Morozov's Discrepancy Principle for Inverse Problems

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Outline

Problem Setting

Definition

Let $F : \mathcal{D}(F) \subset X \to Y$ be a weakly continuous (non-linear) operator between a reflexive Banach space X and a Banach space Y. We are concerned with finding approximate solutions x of ill-posed problems of the form

$$F(x) = y, \tag{1}$$

where only noisy data y^δ with $\|y - y^\delta\| \leq \delta$ is available.

Ill-posedness ...

- The solution x might not be unique.
- The solution x does not depend continuously on the data y (even a small error in y^δ may have huge effects on x).

Classical Tikhonov Regularization

For $\alpha > 0$ a regularized solution x_α^δ is a minimizer of the Tikhonov functional

$$J_\alpha(x) = \left\|F(x) - y^\delta\right\|^2 + \alpha \left\|x\right\|^2 \to \min .$$

- We assume to know a-priori that $\left\|x\right\|^2$ is small for the solution we are interested in. $\longrightarrow$ Oversmoothing of edges/discontinuities.

Tikhonov-type Regularization

For $\alpha > 0$ a regularized solution x_α^δ is a minimizer of the Tikhonov functional

$$J_\alpha(x) = \|F(x) - y^\delta\|^2 + \alpha\Psi(x) \to \min .$$

- We assume to know a-priori that $\Psi(x)$ is small for the solution we are interested in.

Condition

Let $\Psi : \mathcal{D}(\Psi) \subset X \to \mathbb{R}^+$ be a convex functional such that

(i) $0 \in \mathcal{D}(\Psi)$ and $\Psi(x) = 0$ if and only if $x = 0$,

(ii) Ψ is weakly lower semi continous (wrt $\|.\|$ on X),

(iii) Ψ is weakly coercive (ie. $\|x_n\| \to \infty \implies \Psi(x_n) \to \infty$).

Generalized Tikhonov Regularization

Notation

Our regularized solutions $x_\alpha^\delta \in \mathcal{R}_\alpha$ are the minimizers of the Tikhonov-type functionals

$$J_\alpha(x) = \|F(x) - y^\delta\|^2 + \alpha\, \Psi(x),$$

where $\alpha > 0$.
We call $x^\dagger \in \mathcal{S} \neq \emptyset$ a Ψ-minimizing solution if $F(x^\dagger) = y$ and

$$\Psi(x^\dagger) = \min\, \{\Psi(x) \mid x \text{ s.t. } F(x) = y\}.$$

Questions

- How do we choose α?
- What happens as $\delta \to 0$?

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Questions

- How do we choose α? A-priori: $\alpha(\delta) \sim \delta$?
- What happens as $\delta \to 0$? Convergence (in which topology)?

MDP

Definition

When using Morozov's discrepancy principle (MDP) as a parameter choice rule, we choose $\alpha = \alpha(\delta, y^\delta)$ and $x_\alpha^\delta \in \mathcal{R}_\alpha$ such that for constants $1 < \tau_1 \leq \tau_2$,

$$\tau_1 \delta \leq \|F(x_\alpha^\delta) - y^\delta\| \leq \tau_2 \delta.$$

Theorem

Let $\alpha = \alpha(\delta, y^\delta)$ and $x_\alpha^\delta \in \mathcal{R}_\alpha$ be chosen according to MDP. Then

$$x_\alpha^\delta \xrightarrow{\Psi} \mathcal{S} \qquad \text{as} \quad \delta \to 0,$$

where

$$x_n \xrightarrow{\Psi} x \iff \Psi(x_n - x) \to 0.$$

Regularization properties

Condition

For all $x^\dagger \in \mathcal{S}$ we assume that

$$\liminf_{t \to 0^+} \frac{\left\| F((1-t)x^\dagger) - y \right\|^2}{t} = 0.$$

Note. We need $(1-t)x^\dagger \in \mathcal{D}(F)$, which is satisfied if $\mathcal{D}(F)$ is star-shaped (esp. convex), $B_\varepsilon(x^\dagger) \subset \mathcal{D}(F)$, ...

Lemma

Let X be a Hilbert space. If $F(x)$ is continuously Fréchet differentiable, then the above condition is always satisfied.

Lemma

Assume that there exist $\alpha > 0$ and a solution $x^\star$ of $F(x) = y$ such that

$$x^\star = \arg\min_{x \in X} \left\{ \|F(x) - y\|^2 + \alpha \Psi(x) \right\},$$

then $x^\star = 0$.

Theorem

To every $\delta \in (0, \delta^*)$ let y^δ be such that MDP is applicable, then

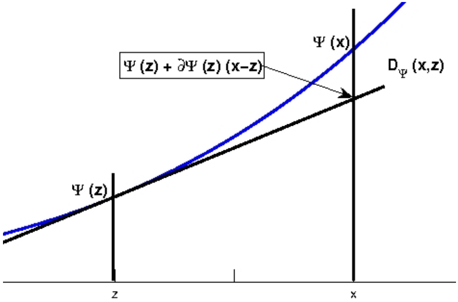
$$\alpha(\delta, y^\delta) \to 0 \qquad \text{and} \qquad \frac{\delta^2}{\alpha(\delta, y^\delta)} \to 0 \qquad \text{as} \quad \delta \to 0.$$

Bregman Distance

Definition

Let $\partial\Psi(x) \subset X^*$ denote the subdifferential of Ψ at $x \in X$. The generalized Bregman distance of $x, z \in X$ wrt Ψ and $\xi \in \partial\Psi(z)$ is defined as

$$D_\Psi^\xi(x,z) = \Psi(x) - \Psi(z) - \langle \xi, x - z \rangle$$



Convergence Rates

Theorem

To all $0 < \delta < \delta^*$ *let* y^δ *be suitable and* $\alpha = \alpha(\delta, y^\delta)$ *according to MDP. Then, under source and nonlinear conditions it holds that*

$$\|F(x_\alpha^\delta) - F(x^\dagger)\| = \mathcal{O}(\delta), \qquad D_\Psi^\xi(x_\alpha^\delta, x^\dagger) = \mathcal{O}(\delta) \quad \text{as} \quad \delta \to 0.$$

[3] M. Burger, S. Osher. Convergence Rates of Convex Variational Regularization. (2004)

[4] E. Resmerita, O. Scherzer. Error estimates for non-quadratic regularization and the relation to enhancement. (2006)

Thank you for your attention!