\phi^{(n+1/2)}$, and the updated $\mathbf{u}$ by $\mathbf{u}^{(n+1)}$.

Step 3. Construct a new distance function by solving

$$\phi_t = S(\phi^{(n+1/2)})(1 - |\nabla\phi|), \quad \phi(\mathbf{x}, 0) = \phi^{(n+1/2)}(\mathbf{x}),$$

to steady state. We denote the steady state solution by $\phi^{(n+1)}$.

Step 4. We have now advanced one time step. The zero level set of $\phi^{(n+1)}$ gives the new interface position and $\phi^{(n+1)}$ is a distance function. Repeat Steps 2 and 3.

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to steady state. We denote the steady state solution by $\phi^{(n+1)}$.

Step 4. We have now advanced one time step. The zero level set of $\phi^{(n+1)}$ gives the new interface position and $\phi^{(n+1)}$ is a distance function. Repeat Steps 2 and 3.

Summary

Step 1. Initialize $\phi(\mathbf{x}, 0)$ to be signed normal distance to the front.

Step 2. Solve

$$\mathbf{u}_t = P_\rho(L\mathbf{u}), \quad \phi_t + \mathbf{u} \cdot \nabla\phi = 0.$$

for one time step with $\rho(\phi)$ given by (5). Denote the updated ϕ by $\phi^{(n+1/2)}$, and the updated $\mathbf{u}$ by $\mathbf{u}^{(n+1)}$.

Step 3. Construct a new distance function by solving

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